A Mixed Reality Environment for Mathematics

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***Abstract*—Throughout history, mathematicians and geometricians have relied on visualizations to convey abstract concepts effectively, including describing, discussing, studying, and teaching Mathematics. In Mathematics education, visualizations are still used whenever possible to support teaching, inspire students, and satisfy their need to see abstract Mathematical facts. In recent years, the integration of virtual reality (VR) and augmented reality (AR) in contemporary Mathematics education offers a fascinating and extremely motivating new tool for teachers, allowing students to visualize Mathematics in three dimensions. This article provides an overview of immersive environments developed in recent years to support geometry education, focusing on the combined use of VR and AR. By examining various methodologies and outcomes, we explore the potential of mixed reality in transforming how students engage with mathematical concepts. The procedures and outcomes of the methods of applications are discussed and described below.**

Keywords—Augmented Reality, Virtual Reality, computer generated environment, geometry education, sentenced-based math problems, interactive playgrounds

I. INTRODUCTION

Mathematics concepts and principles are a very important cornerstone of technology and science education, and also a kind of regular logic science. However, despite its great relevancy, mathematics does not enjoy a great popularity as a teaching subject in general. Studies estimate 17% of the population to be high in mathematical anxiety. The reason for the partial unpopularity of the mathematics lessons and the teaching content lies in the abstract and theoretical way of teaching, which demands students to memorize but not necessarily to understand the subject. If students are exposed to mathematics at the early age and are not excluded from the operation of mathematical concepts, it will help develop their logic. This logic and knowledge is often fundamental when solving real life problems, especially problems situated in the two- or three-dimensional domain that require spatial skills.

Numerous students encounter challenges in conceptualizing space and exhibit deficiencies in spatial reasoning skills. Spatial abilities, in contrast, present an important component of human intelligence, as well as logical thinking. Several studies have demonstrated that well-designed training programs can lead to improvements in spatial abilities. To build this logic and the spatial ability, it is important, for the students, to receive efficient educational activities and tools for mathematics that help them overcome anxiousness.

Virtual environments, inherently three-dimensional, offer interactive playgrounds with a level of interactivity that surpasses reality's constraints. By the term virtual reality, we are referring to a computer-generated environment with scenes and objects that appear to be real, making the user feel they are immersed in their surroundings. This environment is perceived through a virtual reality device which is called a VR Headset and such a device is displayed in Figure 1. If using VR as a tool for mathematics education, it ideally offers an added benefit to learning in a wide range of mathematical domains such as 3D geometry, vector algebra, trigonometry, analysis, as well as other three-dimensional applications and problems.

Fig. 1. A Virtual Reality headset

Augmented Reality (AR) is the real-time enhancement of a physical real-world environment by overlaying virtual computer-generated information onto it, either directly or indirectly. In mathematics education, the studies reported that the use of AR has a positive effect on students' learning performance and attitude toward topics such as solid geometry and more abstract geometric concepts. It occurs particularly when students' curiosity is sparked and they are challenged to create their own projects, thus making learning more enjoyable for all the students.

II. LITERATURE REVIEW

Mathematics constitutes a fundamental aspect of human cognition and logic, crucial to endeavors aimed at comprehending both the world and ourselves. It serves as a potent tool for cultivating mental discipline while fostering logical reasoning and mental rigor. Many recent studies have focused on the problem faced by the students when learning mathematics in the traditional way, which includes the use of standardized curricula, textbooks, and chalkboards. These problems include the students having difficulty understanding sentence-based math problems, lacking knowledge about basic mathematical concepts, and not transforming the sentence-based mathematics problems into an operational form. [1][2]

Schools should forge essential connections between geometry lessons and real-world content that children can relate to. It is recommended that geometry be presented as a dynamic, spatial, and imaginative concept for children from an early age, rather than primarily focusing on labeling and shape classification.

