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On characterization of integral curves of a linear vector field

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Abstract

Karger and Novak [2] have shown that the integral curves of a linear vector field X in E^3 described by a matrix of the form $\begin{bmatrix} A & C \\ 0 & 1 \end{bmatrix}$ can be:

i) Helices with common axes & the same parameter, ifrank [AC] = 3.
ii) Circles which lie in planes parallel to each other and which have centers on the axis perpendicular to those parallel planes, if rank [AC] =2.
iii) Parallel straight lines, if rank [AC] =1. The results of Kargerand Novak are extendedby us to E^{2n+1}. It is shown that all the results are also valid in the general case.

Key words: integral curves, linear vector field.

§1. Introduction

The integral curves of a linear vector field on E^3 are dependent on the rank of the matrix, which defines the linear vector fields. They are circles or helices in the cases when the matrix of the linear vector field has even or respectively odd rank. In recent years the theory of helices in higher dimensions has been extensively studied. In the present paper, we investigate the theory of integral curves of a linear vector field and show that the theory of integral curves of a linear vector field in the (2n + 1)-dimensional Euclidean space (n ≥ 1) is the same as in the case n = 1.

§2. Preliminaries

Let $\alpha: I \rightarrow E^n$, $t \rightarrow \alpha(t)$ be a parameterized curve and let X be a vector field in E^n ([1], [3]). If $\frac{d\alpha}{dt} = X(\alpha(t))$, $\forall t \in I$ holds true, then the curve α is called an integral curve of the vector field X.

Let V be a vector space over R of dimension 2n+1. A vector field X on V is called linear if X(v) = A(v), ∀v ∈ V , where A is a linear mapping from V into V . Let $A \in M_{2n+1,2n+1}(R)$ be a skew - symmetric matrix. Then we can choose an orthonormal basis ψ in R^{2n+1}, such that the matrix A reduces to the form

$$\begin{pmatrix} 0 & \lambda_1 & 0 & \dots & 0 & 0 & 0 \\ -\lambda_1 & 0 & \lambda_2 & \dots & 0 & 0 & 0 \\ 0 & -\lambda_2 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \lambda_{2n-1} & 0 \\ 0 & 0 & 0 & \dots & -\lambda_{2n-1} & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix},$$

Where $\lambda \in R \setminus \{0\}$. If $C = {}^t(a_1, a_2, a_3, \dots, a_{2n}, a_{2n+1}) \in M_{1,2n+1}(R)$,is a column matrix, then we showed that the value of X at any point P of E^{2n+1} can be written as

$$\begin{pmatrix} X(P) \\ 1 \end{pmatrix} = \begin{pmatrix} A & C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} P \\ 1 \end{pmatrix},$$

where $\begin{pmatrix} A & C \\ 0 & 1 \end{pmatrix}$ is called the matrix of the linear vector field X.

§3. Integral curves. Linear vector fields in E^3

Let X be a linear vector field in E^3 and let $\{o; u_1, u_2, u_3\}$ be an orthonormal frame of E^3; then the matrix in this frame can be written as

$$\begin{pmatrix} A & C \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \lambda & 0 & a \\ -\lambda & 0 & 0 & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathrm{rank}[AC] = 3$$

Then the value of X at a point P= (x,y,z) of E^3 is

$$\begin{pmatrix} X(P) \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & \lambda & 0 & a \\ -\lambda & 0 & 0 & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

or $X(P)=(\lambda y+a, -\lambda x+b, c)$. On the other hand if the curve

$\alpha :\to E^3$, $t \to \alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ is an integral curve of X, then denoting by dot the derivative w.r.t. the variable t, we can write the differential equation

(3.1) $\dot{\alpha}(t) = X(\alpha(t))$, $\forall t \in I$ as the system of differential equations

