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Thermal instability of a heterogeneous Fluid layer with freeBoundaries in the presence of a magnetic field

(**Key words**: MHD/thermal instability/heterogeneous fluid / free boundaries)

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Abstract

The thermal instability of an electrically conducting, viscous, incompressible and heterogeneous fluid layer with free boundaries in the presence of a transverse magnetic field under Boussinesqapproximation has been studied. A discussion of the existence of the marginal states, over stability and the validity of the principle of the exchange of stabilities,has been made; and the frequency of oscillations and the Rayleigh number have been determined as function of the magnetic field and the densitydistribution, which give the effects of the magnetic field of the heterogeneity on the onset of instability in various cases of interest. Further, the problem has been solvedby using variational principle and the effects of various parameters have been discussed through numerical computations in the case when the density varies exponentially.

Introduction

The magnetohydrodynamical equations of a conducting fluid allow some patterns of flow which can be realized only for certain range of values of the parameters characterizing them due to their inherent instability to sustain themselves against small perturbations. The problem of onset of instability in a horizontal fluid layer, heated from below and first discussed by Benard[1]in 1900, gives somestriking features of instability and is known as Benard problem.

Rayleigh[2] laid down the theoretical foundations for the study of the thermal Instability of such fluid layer and showed that the Rayleigh numbers $R = g\,\alpha\,\beta\,d^4 / K\nu$Decides the stability of the configuration, where g, α, β,K,ν and$d$ acceleration due to gravity, the coefficient of the volume expansion, the adverse temperature gradient, the thermal diffusivity, the kinematic viscosity and the thickness of the Medium, respectively. He showed that the instability sets in when R *exceeds* certain Critical value$R_c$and that when R just exceedsR_c, stationary pattern of motioncome to prevail. The theory was further developed by Jeffreys[3], Low[4], Pellew and Southwell[5] and several others. Later on, the stability of the layer of viscous fluid of Variable density was discussed by Hide6, Chandrasekhar7, Drazin8 and others.

The inhibiting effect of the magnetic field on the stability of a homogeneous fluid layer heated from below has been discussed by Thompson9, chandrasekhar10and Nakagawa11, Chandrrasekhar10, in particular, has analyzed the thermal instabilityof a homogeneous fluid layer in the presence of a magnetic field, in detail.

As in many situations, the fluid may not be homogeneous and moreover,the Magnetic field may be prevalent, it is interesting to study the effect of the magneticfield on the stability of heterogeneous fluid layer heated from below.The problem may have applications in atmospheric studies, oceanography, geology, and various other fields.

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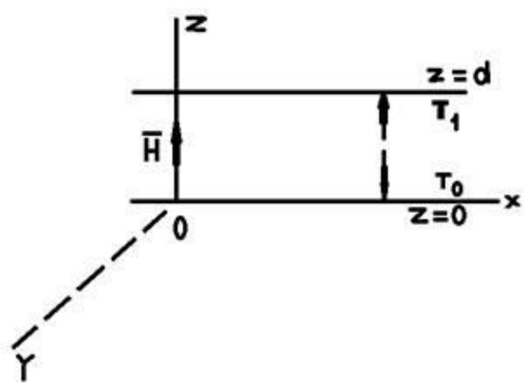
In the present paper, we therefore, study the stability of a viscous, incomprehensible heterogeneous and conducting fluid layer with free boundaries in the presence of a magnetic field acting in the transverse direction. The density of the fluid is taken as a function of the vertical co-ordinate and the surroundings are assumed to be non-conducting. We investigate the stability in the Bousssineq approximation and discuss the existence of the marginal states and the over stability; and the validity of the principle of the exchange of stabilities. We also obtain the frequency of oscillations and Rayleigh number in terms of the magnetic field, density etc., and thus discuss the effect of the magnetic field and the heterogeneity on the stability of the system. We further solve the problem by using the variational principle and discuss the effect of various parameters through numerical computation when the density varies exponentially.

The Problem

Consider electrically conducting, viscous and incompressible fluid layers confined between two boundaries z = 0and z= d and let the boundaries be free. Let the density of the fluid, apart from its variation due to the temperature, be $\rho_0 f(z)$. Where ρ_0 is the density a the lower boundaries (z = 0), so that f(0)=1 and f(z) is a monotonic function of the vertical co-ordinate z.



Let the lower and upper boundaries be maintained at uniform temperaturesT_0andT_1 respectively with $T_0 > T_1$. Further let a uniform magnetic field$\boldsymbol{H}$= *(0, 0, H)* Pervade the whole medium in the transverse direction to the layer.

