ITERATIVE TECHNIQUE FOR NON LINEAR INTEGRAL EQUATION WITH ERROR TERMS IN BANACH SPACE

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ABSTRACT

In this paper, we use partial order theory to study nonlinear integral equations in general Banach spaces, existence and uniqueness of solution are obtained and error estimate of the iterative sequences of approximation solution is given. As an application, we utilize main results presented in this paper to study the existence and uniqueness problems of weakly Caratheodory's solution for a class of nonlinear differential equations

KEY WORDS

Banach space, Partial ordering theory, Integral equations,Differential equations, Iterative technique.

INTRODUCTION

Consider the non linear Integral equations of the form

$$u(t) \leq h(t) + \int_0^t k(t,s)\left[f(s,u(s)) + M\,u(s)\right]ds \quad ------\rightarrow (1.1)$$

$$u(t) \leq h(t,u) + \int_0^t k(t,s)\left[f(s,u(s)) + M\,u(s)\right]ds \; ------\rightarrow (1.2)$$

Where M > 0, $k(t,s) \epsilon C[D,R^+]$, $h(t) \epsilon C[I,E]$, I[1,0], $R^+ = [0;+\infty]$,

$R=(-\infty,\infty)$,

$D=\{ (t,s) \epsilon R^2 / \; 0 \leq s \leq t \leq 1\}$, $h(t,u) \epsilon C[I\times E, E]$, E is a real Banach space with $\| \, . \, \|$and f(t,u) satisfies the weak carathedory conditions.

Solution of the integral equation (1.1) can be obtained by the iterative sequences and the error estimate of the iterative sequences of approximation solution is given. As an application we prove that the existence and uniqueness problem of weakly Caratheodory's solution for a class of non linear differential equation in Banach space.

$$\begin{aligned} u' &= f(t,u) \\ u(0) &= x_0 \end{aligned} \quad ------\rightarrow \quad (1.3)$$

In control theory,an open loop input – output system is described in many circumstances by the integral relationship

$$x(t) = x_0(t) + \int_0^t k(t,s)u(s)ds \quad t \geq 0 \quad ------\rightarrow (1.4)$$

Where u stands for the input and x stands for the output.

The system (1.4) is controlled by the feedbacklaw,

$$u(t) = f(t, x(t)) ------\rightarrow (1.5)$$

Implies

$$x(t) = x_0(t) + \int_0^t k(t,s) f(s, x(s))ds \quad ------\rightarrow (1.6)$$

PRELIMINARIES

Let E be a real Banach space and P a cone in E. The order is introduced by cone P,

A cone P is said to be normal if there exists a constant N > 0 such that x, y E . N is called the normal constant of P.

Let the real Banach space E be partially ordered by a cone P of E, i.e. x_< y iff y-x E P. Recall that cone P is said to be solid if its interior int(P) is not empty. In this case, we write x<<tl<...<tk<...< t.

MAIN RESULTS

Theorem: 1

$Let\ u_0, v_0 \in [I, E] such\ that\ u_0 \prec v_0\ and\ if\ f : I \times \Omega \rightarrow E.$ Suppose that the Conditions (i),(ii) and (iii) are satisfied . Then the integral equation (1.1) has a unique solution

$\bar{u}\ in\ [u_0, v_0]$and the iterative sequence for any $x_0(t) \in [u_0, v_0]$.

$$x_n(t)=h(t)+\int_0^t k(t,s)[f(s,x_{n-1}(s))+M\,x_{n-1}(s)]ds \quad n=1,2,3\ldots\ldots------\to (1.7)$$

Theorem:1.1

Let $u_0,v_0\in C[I,E]$ *such that* $u_0\prec v_0$ *and if* $f:I\times\Omega\to E.$

Suppose that

(i) $f(t,s):I\times\Omega\to E$ satisfies the weak caratheodory conditions and suppose that $f(t,u_0(t)),f(t,v_0(t))\in L_P\,[I,E]$;

(ii) There exist constants $L>0$ $\beta\in\lfloor I,E\rfloor$.such that

$$-\beta(v-u)\le f(t,v)-f(t,u)\le L(v-u)\,\text{for}\,u_0(t)\le u\le v\le v_0(t),t\in I \text{ and}$$