$\dot{x} = \lambda y + a$

$\dot{y} = -\lambda x + b$

(3.2) $\frac{dy}{dt} = c$

For the shake of shortness, we set λ = 1 and then (3.2) reduces to

$x^{\cdot} = y + a$

$y^{\cdot} = -x + b$

(3.3) $\frac{dy}{dt} = c$

Then the solution of the last equation of this system is

(3.4) $z = ct + d$

For the solutions of the first two equations we derivate the second equation and obtain that

(3.5) $\ddot{y} + y = -a$,

Which is the first order linear differential equation with constant coefficient. We know the the solution of this equation is

(3.6) $y = A \cos t + B \sin t - a.$

On the other hand the derivation of (3.5) and the second of (3.3) give us that

(3.7) $x = A \sin t - B \cos t + b.$

Thus the integral curves of X can be written as

(3.8) $\alpha(t) = (A \sin t - B \cos t + b, A \cos t + B \sin t - a, ct + d).$

This is a family of inclined curves with common axes and the same parameter, since we have

(3.9 $H = \frac{K_1}{K_2} - \frac{1}{e}\sqrt{A^2 + B^2}$

Where K_1 and K_2 are the curvatures of the curve and H is constant for each one of the curves. Now we assume the case that rank [AC] = 2.

In this case c = 0 we know that $\lambda \neq 0$. Hence (3.7) gives us the equation of integral curves as
3.10) **$\alpha(t) = (A \sin t - B \cos t + b, A \cos t + B \sin t - a, d).$**

The curves are the circles each one of which lies on the parallel planes and the centers of these circles are located on an axis perpendicular to those parallel planes. Finally, assume that rank [AC] = 1. In this case we have that λ = 0 and the system (3.10) reduces to the system

$\dot{x} = a$,

$\dot{y} = b$

(3.11) $\dot{z} = c$

Then the solution of this system is

(3.12) $\boldsymbol{\alpha(t) = (at + d_1, bt + d_2, ct + d_3)}$.

§4. Linear vector fields in E^{2n+1}.The general case

Theorem. Let X be a linear vector field on E^{2n+1} determined by the matrix

$$\begin{pmatrix} A & C \\ 0 & 1 \end{pmatrix}$$

where, with respect to an orthonormal frame, A is a skew-symmetric and C is a column matrix. Then the integral curves of this vector field X are as follows:

i) The integral curves of X are the inclined curves whose axes coincide
ii) Each of the integral curves of X lie on a rank A-dimensional right hypercylinder.

Proof. Let $A \in M_{2n+1,2n+1}(\mathbb{R})$ be a skew-symmetric matrix and $C \in M_{1,2n+1}(\mathbb{R})$ be a column matrix such that A=

$$\begin{pmatrix} 0 & \lambda_1 & 0 & \dots & 0 & 0 & 0 \\ -\lambda_1 & 0 & \lambda_2 & \dots & 0 & 0 & 0 \\ 0 & -\lambda_2 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \lambda_{2n-1} & 0 \\ 0 & 0 & 0 & \dots & -\lambda_{2n-1} & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix},$$

and $C = {}^t(a_1,a_2,a_3,\dots,a_{2n},a_{2n+1})$. For a linear vector field X and all the points $p = (x_1, x_2, \dots, x_{2n+1}) \in E^{2n+1}$, we have

$$\begin{pmatrix} X(P) \\ 1 \end{pmatrix} = \begin{pmatrix} A & C \\ 0 & 1 \end{pmatrix}\begin{pmatrix} P \\ 1 \end{pmatrix},$$

where rank[AC] = 2n + 1 and

$\boldsymbol{X(p) = (\lambda_1X_2 +a_1,-\lambda_1X_1 +\lambda_2X_3 +a_2,\dots,-\lambda_2n-1X_2n-1+a_2n+a_2n+1,a_2n)}$.

In addition, if the curve $\alpha : I \rightarrow E^{2n+1}$ is an integral curve of the linear vector field X, then it satisfies the differential equation

(4.1) $\dot{\alpha}(t) = X(\alpha(t)), \forall t \in I$

and (4.3) provides the system of differential equations

$\dot{x}_1 = \lambda_1 x_2 + a_1$

$\dot{x}_2 = -\lambda_1 x_1 + \lambda_2 x_3 + a_2$

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$\dot{x}_{2n} = -\lambda_{2n-1} x_{2n-1} + a_{2n}$

(4.2) $\dot{x}_{2n+1} = a_{2n+1}$.