Basic equations: The basic equations governing our system are

$$\rho 0[\frac{\partial V}{\partial t} + (V.\nabla)v] = \text{grad}\,[P+\frac{H^2}{8\Pi}] + \pi\nabla^2 V + \frac{1}{4\pi}(H.\nabla)H \quad (1)$$

$$\text{div}\, V=0 \quad (2)$$

$$\frac{\partial T}{\partial t} + (V.\nabla)\, T = \frac{K}{P_0 C_U}\nabla^2\, T - \frac{p}{P_0 C_U}\, \text{div}\, V + \frac{\emptyset}{P_0 C_0} \quad (3)$$

$$\frac{\partial \rho}{\partial t} + (V.\nabla)\, p=0 \quad (4)$$

$$\text{div}\, H=0 \quad (5)$$

$$\frac{\partial H}{\partial t} = \text{curl}\,(V \times H) + \eta\nabla^8\, H \quad (6)$$

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$$\rho = \rho_0[f(Z) + \alpha\,(T_0 - T)] \qquad (7)$$

Where V,p,ρ_0,T,H denote velocity, hydrostatic pressure, density, temperature and applied magnetic field: $\emptyset$,$c_U$denote dissipation function, specific heat of the fluid at constant volume; and μ, α and k denote the coefficient of viscosity ,coefficient of Volume expansion and thermal conductivity of the medium, respectively.

Let the initial state be characterized by

$$V = 0 \qquad (8)$$

$$T = T_0 - \beta_z\,,\ \beta = (T_0 - T_1)\,/\,d \qquad (9)$$

$$P = p + H^2/8_{\prod} = -\int g\rho\, dz \qquad (10)$$

$$H = (0, 0, H) \qquad (11)$$

where $\rho_0$is the density at the lower boundary z=0 and β is the adverse temperaturegradient.
Perturbation equations: Let the initial state be slightly perturbed such that

$V \rightarrow 0 + \delta V,\ T \rightarrow T + \emptyset\ ,\ P \rightarrow P + \delta P$

$$H \rightarrow H + \delta H,\ \rho \rightarrow \rho_0[f(z) + \alpha(T_0 - T - \emptyset)] + s_p \qquad (12)$$

Then, the liberalized perturbation equations for our system in the Boussinesqapproximations are

$$\rho_0\frac{\partial u}{\partial_t} = \frac{\partial}{\partial_t}\delta P + \mu\,\nabla^2\,\mu + \frac{H}{4\mu}\frac{\partial h_x}{\partial_z} \qquad (13)$$

$$\rho_0\frac{\partial \upsilon}{\partial_t} = \frac{\partial}{\partial_y}\delta P + \mu\,\nabla^2\,\upsilon + \frac{H}{4\mu}\frac{\partial h_y}{\partial z} \qquad (14)$$

$$\rho_0\frac{\partial_w}{\partial_t} = \frac{\partial}{\partial z}\delta P + \mu\,\nabla^2\,w + \frac{H}{4\mu}\frac{\partial h_z}{\partial_z} - g(\delta\rho - \alpha\rho 0^{\emptyset}) \qquad (15)$$

$$\frac{\partial \mu}{\partial_x} + \frac{\partial u}{\partial_y} + \frac{\partial w}{\partial_z} = 0 \qquad (16)$$

$$\frac{\partial h_x}{\partial_x} + \frac{\partial h_\mu}{\partial_y} + \frac{\partial h_z}{\partial_z} = 0 \qquad (17)$$

$$\frac{\partial \delta_p}{\partial_t} = -\rho_0\, w\,\frac{df}{d_z} \qquad (18)$$

$$\frac{\partial_\emptyset}{\partial_t} - \beta w = K\ \nabla^2 \emptyset \qquad (19)$$

$$\frac{\partial h_x}{\partial_t} = \eta\nabla^2 h_x + H\frac{\partial u}{\partial z} \qquad (20)$$

$$\frac{\partial h_y}{\partial_t} = \eta \nabla^2 h_y + H\frac{\partial v}{\partial z} \quad (21)$$

$$\frac{\partial h_z}{\partial_t} = \eta \nabla^2 h_z + H\frac{\partial w}{\partial z} \quad (22)$$

where, $\delta V = (\mu, \upsilon, w)$. $\varnothing$, δP,$\delta H = (h_x, h_y, h_z)$ and δp respectively denote small perturbations in velocity, temperature, pressure, magnetic field and $K = k/\rho_0 c_\upsilon$ the thermal diffusivity of the medium.

Analysis into normal modes: Now, we analyze an arbitrary disturbance into complete set of normal modes and examine the stability of each of these modes, separately. For the present problem, the analysis can be made in terms of two-dimensional periodic waves of assigned wave numbers. Hence, we assume that all the perturbations have the space-time dependence as

$$F(z) \exp[i (k_x x + k_y y) + pt] \quad (23)$$

Where $k = \sqrt{k_x^{\ 2} + k_y^{\ 2}}$ is the wave number of them,disturbance, k_x, k_y are real and p is a constant which can be complex and F(z) is some function of the vertical co-ordinate z.

