ITERATIVE TECHNIQUE FOR NON LINEAR INTEGRAL EQUATION WITH ERROR TERMS IN BANACH SPACE

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ABSTRACT

In this paper, we use partial order theory to study nonlinear integral equations in general Banach spaces, existence and uniqueness of solution are obtained and error estimate of the iterative sequences of approximation solution is given. As an application, we utilize main results presented in this paper to study the existence and uniqueness problems of weakly Caratheodory’s solution for a class of nonlinear differential equations

KEY WORDS

Banach space, Partial ordering theory, Integral equations, Differential equations, Iterative technique.

INTRODUCTION

Consider the non linear Integral equations of the form

\[ u(t) \leq h(t) + \int_0^t k(t,s)[f(s,u(s)) + Mu(s)]ds \rightarrow (1.1) \]

\[ u(t) \leq h(t,u) + \int_0^t k(t,s)[f(s,u(s)) + Mu(s)]ds \rightarrow (1.2) \]

Where \( M > 0, k(t,s) \in C[D,R^+] \), \( h(t) \in C[I,E] \), \( I[1,0] \), \( R^+ = [0;+\infty] \), \( R=(-\infty,\infty) \),

\( D= \{ \ (t,s) \in R^2 \ / \ 0 \leq s \leq t \leq 1 \} \), \( h(t,u) \in C[I\times E, E] \), \( E \) is a real Banach space with \( I \). And \( f(t,u) \) satisfies the weak Carathéodory conditions.
Solution of the integral equation (1.1) can be obtained by the iterative sequences and the error estimate of the iterative sequences of approximation solution is given. As an application we prove that the existence and uniqueness problem of weakly Caratheodory’s solution for a class of non-linear differential equation in Banach space.

\[
\begin{align*}
  u' &= f(t,u) \\
  u(0) &= x_0
\end{align*}
\]

(1.3)

In control theory, an open loop input–output system is described in many circumstances by the integral relationship

\[
  x(t) = x_0(t) + \int_0^t k(t,s)u(s)ds \geq 0
\]

(1.4)

Where \( u \) stands for the input and \( x \) stands for the output.

The system (1.4) is controlled by the feedback law,

\[
  u(t) = f(t,x(t))
\]

(1.5)

Implies

\[
  x(t) = x_0(t) + \int_0^t k(t,s)f(s,x(s))ds
\]

(1.6)

PRELIMINARIES

Let \( E \) be a real Banach space and \( P \) a cone in \( E \). The order is introduced by cone \( P \).

A cone \( P \) is said to be normal if there exists a constant \( N > 0 \) such that \( x, y \in E \). \( N \) is called the normal constant of \( P \).

Let the real Banach space \( E \) be partially ordered by a cone \( P \) of \( E \), i.e. \( x < y \) iff \( y-x \in P \). Recall that cone \( P \) is said to be solid if its interior \( \text{int}(P) \) is not empty. In this case, we write \( x << t \) and \( t \).

MAIN RESULTS

Theorem: 1

Let \( u_0, v_0 \in [I,E] \) such that \( u_0 < v_0 \) and if \( f : I \times \Omega \rightarrow E \). Suppose that the Conditions (i),(ii) and (iii) are satisfied. Then the integral equation (1.1) has a unique solution \( \bar{x} \) in \( [u_0,v_0] \) and the iterative sequence for any \( x_0(t) \in [u_0,v_0] \).
\[ x_n(t) = h(t) + \int_0^t k(t, s) [f(s, x_{n-1}(s)) + M x_{n-1}(s)] ds \quad n = 1, 2, 3, \ldots \rightarrow (1.7) \]

**Theorem 1.1**

Let \( u_0, v_0 \in C[I, E] \) such that \( u_0 < v_0 \) and if \( f : I \times \Omega \rightarrow E \).