If we rewrite the matrix A by renumbering non-zero elements λ_i, we obtain the following

(4.3) $\begin{pmatrix} X\ (p) \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 & A \\ 0 & 0 & B \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P \\ 1 \end{pmatrix}$,

where $\lambda = \begin{pmatrix} 0 & \lambda_1 & 0 \\ -\lambda_1 & 0 & \lambda_m \\ 0 & -\lambda_m & 0 \end{pmatrix} \in M_{m+1,m+1}(R)$

$$C' = \begin{pmatrix} A' \\ B \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ . \\ a_{2n+1} \end{pmatrix}$$

and $A' \in M_{m+1,m+1}(R)$, $B \in M_{1,2n-m}(R)$.

If we use the notation $\wedge = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}$, then (4.3) and (4.5) give us that

(4.4) $$\dot{x} - \Lambda P = C',$$

Which is a first order linear differential equation with constant coefficient. The solutions of this equation are given by

(4.5) $$\alpha(t) = e^{t\Lambda} \int_0^1 C' e^{-i-u\Lambda} du + D$$

Since we have that
$\alpha(t) = \begin{pmatrix} e^{t\lambda} & 0 \\ 0 & I_{2n-m} \end{pmatrix}$;
(4.7) can be written as

(4.6) $$\alpha(t) = \begin{pmatrix} e^{t\lambda} \int_0^t C' e^{-u\lambda} A' du \\ Bt \end{pmatrix} + D$$

or, since the matrix λ is skew-symmetric, the rank λ is even, so m is odd and det λ ≠ 0 0 and λ^{-1} exists. Then

$$(4.7)\alpha(t) = \begin{pmatrix} e^{t\lambda} \int_0^t C' e^{-u\lambda} A' du \\ Bt \end{pmatrix} + D$$

and we have:

I. The last 2n − m components of the curve α are of the form $b_i t + d_i$.

II. $\|\alpha'(t)\|^2 = \|e^{t\lambda} A'\|^2 + \|B\|^2 = \|A'\|^2 + \|B'\|^2$, which means that $\|\alpha(t)\|^2 = constant$

Thus we can say that, in E^{2n+1}, by choosing an orthogonal frame

$\{O; u_1, ...u_{m+1}, u_{m+2}, u_{2n+1}\}$,

$$\text{we have } \langle \alpha'(t), u_r \rangle = \|\alpha'(t)\|.\|u_r\| cos\theta_r, m + 2 \leq r \leq 2n + 1$$

$$= \sqrt{\| A' \|^2 + \| B \|^2} \cos\theta_r.$$

and hence the angle in between each of the curves of the family α(t), and each of the base vectors u_r is constant. Therefore, each of the curves α(t) makes a constant angle with the space. Sp $\{u_{m+2}, ..., u_{2n+2}\}$. On the other hand, since the first m + 1 components of each curve α(t) can be represented by the vector

$-\lambda^{-1}A' + \lambda^{-1}e^{t\lambda} + D_1$, $D_1 \in M_{1,m+1}(R)$; $D = \begin{pmatrix} D1 \\ D2 \end{pmatrix}$

Using the curves $\alpha(t) = -\lambda^{-1}A' + \lambda^{-1}e^{t\lambda}A' + D$

And the points $-\lambda^{-1}A' + D_1 = Q = (q_1), ..., q_{m+1})$

We obtain that $d(Q, \alpha(t)) = constant$

Therefore we can say that all the curves α(t) are the inclined curves and they lie on the right hyper cylinders whose bases are S^m having as centers the points $(q_1, q_2, ...q_{m+1}, 0, ..., 0)$, and the radii $\|\lambda^{-1}A'\|$

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