Taking the perturbations of the form (23), the eqns (13) to (22) reduce to give

$$\rho_0 P^{\ U} = -i k_X \delta P + \mu(D^{\ 2} - k^{\ 2}) U + (H/P4_\pi) DH_1 \quad (24)$$

$$\rho_0 P^{\ v} = -i k_y \delta P + \mu(D^{\ 2} - k^{\ 2}) v + (H/P4_\pi) DH_2 \quad (25)$$

$$\rho_0 P^{\ w} = -D\,\delta P + \mu(D^{\ 2} - k^{\ 2}) W + (H/P4_\pi) DH_3 - gY + g\,\alpha \varnothing \rho 0 \quad (26)$$

$$i k_X U + i k_y V + DW = 0 \quad (27)$$

$$p\varnothing - \beta W = K(D^{\ 2} - k^{\ 2})\,\theta \quad (28)$$

$$p Y = -\rho_0 W (df/dz) \quad (29)$$

$$ik_X H_1 + ik_Y H_2 + DH_3 = 0 \quad (30)$$

$$pH_1 = \eta(d^{\ 2} - k^{\ 2}) H_1 + HDU \quad (31)$$

$$pH_2 = \eta(d^{\ 2} - k^{\ 2}) H_2 + HDV \quad (32)$$

$$pH_3 = \eta(d^{\ 2} - k^{\ 2}) H_3 + HDW \quad (33)$$

Where $D = \frac{d}{dz}$ and U,V,W,H_1,H_2,H_3,Y, δP and H are the values of F(z) for υ,v,w, h_x,h_y,h_z, δP and θ, respectively.

Let ζ and $\xi/4\prod$ denote the z-components of the vorticity. w =curl δV and the current density j=(curl δH) /4 $\prod$, induced by the perturbation, so that

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$\zeta = \frac{\partial u}{\partial_x} - \frac{\partial u}{\partial_y}$ and $\xi = \frac{\partial h_y}{\partial_x} - \frac{\partial h_x}{\partial_y}$. (34)
Let

$\xi = Z(z) \exp[i(k_x x + k_y y) + pt]$

$\zeta = X(z) \exp[i(k_x x + k_y y) + pt]$ (35)

Then, we have from (34)

$Z = ik_X V - ik_y U$ and $X = ik_X H_2 - ik_y H_1$ (36)

Now, in terms of Z and X, (31)and (32) reduce to give

$pX = (D^2 - k^2)X + HDZ$ (37)

while (24) and (25) yield

$pZ = (D^2 - k^2)Z + (H/4_\pi \rho_0) DX$ (38)

Further, eliminating δP from (24), (25) and (26) and using (27),(29) and (30),we obtain

$p(D^2 - k^2) W = -gak^2 ø + \nu (D^2 - k^2) W + (H/4\pi \rho 0)(D^2 - k^2) DH_3 + (g/\rho_0)k^2 Y$ (39)

where $\nu = \mu / P_0$ is the kinematic viscosity.

Hence, the eqns (28),(33), (37),(38) and (39) constitute the relevant equations of our system.

Now using following non-dimensional numbers

$D' = dD, \alpha' = kd, / \sigma = p D^2/\nu, p\_(1=) \nu/k, P\_2 = \nu/\eta$
$R' = (g\alpha\beta d^4/k\nu, R\_2 = (gd^4/k\nu)(df/dz), Q = (H^2 d^2)/(4\pi \rho_0 \nu\eta)$ (40)

and dropping the primes for convenience, these eqns. reduce to

$(D^2 - \alpha^2 - p_1) ø = -(\beta d^2/k)W$ (41a)

$(D^2 - \alpha^2 - p_2\sigma) H_3 = -(Hd/\eta) DW$ (41b)

$(D^2 - \alpha^2 - p_2\sigma) X = -(Hd/\eta) DZ$ (41C)

$(D^2 - \alpha^2 - p_2\sigma) Z = -(Hd/4_\pi \rho_0 \nu) DX$ (41d)

$(D^2 - \alpha^2)(D^2 - \alpha^2 - 0)0W + (Hd0/4_\pi \rho_0 \nu) \times D(D^2 - \alpha^2 H_3 = (g\alpha d^2/\nu).\sigma\alpha^2 ø + (g\alpha d^4 \alpha^2 Df/\nu^2)W$ (41e)

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Operating the eqn. (41e) by ($D^2-\alpha^2-P_1\sigma$) ($D^2-\alpha^2-P_1\sigma$) and using (41α)and (41b), We get following differential equation of order eight in the perturbed velocity W

$$\sigma p_1(D^2-\alpha^2)(D^2-\alpha^2-\sigma p_1)[(D^2-\alpha^2-\sigma)(D^2-\alpha^2-p_2\sigma)-QD^2]W$$

$$=-\sigma p_1 R\alpha^2)(D^2-\alpha^2-\sigma p_2)W+R_2\alpha^2(D^2-\alpha^2-p_1\sigma)(D^2-\alpha^2-p_2\sigma)W \quad (42)$$