(iii) $u_0(t)\le h(t,u(t))+\int_0^t k(t,s)[f(s,u_0(s))+M\,u_0(s)]ds$

$$v_0(t)\ge h(t,v(t))+\int_0^t k(t,s)[f(s,v_0(s))+M\,v_0(s)]ds$$

Then the integral equation (1.1) has a unique solution u in $[u_0, v_0]$and

the iterative technique for any $x_0(t)\in[u_0,v_0]$.

$$x_n(t)=h(t,x(t))+\int_0^t k(t,s)[f(s,x_{n-1}(s))+M\,x_{n-1}(s)]ds \quad n=1,2,3\ldots\ldots------\to (1.8)$$

Converges uniformly to u on I have the following error estimate.

$$\|x_n-u\|_c\le\frac{2N[K+N\,K_0\,(M+L)]^n}{n!}\|v_0-u_0\|c,n=1,2,3\ldots\ldots\ldots\ldots$$

$$where\ k_0=\max\left\{\ k(t,s)\middle/(t,s)\in D\right.$$

roof:

For any $u(t)\in[u_0,v_0]$.

We define the mapping A by

$$Au(t)=h(t,u(t))+\int_0^t k(t,s)[f(s,u(s))+M\,u(s)]ds$$

$$Av(t) = h(t, v(t)) + \int_0^t k(t,s)[f(s, v(s)) + M\, v(s)]ds$$

It is easy to see the condition (i) that

$A:[u_0, v_0] \to C[I, E]$ andu(t) is a solution of equation (1.1) iff u is a fixed

point of A.ie, u = Au.

Now to prove that $A:[u_0, v_0] \to [u_0, v_0]$ is the increasing operator.

Let $u, v \in [u_0, v_0]$. such that $v \geq u$

From (i) and (ii)

By condition (ii) , $Av(t) - Au(t) \geq [h(t, v(t)) - h(t, u(t))] + \int_0^t k(t,s)[(M - \beta)[v(s) - u(s)]]ds \geq 0$

Therefore A is an increasing operator on $[u_0, v_0]$.

By condition (iii), $u_0 \leq Au_0$, $Av_0 \leq v_0$

Thus $A:[u_0, v_0] \to [u_0, v_0]$

Set $u_n = Au_{n-1}$, $v_n = Av_{n-1}$ $n = 1,2,3.......$

By induction and the monotonicity of A,

$$u_0 \leq u_1 \leq .... \leq u_n \leq v_n \leq .... \leq v_1 \leq v_0 ---------\to (1.9)$$

By (3.20) and condition (ii),

$$0 \leq v_n(t) - u_n(t) = Av_{n-1}(t) - Au_{n-1}(t), \quad n = 1,2,3.......$$

$$v_n - u_n \leq [h(t, v_{n-1}) - h(t, u_{n-1})] + \int_0^t k(t,s)[(M + L)[v_{n-1}(s) - u_{n-1}(s)]]ds, n = 1,2,3.....$$

Substituting n= 1, 2, 3…..

$$\|v_n(t)-u_n(t)\| \leq \frac{[\ k+M\,k_0\,(M+L)\ ]^n}{n!}\,t^n\,\|v_0-u_0\|$$

For any positive integer m,

$$\|v_n-u_n\| \leq \frac{[\ k+M\,k_0\,(M+L)\ ]^n}{n!}\,\|v_0-u_0\|\ \ n=1,2,3............$$

$\Rightarrow \{u_n\},\{v_n\}$ are Cauchy sequence in C[I,E], there exist $u^*, v^* \in C[I,E]$ such that

$$u_n(t) \to u^*(t), v_n(t) \to v^*(t)\, as\, n \to \infty\, uniformly\, on\ t \in I$$

$$obviously\, u_n \leq u^* \leq v^* \leq v_n\ (n=1,2...)$$

$$\Rightarrow u^* = v^* = \overline{u} \in [u_0, v_0]$$

By the monotonicity of A , $\overline{u} = A\overline{u}$

This implies that $\overline{u}$ is a solution of (1.1)

Similarly $\overline{v}$ is also a solution.

$$\Rightarrow \overline{v} = \overline{u}$$

Therefore $\overline{u}(t)$ is a unique solution, by induction $u_n \leq x_n \leq v_n$

$$\|x_n-\overline{u}\|_c \leq \ \|x_n-u_n\|_c\ \ \|u_n-\overline{u}\|_c$$

$$\leq\ \|v_n-u_n\|_c + \|v_n-u_n\|_c\ \ \|x_n-\overline{u}\|_c \leq \frac{2N\,[\ K+M\,k_0\,(M+L)\ ]^n}{n!}\,\|v_0-u_0\|_c\ \ n=1,2,3............$$

$$\leq 2N\,\|v_n-u_n\|_c$$

This is the error estimate.