Suppose that

(i) \( f(t, s) : I \times \Omega \rightarrow E \) satisfies the weak caratheodory conditions and suppose that

\[ f(t, u_0(t)), f(t, v_0(t)) \in L_p[I, E] ; \]

(ii) There exist constants \( L > 0, \beta \in [I, E] \) such that

\[ -\beta (v - u) \leq f(t, v) - f(t, u) \leq L(v - u) \text{ for } u_0(t) \leq u \leq v \leq v_0(t), t \in I \text{ and} \]

(iii) \( u_0(t) \leq h(t, u(t)) + \int_0^t k(t, s) [f(s, u_0(s)) + M u_0(s)] ds \)

\[ v_0(t) \geq h(t, v(t)) + \int_0^t k(t, s) [f(s, v_0(s)) + M v_0(s)] ds \]

Then the integral equation (1.1) has a unique solution \( u \) in \([u_0, v_0]\) and the iterative technique for any \( x_0(t) \in [u_0, v_0] \).

\[ x_n(t) = h(t, x(t)) + \int_0^t k(t, s) [f(s, x_{n-1}(s)) + M x_{n-1}(s)] ds \quad n = 1, 2, 3, \ldots \rightarrow (1.8) \]

Converges uniformly to \( u \) on \( I \) have the following error estimate.

\[ \| x_n - u \|_c \leq \frac{2N[K + N K_0(M + L)]}{n!} \| v_0 - u_0 \| c, n = 1, 2, 3, \ldots \]

where \( k_0 = \max \{ k(t, s) / (t, s) \in D \} \)

**Proof:**

For any \( u(t) \in [u_0, v_0] \).

We define the mapping \( A \) by

\[ Au(t) = h(t, u(t)) + \int_0^t k(t, s) [f(s, u(s)) + M u(s)] ds \]
\[ A v(t) = h(t, v(t)) + \int_{0}^{t} k(t, s)[f(s, v(s)) + M v(s)] ds \]

It is easy to see the condition (i) that

\[ A : [u_0, v_0] \rightarrow C[I, E] \text{ and } u(t) \text{ is a solution of equation (1.1) iff } u \text{ is a fixed point of } A. \text{ i.e., } u = Au. \]

Now to prove that \( A : [u_0, v_0] \rightarrow [u_0, v_0] \) is the increasing operator.

Let \( u, v \in [u_0, v_0] \) such that \( v \geq u \)

From (i) and (ii)

\[ \text{By condition (ii), } A v(t) - A u(t) \geq [h(t, v(t)) - h(t, u(t))] + \int_{0}^{t} k(t, s)[(M - \beta)v(s) - u(s)] ds \geq 0 \]

Therefore A is an increasing operator on \( [u_0, v_0] \).

By condition (iii), \( u_0 \leq Au_0, Av_0 \leq v_0 \)

Thus \( A : [u_0, v_0] \rightarrow [u_0, v_0] \)

Set \( u_n = Au_{n-1}, v_n = Av_{n-1} n = 1, 2, 3 \ldots \)

By induction and the monotonicity of A,

\[ u_0 \leq u_1 \leq \ldots \leq u_n \leq \ldots \leq v_n \leq \ldots \leq v_1 \leq v_0 \rightarrow (1.9) \]

By (3.20) and condition (ii),

\[ 0 \leq v_n(t) - u_n(t) = Av_{n-1}(t) - Au_{n-1}(t), n = 1, 2, 3 \ldots \]

\[ v_n - u_n \leq [h(t, v_{n-1}) - h(t, u_{n-1})] + \int_{0}^{t} k(t, s)[(M + L)[v_{n-1}(s) - u_{n-1}(s)]] ds, n = 1, 2, 3 \ldots \]

Substituting \( n = 1, 2, 3 \ldots \)
For any positive integer m,
\[
\|v_n(t) - u_n(t)\| \leq \left[ \frac{k + M k_0 (M + L)}{n!} \right]^n t^n \|v_0 - u_0\|
\]

\[
\|v_n - u_n\| \leq \left[ \frac{k + M k_0 (M + L)}{n!} \right]^n \|v_0 - u_0\| \quad n = 1, 2, 3, \\
\Rightarrow \{u_n\}, \{v_n\} \text{ are Cauchy sequence in } C[I, E], \text{ there exist } u^*, v^* \in C[I, E] \text{ such that}
\]

\[
u_n(t) \to u^*(t), v_n(t) \to v^*(t) \text{ as } n \to \infty \text{ uniformly on } t \in I
\]

\[
\text{obviously } u_n \leq u^* \leq v_n \quad (n = 1, 2, \ldots)
\]

\[
\Rightarrow u^* = v^* = \bar{u} \in [u_0, v_0]
\]

By the monotonicity of A, \( \bar{u} = A \bar{u} \)

This implies that \( \bar{u} \) is a solution of (1.1)

Similarly \( \bar{v} \) is also a solution.

\[
\Rightarrow \bar{v} = \bar{u}
\]

Therefore \( \bar{u}(t) \) is a unique solution, by induction \( u_n \leq x_n \leq v_n \)

\[
\|x_n - \bar{u}\| \leq \|v_n - u_n\| + \|u_n - \bar{u}\|
\]

\[
\leq \|v_n - u_n\| + \|v_n - u_n\| \leq \left[ \frac{2N \left[ K + M k_0 (M + L) \right]}{n!} \right] \|v_0 - u_0\| \quad n = 1, 2, 3, \\
\leq 2N \|v_n - u_n\|
\]

This is the error estimate.