We seek the solution of this equation satisfying certain boundary conditions which we enumerate below

Boundary conditions: Since the boundaries are free and non-conducting, followingChandrasekhar [12], the relevant boundary conditions for our problem are

(i) W= D^2W =0

(ii) Ø=0

(iii) DZ=0

(iv) X=0 (43)

at z=0 and z=1

Results and Discussion

Now, we deduce some interesting results from the eqn (41) and (42) and the conditions(43).

*Stationary convection and the principle of the exchange of stabilities:*First, weexamine whether the instability can set in as stationary convectionand the principle of the exchange of stabilities is valid

Supposing that the instability sets in as ordinary convection,the marginal states are characterized by $\sigma=0$ and eqns (41) reduce to

$$(D^2-\alpha^2)Ø=-\beta d^2/k)W$$

$$(D^2-\alpha^2)H_3=-(Hd/\eta)DW \quad (44)$$

$$(D^2-\alpha^2)X=-(Hd/\eta)DZ$$

$$(D^2-\alpha^2)Z=-(Hd/)/4\pi p_0\nu)DX$$

$$(\frac{D^2gd^4}{\nu}Df)W=0(44)$$

Solving the eqns (44) and using the boundary conditions (43), we find that W=0, ø=0, U=0,V=0,$H_1=0$,$H_2=0$etc. are the only solutions. Thiscontradicts the hypothesis that the initial stationary solutions are perturbed.

Consequently, the instability cannot set in as stationary convection and the principle of the exchange of stabilities is not valid.

However, in the special case when *Df=0, the eqn. (42),* in the marginalstate, reduces to

$$(D^2-\alpha^2)(D^2-\alpha^2)^2-QD^2]W=-R\alpha^2W \qquad (45)$$

And , the boundary conditions W= D^2 W=0 etc. suggest that the proper solution forW for the lowest mode is W=W_0sin πz. Then, the instability sets in as stationary convection and (45) yields the characteristic equation

$$R=\frac{\pi^2+\alpha^2}{\pi^2}[(\pi^2+\alpha^2)^2+\pi^2Q](46)$$

which gives the Rayleigh number and shows the inhibiting effect of the magnetic field on the onset of instability on further analysis. This agrees with Chandrasekhar's [12]result.

Frequency of oscillation at the marginal state: Now, we examine whether instability can set in as over stability i.e. oscillations of increasing amplitude and determine the frequency of oscillations in the marginal state which is characterized by $\sigma=(i\sigma_2)$ where σ_2 is real. Further, we notice that the proper solutions for *W*belonging to the lowest mode can be taken as $W=W_0\ sin\,\pi z$.

Substituting, therefore, $\sigma=i\sigma_2$ and W=W_0 sin πz in the eqn (42) and equating real and imaginary parts, and setting.

R_1= R/ π^4, $R_2{}'=R_2/\pi^4 Q_1=Q/\pi^2$, $\alpha^2=\pi^2$x(47)

We obtain

$p_1{}^2p_2(1+\chi)\sigma_2{}^4+[p_1p_2\,\chi(R_1+R_2{}')-p_1{}^2Q_1(1+x)-(1+x)$

$-(1+x)^2(p_1{}^2+p_1+p_1p_2)]\,\sigma_2{}^2-R_2{}'\,x(1+x)^2=0$ (48)

And

$xR_1p_1=p_1((1+x)^3-\sigma_2{}^2(1+x)\,[P_1P_2+P_1{}^2(1+P_2)]$
$+Q_1P_1(1+\chi)-R_2{}'\,x(P_1+P_2)$ (49)

Now , eliminating R_1 from (48) and (49) and solving the resultant equation forσ_2,we get

$\sigma_2{}^2=(-M\pm\sqrt{M^2-4LN})/2L$ (50)
Where
$L=p_1p_2{}^2(1+x)\,(p_1+1)$

$M=R_2{}'\,x+Q_1(1+x)\,p_1(p_1-p_2)+p_1(1+p_1)(1+x)^3$ (51)

$N=R_2{}'x(1+x)^2$

The eqn (50) gives the required expression for the frequency of oscillation in the marginal state.

Analysing (50) further, we notice that :When $R_2' > 0$ *and* $M > 0$, then $\sigma_2^{\ 2}$ is negative. Hence the marginal state and the over stability cannot occur in this case.