Theorem:1.3

$Let\ u_0, v_0 \in [I, E] such\ that\ u_0 \prec v_0$, Suppose that the conditions (i),(ii) and (iii) are satisfied and suppose that

(H) $u_0'(t) \le f(t, u_0(0)) \le x_0$, $v_0'(t) \le f(t, v_0(0)) \le x_0$

Then the initial value problem (3.3) has a unique weak caratheodory's solution w^* in $[u_0, v_0]$

$$w_n(t) = e^{-Mt}[x_0 + \int_0^t [f(s, w_{n-1}(s)) + M\, w_{n-1}(s)]e^{-Ms} ds\ ]\ n = 1,2,3...... ------ \rightarrow (2)$$

Converges uniformly to w^* onI and have the following error estimate

$$\|w_n - w^*\|_c \le \frac{2N\,[\ N(M+L)\ ]^n}{n!}\|v_0 - u_0\|_c\ \ n = 1,2,3.......... ------ \rightarrow (2.1)$$

Proof:

The IVP $u'(t) = f(t, u)$, $u(0) = x_0$ is equivalent to the integral equation

$$u(t) = e^{-Mt}\, x_0 + \int_0^t [f(s, u(s)) + M\, u(s)]e^{-M(t-s)} ds\ ] \qquad ------ \rightarrow (2.2)$$

$$u(t)e^{Mt} = x_0 + \int_0^t [f(s, u(s)) + M\, u(s)]e^{Ms} ds\ ]$$

$$[u(t)e^{Mt}]' = [f(t, u(t)) + M\, u(t)]e^{Mt}$$

$$[u(t)e^{Mt}]' \le [f(t, u_0(t)) + M\, u_0(t)]e^{Mt}, u_0(0) \le x_0 \qquad .------ \rightarrow (2.3)$$

Similarly

$$[v(t)e^{Mt}]' \le [f(t, v_0(t)) + M\, v_0(t)]e^{Mt}, v_0(0) \le x_0 \quad ------ \rightarrow (2.4)$$

$\Rightarrow$the conditions (i) and (ii) are satisfied.

For any $u(t) \in [u_0, v_0]$

$$Au(t) = e^{-Mt} x_0 + \int_0^t [f(s,u(s)) + M u(s)] e^{-M(t-s)} ds \;]$$

$\Rightarrow$ Au=u

Then by condition (ii) of theorem 1.3,

$$Av(t) - Au(t) = \int_0^t [[f(s,v(s)) + f(s,u(s))] + M [v(s) - u(s)]] e^{-M(t-s)} ds$$

$$Av(t) - Au(t) \geq (M - \beta) \int_0^t e^{-M(t-s)} [v(s) - u(s)] ds \geq 0$$

$\Rightarrow$ A is the increasing operator on $[u_0, v_0]$.

Then by condition (iii) of theorem (1.3)

$$v_n(t) - u_n(t) \leq (M+L) \int_0^t [v_{n-1}(s) - u_{n-1}(s)] ds \; , n = 1,2,3.....$$

By the normality of the cone P,

$$\| v_n - u_n \| \leq \frac{[N(M+L)]^n}{n!} \| v_0 - u_0 \|, n = 1,2,3.....$$

As $n \to \infty$,

$$u_n = Au_{n-1} \leq A w^* \leq Av_{n-1} = v_n \, , \quad n = 1,2,3.......$$

$$w^* \leq A w^* \leq w^* \Rightarrow v^* = w^*$$

$$\| w_n - w^* \| \leq 2N \frac{[N(M+L)]^n}{n!} \| v_0 - u_0 \|_c \; n = 1,2,3....$$

This is the error estimate.

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