Theorem:1.3
Let \( u_0, v_0 \in [I, E] \) such that \( u_0 \prec v_0 \), Suppose that the conditions (i),(ii) and (iii) are satisfied and suppose that

\[
\text{(H)} \quad u_0'(t) \leq f(t, u_0(t)) \leq v_0'(t) \leq f(t, v_0(t)) \leq x_0
\]

Then the initial value problem (3.3) has a unique weak caratheodory’s solution \( w^* \) in \([u_0, v_0]\)

\[
w_n(t) = e^{-M t} x_0 + \int_0^t \left[ f(s, w_{n-1}(s)) + M \, w_{n-1}(s) \right] e^{-M s} ds \quad n = 1, 2, 3, \ldots \rightarrow (2)
\]

Converges uniformly to \( w^* \) on\( I \) and have the following error estimate

\[
\|w_n - w^*\| \leq \frac{2N \left[ N(M + L) \right]^n}{n!} \|v_0 - u_0\|, \quad n = 1, 2, 3, \ldots \rightarrow (2.1)
\]

Proof:

The IVP \( u'(t) = f(t, u), \quad u(0) = x_0 \) is equivalent to the integral equation

\[
u(t) = e^{-M t} x_0 + \int_0^t \left[ f(s, u(s)) + M \, u(s) \right] e^{-M (t-s)} ds \quad \rightarrow (2.2)
\]

\[
u(t)e^{M t} = x_0 + \int_0^t \left[ f(s, u(s)) + M \, u(s) \right] e^{M s} ds
\]

\[
[u(t)e^{M t}] = [f(t, u(t)) + M \, u(t)] e^{M t}
\]

\[
[u(t)e^{M t}] \leq [f(t, u_0(t)) + M \, u_0(t)] e^{M t}, \quad u_0(0) \leq x_0 \quad \rightarrow (2.3)
\]

Similarly

\[
[v(t)e^{M t}] \leq [f(t, v_0(t)) + M \, v_0(t)] e^{M t}, \quad v_0(0) \leq x_0 \quad \rightarrow (2.4)
\]

\Rightarrow \text{the conditions (i) and (ii) are satisfied.}
For any \( u(t) \in [u_0, v_0] \)

\[
Au(t) = e^{-Mt} x_0 + \int_0^t \left[ f(s, u(s)) + M u(s) \right] e^{-M(t-s)} ds
\]

\( \Rightarrow Au = u \)

Then by condition (ii) of theorem 1.3,

\[
Av(t) - Au(t) = \int_0^t \left[ f(s, v(s)) + f(s, u(s)) \right] + M [v(s) - u(s)] e^{-M(t-s)} ds
\]

\[
Av(t) - Au(t) \geq (M - \beta) \int_0^t e^{-M(t-s)} [v(s) - u(s)] ds \geq 0
\]

\( \Rightarrow A \) is the increasing operator on \([u_0, v_0]\).

Then by condition (iii) of theorem (1.3)

\[
v_n(t) - u_n(t) \leq (M + L) \int_0^t [v_{n-1}(s) - u_{n-1}(s)] ds, n = 1, 2, 3, ...
\]

By the normality of the cone \( P \),

\[
\left\| v_n - u_n \right\| \leq \frac{[N(M + L)]^n}{n!} \left\| v_0 - u_0 \right\|, n = 1, 2, 3, ...
\]

As \( n \to \infty \),

\[
u_n = Au_{n-1} \leq Aw^* \leq Av_{n-1} = v_n, n = 1, 2, 3, ...
\]

\[
w^* \leq Aw^* \leq w^* \Rightarrow v^* = w^*
\]

\[
\left\| v_n - w^* \right\| \leq 2N \left\| v_0 - u_0 \right\|, n = 1, 2, 3, ...
\]
This is the error estimate.

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