When $R_2' > 0$ *and* $M < 0$, then marginal states exist and over stability occurs and σ_2 is given by

$\sigma_2^{\ 2} = (M' + \sqrt{M'^{\ 2} - 4LN})/2L$ (52)

Provided $M'^{\ 2} - 4LN \geq 0$ where $M' = -M$

When $R_2' < 0$ and $M > 0$, over stability may occur and than, σ_2 will begiven by

$\sigma_2^{\ 2} = (-M + \sqrt{M^{\ 2} - 4LN})/2L$ (53)

When $R_2' < 0$ and $M < 0$, overstability may occur and than, σ_2 will be given by

$\sigma_2^{\ 2} = (M + \sqrt{M'^{\ 2} - 4LN})/\ 2L$ (54)

Rayleigh number: Considering the situation as described above, we find that The eqn.(49) yields the Rayleigh number in the marginal state as

$R = \pi^{\ 4}R_2 = \pi^{\ 4}[\frac{(1+x)^{\ 3}}{x} + Q_1\frac{(1+x)}{X} - R_2'\frac{(p_1+p_2)}{p_1}$ - $\{p_1 + p_2(1+p_2)\}$
$(\frac{(1+X)}{X})\sigma_2^{\ 2}]$(55)

Where $\sigma_2^{\ 2}$ is given by (50).

Marginal state: From the analysis of (50), we conclude that marginal stateand hence the solution describing over stability cannot occur if

$R_2' > 0$ *and* $p_1 > p_2$ (56)

For ,in such a case,L,M,N will be positive and consequently $\sigma_2^{\ 2}$ will be negative *,contradicting the hypothesis that* σ_2 *is real.*

From (40) and (47), we see that the condition (56) is equivalent to

(df/dz) > 0 and k <η (57

In particular, when R_2'*=0, the condition (57) reduces to simply*

$k < \eta$ (58)

Which agrees with Chandrasekhar's result [12].

Hence, in order that marginal states may exist and the over stability may occur for $R_2' > 0$, we must have
$p_1 < p_2$ (59)

Even when this is the case, analysis of (50) further suggests that the over stable Solutions are possible only when

$$R'_2 p_2{}^2 x + p_1 (1+x)^3 + Q_1 p_1{}^2 (1+x) < Q_1 p_1 p_2 (1+x) \quad (60)$$

and

$$[R'_2 p_2{}^2 x + p_1 (1+p_1)(1+x)^3 + Q_1 p_1 (p_1 - p_2)(1+x)]^3 \geq 4 p_1 p_2{}^2 R'_2 x (p_1+1)(1+x)^3 \quad (61)$$

Thus, for the marginal states to exist and over stability to occur for $R'_2 > 0$, the magnetic field and the wave number should be such that

$$Q_1 > \frac{R'_2 p_2{}^2 x}{p_1 (p_2 - p_1)(1+x)} + \frac{1+p_1}{(p_2 - p_1)} (1+x)^2 \quad (62)$$

Besides that $k < \eta$.

Further, we notice that, for a given Q_1 (the magnetic field), the over stable solutions are possible only when $x < x^*$, where x^* is given by

$$p_1 (1+p_1)(1+x^*)^3 + R'_2 p_2{}^2 x^* - Q_1 p_1 (p_2 - p_1) = 0 \quad (63)$$

In particular when $R'_2=0$, the condition (62) simply reduces to

$$Q_1 > (1+x^2)\frac{(1+p_1)}{(p_2 - p_1)} \quad (64)$$

Which agrees with a similar result obtained by Chandrasekhar [12] for a homogeneous fluid.

Similarly, when $R'_2 < 0$, we observe that the marginal states exist and the overstability occurs.

*Nature of non-oscillatory modes :*when $R'_2>0$ *and* $M>0$, only non-oscillatory modes can exist for which $\sigma_2=0$ and $\sigma=\sigma_1$ (σ_1 is real). Hence substituting $\sigma=\sigma_1$ and $W=W_0 \sin \pi z$ in (42), we obtain the characteristic equation as

$$F(\sigma_1) \equiv A\sigma_1^4 + B\sigma_1^3 + C\sigma_1^2 + d\sigma_1 + E = 0 \quad (65)$$

Where

$$A = p_1^2 p_2 (\pi^2 + \alpha^2)$$

$$B = p_1 p_2 (\pi^2 + \alpha^2)^2 + p_1^2 (1+p_2)(\pi^2 + \alpha^2)^2$$

$$C = p_1 (1+p_2)(\pi^2 + \alpha^2)^3 + p_1^2 (\pi^2 + \alpha^2)^3 + p_1^2 Q\pi^2 (\pi^2 + \alpha^2)$$

$$-\alpha^2 p_1 p_2 (R+R_2) \quad (66)$$

$$D = p_1 (\pi^2 + \alpha^2)^4 + p_1 Q\pi^2 (\pi^2 + \alpha^2)^2 - \alpha^2 (\pi^2 + \alpha^2)\{Rp_1 + R_2 (P_1 + P_2)\}$$

$$E = -R_2 \alpha^2 (\pi^2 + \alpha^2)^2$$

Now, since $F(+\infty)$ = positive and $F(0)$ =negative, it is obvious that the equation (65) willpossess at least one positive real root. Hence the system is unstable

.

Numerical Computation

Now, we discuss the effects of various parameters on the stability of our system by makingnumerical calculations by using the variational principle. For thispurpose, we will compute the values of **σ** for different values of the parameters and derive the conclusions.

Following Chandrasekhar[12], let p_i bea characteristic value and let the solutions corresponding to p_i be distinguished by a subscript **i** .Then, from the eqn (26), we get

$$D\delta P_i = -\rho_0 - p_i W_I + \mu D^2 - k^2) W_i + g\alpha\rho_0\theta_j + (H/4\pi) DH_{3i} + (g\rho_0 W_i / P_i)(df/dz) \quad (67)$$

Let p_j be another characteristic value and the solution corresponding to it be denoted by the subscript j. Then, multiplying (67) by W_j and integrating w. r.t.zfrom z=0 to z=d, using the boundary conditions (43), we obtain

$$-\int_0^d \delta P_i - DW_j\,dz = \int_0^d \rho 0\, p_i - \mu k^2 + ((g\,p_0/p_i)(df/dz) W_i W_j dz + \int_0^\alpha g\alpha\ P_0\theta_j\ W_j\,dz - \int_0^\alpha \pi DW_i - DW_j\,dz - \int_0^\alpha H/4_\pi) - H_{3i} DW_j\,dz \quad (68)$$

Now, for the characteristic valueP_j,the eqns. (28) and (33) become

$$P_j\theta_j - \beta W_j = K(D^2 - k^2)\theta_j$$

$$P_j H_{3i} =- \eta k^2 H_{3j} + \eta D^2 H_{3j} + HDW_j$$

Multiplying the first equation by θ_i and the second by H_{3i} and then, integratingw.r.t.z from z=0 to z=d, using the boundary condition (43), we get

$$\int_0^d p_i\ \theta_i\theta_j dz + \int_0^d k\ k^2\theta_i\theta_j\,dz + -\int_0^d k\,d\,\theta_j\,dz = \int_0^d \beta W_j\ \theta_i dz$$

$$-\int_0^d (\tfrac{H}{4}\pi)\,DW_j H_{3i}\,dz = -\int_0^d (p_j/4\pi) H_{3j}\ H_{3i}\,dz - \int_0^d \eta k^2/4\pi) H_{3i} H_{3j}\,dz -$$

$$-\int_0^d \eta/4\pi)\,DH_{3j}\ DH_{3i}\,dz$$

Substituting these results in eqn. (68), we obtain

$$-\int_0^d \delta P_i\ DW_j\,dz = -\int_0^d (-p_i\ p_0 - \mu k^2 + (gp_0/P_i)\,(df/dz) W_\xi\ W_j\,dz - \int_0^\alpha \pi DW_\xi\ DW_j\,dz$$

$$+(g\alpha p_0/\beta)\int_0^d pj\ \theta_i\theta_j\,dz (g\alpha p_0/\beta) kk^2) \int_0^d \theta_i\theta_j\,dz (g\alpha p_0/\beta) k \int_0^\alpha D\theta_i D\theta_j\,dz -$$

$$-\int_0^d (pj/4\pi) H_{3i}H_{3j}\,dz - (\eta k^2/4\pi) - \int_0^\alpha H_{3i}H_{3j}\,dz - (\eta/4\pi) - \int_0^\alpha DH_{3j}\ DH_{3i}\,dz$$

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(69)

Similarly, from the eqn(24) and (25) from P_3 we get

$k^2 \delta p_i = - p_i p_0 DW_i + \pi(D^2 - k^2) DW_i + (H/4\pi) D^2 H_{3i}$ (70)

Multiplying it by DW_j and integrating w.r.t.z from z=0 to z=d, we get

$k^2 \int_0^d \delta P_i\ DW_j\ dz = -\int_0^\alpha p_i\ p_0\ DW_i DW_j\ dz + \int_0^d (D^2 - k^2) DW_i DW_j\ dz + (H/4\pi)$
$+ \int_0^\alpha D^2 H_{3i}\ DW_j\ dz$ (71)

Further considering the eqn. (33) for p_j and multiplying it by $D^2 H_{3i}$ and Then integrating from z=0 to z=d , using boundary conditions (43) and then substituting the result in (71), we obtain

$\int_0^d \delta P_i\ DW_j\ dz = -\int_0^\alpha (p_i p_0 / k^2) DW_i\ DW_j\ dz - \int_0^d \mu DW_i DW_j\ dz - \int_0^\alpha (\mu/k^2)\ D^2 W_j\ dz - -\int_0^\alpha (\eta/4\pi k^2)$
$D^2 H_{3i}\ dz - (\eta/4\pi) \int_0^\alpha DH_{3j}\ DH_{3i}\ dz - (/P_\xi 4\pi k^2) \int_0^\alpha DH_{3j}\ DH_{3i}\ dz$ (72)

Hence, equating (69) and (72), setting $i = j$ and suppressing the subscripts, we find that

$-p \int_0^\alpha p_0\ [W^2 + (DW)^2 / k^2]\ dz + (g/p) \int_0^\alpha p_0\ (df/dz)\ W^2\ dz + p$
$\int_0^\alpha g\,\alpha p_0 / \beta) \o^2 dz + \int_0^\alpha g\,\alpha p_0\ k\beta)\ [k^2 \o^2 + (D\o)^2]\ dz =$
$\int_0^d \pi[k^2 W^2 + 2(DW)^2 + (D^2 W)^2 / k^2)] dz + \int_0^\alpha (p/4\pi)\ [(DH_3)^2 / k^2 + H_3^2]^2\ dz + P$
$\int_0^\alpha (\eta/4\pi)\ [D^2 H_3)^2 / k^2 + 2(DH_3)^2 + k^2 H_3^2 \}\ dz$ (73)

Which forms a basis for the variation formulation of our problem.

To see that the eqn. (73) provides the basis for the variation formulation,We consider the effect on *p*, determined in accordance with (73), of arbitrary varia-tins δw , $\delta\theta$, θH_2 in W, θ,H_3 respectively, compatible with the boundary conditions on W, θ and H_3.

Let

$I_1 = \int_0^\alpha p_0\ [W^2 + (DW)^2 / k^2]\ dz$

$I_2 = \int_0^\alpha P_0\ (df/dz)\ W^2\ dz$

$I_3 = \int_0^\alpha \mu[k^2 W^2 + 2(DW)^2 + (D^2 W)^2 / k^2]\ dz$ (74)

$I_4 = \int_0^\alpha g\,\alpha p_0 / \beta(k[(D\theta)^2 + k^2 \theta^2]\ dz$

$I_5 = \int_0^\alpha g\,\alpha p_0 / \beta)\ \theta^2\ dz$

$I_6 = \int_0^{\alpha} (1/4\pi)\,[(D\,H_3)^2/k^2 + H_3^2]\,dz$

$I_7 = \int_0^{\alpha} (\eta/4\pi)\,[D^2 H_3)^2/k^2 + 2(DH_3)^2 + k^2 H_3^2\}\,dz$

Substituting (74) in (73) and considering first order variations only, we get

$-(I_1 + \frac{g}{p^2} I_2 - I_5 - I_6)\frac{\delta p}{2} = (\frac{p\delta I_1}{2} - \frac{g}{p}\frac{\delta I_2}{2} - \frac{\delta I_5}{2} - \frac{\delta I_4}{2} + \frac{\delta I_3}{2} + \frac{p\delta I_6}{2} + \frac{\delta I_7}{2})$ (75)

Where

$\delta I_1 = 2\int_0^d \delta W\,[p_0 - (p_0/k^2) D^2 W]\,dz$

$\delta I_2 = 2\int_0^d (df/dz)\,W\delta W\,dz$

$\delta I_3 = 2\int_0^d \delta W\,(k^2 \pi W - 2\pi D^2 W + (k^2 D^2)(\mu D^2 W)]\,dz$

$\delta I_4 = 2\int_0^d g\,\alpha p_0/\beta)\,K\,\delta\theta\,[D^2\theta - k^2\theta^2)dz$(76)

$\delta I_5 = 2\int_0^{\alpha} g\,\alpha p_0/\beta)\,\theta\delta\theta\,dz$

$\delta I_6 = \int_0^{\alpha} (1/4\pi)\,[-k^2 D^2 H_3\delta H_3 + H_3\delta H_3]\,dz$

$\delta I_7 = \int_0^{\alpha} (\eta/4\pi)[k^2 D^4 H_3\delta H_3 - 2D^3\delta H_3 + k^2 H_3\delta H_3]\,dz$

Now simplifying (75)and (76) forδW,we get

$-(I_1 + \frac{g}{p^2} I_2 - I_5 - I_6)\frac{\delta p}{2} = \int_0^d \delta W\,[Pp_0 W - \frac{Pp_0}{k^2} D^2 W - \frac{g}{p} p_0 \frac{df}{dz} - g\alpha p_0 \theta$
$+ k^2 \pi W 2\pi D^2 W + \frac{D^2}{k^2}(\pi D^2 W) + \frac{H}{4\pi k^2} D^3 H_3 - \frac{H}{4\pi} DH_3]\,dz$(77)

Hence, the coefficient of δW in the above integral vanishes

If $Pp_0 W - \frac{Pp_0}{k^2} D^2 W - \frac{g}{p} p_0 \frac{df}{dz} - g\alpha p_0\,\theta + k^2 \mu W - 2\mu D^2 W + \frac{D^2}{k^2}(\mu D^2 W)$

$+\frac{H}{4\pi k^2} D^3 H_3 - \frac{H}{4\pi} DH_3 = 0$ (78)

Thus, a necessary and sufficient condition that $p$be zero for first order variationsδW_3 ,δH_3, $\delta\theta$, compatible with the boundary conditions, is that W,H_3, θ are the solutions fo the characteristic value problem. Hence, it is possible to solve the present problem by using variation principle.

Now,we return to eqn. (73). Let the density distribution be governed by f(z)=exp(yz), where y is small.*Let the solutions for W and θ be*

$$W = W_0 \sin(\eta\pi z/d) \text{ and } \theta = \theta_0 \sin(\eta\pi z/d)$$

Where η is an integer. Let further

$$l = \eta\pi/d,\ y = k/l,\ b = y/l,\ \sigma = pd^2/\nu,\ p_1 = \nu/k$$

$p_2 = \nu/\eta$, $R = g\alpha\beta d^4/k\nu$, $Q = H^2 d^2/4\pi p_0 \nu\eta$ and $R_3 = gd^3/\nu^2$ (79)

Substituting above in the eqn. (73) and simplifying, we obtain following fourth degree equation in σ

$$A\sigma^2 + B\sigma^2 + C\sigma^2 + D\sigma + E = 0 \quad (80)$$

where

$$A = p_1 p_2 (1+y^2)(b^2+4)$$

$$B = (p_1 + p_2 + p_1 p_2)(1+y^2)(b^2+4)\eta^2\pi^2$$

$$C = (1 + p_1 + p_2)(1+y^2)(b^2+4)n^4\pi^4 - 4R_3(e^{b\eta\pi}-1)y^3 p_1 p_2 + Q p_1 n^2\pi^2(b^2+4)(1+y^2) - Ry^2 p_2 (b^2+4)$$

$$D = n^6\pi^6(1+y^2)^4(b^2+4) - 4R_3(e^{b\eta\pi}-1)y^2(1+y^2)n^2\pi^2(P_1 + P_2)$$

$$+Q(1+y^2)^2 n^4\pi^4(b^2+4) - Ry^2(1+y^2)(b^2+4)n^2\pi^2$$

$$E = -4R_3 e^{b\eta\pi}-1)y^2(1+y^2)^2 n^2\pi^2 \quad (81)$$

Now, considering *(80)* for $n=1$ and making numerical calculations, we haveobtained the roots o*f (80)* for different values of *Q,y and b* as shown in the following tables*:*

$P_1 = 2.47\times 10^{-2}$ $P_2 = 1.46\times 10^{-7}$ $R=10$ $R_3=1$

TABLE 1 TABLE 2 TABLE 3

y=1, b= 0.1 $Q=10^6$ $b=0.1$ $y=1, Q=10^6$

Q	σ		y	σ		b	σ
0	9.48×10^{-3}		.001	3.69×10^{-13}		1.0	1.77×10^{-5}
10	2.65×10^{-3}		.01	3.69×10^{-11}		2.0	2.67×10^{-4}
10^3	3.55×10^{-4}		0.1	3.69×10^{-9}		2.5	1.002×10^{-3}
10^4	3.66×10^{-5}		1.0	3.69×10^{-7}		3.0	3.80×10^{-3}
10^5	3.69×10^{-6}					3.5	1.46×10^{-2}
10^6	3.69×10^{-7}					4.0	5.71×10^{-2}

It is seen, from Table 1, that σ decreases as Q increases, showing that the magnetic field has a stabilizing effect on the system. Further, the Table 2 reveals that σincreases as *y* increases and, therefore, the perturbations of high wave number have adestabilizing value of *b* helps to destabilize the system.

We conclude that the principle of the exchange of stabilities is not valid for our system and that frequency of oscillations in the marginal state are given by (50) and The Rayleigh number by (55). It is seen that for *(df/dz)>0 and* $K < \eta$, the marginal states do not exist and we have only non-oscillatory modes which make the system unstable. However, the marginal states exist and over stability occurs when $K > \eta$, and the magnetic field and the wave number are such that they satisfy (62). Further, analyzing the problem by variational procedure and making the numerical computation, we observe that the magnetic field has a stabilizing influence, while the wave Number and heterogeneity have a destabilizing effect on the system, when density varies exponentially.

The various results obtained may find applications in many geophysical and terrestrial conditions, atmospheric studies, oceanography and related fields.

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