The advent of technology allows lesson scenarios to be created using VR and AR for the children. These contain the most important information on flat or spatial figures being the subject of a lesson. [3]

Virtual reality can provide interactive playgrounds with a degree of interactivity that goes far beyond what is possible in reality. However, since costs of a VR setup will always exceed costs of a standard desktop computer, justifications for the higher expenses must be given to those funding the implementation. [4]

III. TEACHING MATHEMATICS WITHOUT MIXED REALITY

Traditional math teaching, though established, comes with limitations that can hinder student learning and engagement. Sentence-based problems, while integrating language and math, can create obstacles for students struggling with either area. The heavy focus on lectures and textbooks can lead to students memorizing formulas without truly understanding the underlying concepts. This "rote learning" approach makes it difficult to apply knowledge to new situations. Furthermore, traditional methods often rely on passive learning through lectures and textbooks, limiting opportunities for critical thinking and problem-solving. Students may also struggle to see the relevance of math if it's not connected to real-world applications. This disconnect between theory and practice can make it difficult for students to grasp the importance and usefulness of math. In essence, traditional methods, while familiar, can unintentionally promote memorization over understanding, limit student engagement, and hinder the development of critical skills.

IV. MIXED REALITY FOR GEOMETRY TEACHING

Geometry is a cornerstone of mathematics, delving into the intricacies of shapes, sizes, dimensions, and spatial arrangements. Ever since people have interacted with nature and its phenomena, Geometry has been of core relevance. Studying geometry not only facilitates practical applications in areas such as architecture, engineering, and art but also plays a crucial role in nurturing students' logical reasoning and problem-solving skills. This implies analyzing and elaborating arguments about spatial forms, shapes, and abstract math concepts. By engaging in geometry, students develop spatial awareness, foster critical thinking skills, and a deeper appreciation for the intricacies of the world around them.

However, geometry often presents abstract challenges, leading many students to struggle and exhibit poor performance. Geometric equations frequently yield complex three-dimensional shapes that can be challenging to visualize, particularly when the geometric object:

- Is depicted in a space that does not rotate in three dimensions, like a book.
- Is depicted only from one facet (e.g. frontside).
- It results from incisions of objects.
- It tends to infinity, such as lines or layers.

Augmented Reality (AR) and Virtual Reality (VR) technologies are revolutionizing geometry education, offering immersive, interactive experiences that surpass conventional textbooks.

In AR, students can visualize geometric shapes within their real-world surroundings, augmenting their grasp of spatial concepts. VR takes this a step further by enabling learners to interact with and manipulate objects within immersive virtual environments, facilitating intuitive comprehension. These technologies also offer gamification and collaborative features, making geometry learning engaging and accessible.

By leveraging AR and VR, educators can overcome the constraints of traditional teaching methods, transforming geometry into a more interactive and impactful learning experience for students. Through a VR device and a combination of methods, the training aim could be both easily perceptive and fun at the same time as:

- Teachers can prepare for their lesson through a virtual classroom.
- Students can comprehend concepts and practice in a safe and comfortable learning environment.
- Teachers are given the ability to analyze a three-dimensional object associated with a concept, from different perspectives, in real time.
- Trainees can remember and become familiar with new knowledge in their personal space.

The ability to visualize brings the greatest advantage in the fields of mathematics and geometry. Virtual Reality empowers students to create and explore complex concepts from any angle, offering a three-dimensional, 360-degree view of objects they encounter.

V. EXPLORING AR/VR GEOMETRY TEACHING TOOLS

Utilizing Augmented Reality (AR) and Virtual Reality (VR), various software solutions enhance geometry education. Through interactive simulations and 3D models, these tools elevate spatial reasoning and problem-solving skills. This paper aims to explore the features and benefits of these AR/VR-based geometry teaching platforms, shedding light on their potential contributions to educational practices.

A. NeoTrie VR's Immersive Virtual Reality

Neotrie, a Unity 3D-based software, provides immersive VR experiences for 3D geometry. It enables creation, manipulation, and interactive exploration of geometric objects in a fully immersive environment. Neotrie aids students in mastering stereometry, enhancing 3D visualization, and transitioning between 2D and 3D representations, among other educational goals.

1) Technical Requirements, Interface, and Basic Actions

The Neotrie utilizes Virtual Reality equipment, including a VR visor and controllers, alongside a standard videogame computer. The default scenario is set within a neoclassical Greek temple surrounded by natural landscapes. Students can freely move within this environment through walking, flying, or teleporting. The software integrates a voice recognition system based on Microsoft's Cortana assistant, enabling users to summon shapes by name, such as graphs, polyhedra, and round bodies. Basic actions are executed using physical controllers, triggering a menu of options upon button press (Figure 2a). These actions include creating and modifying vertices, edges, and faces, erasing elements, adding 3D annotations, performing shape translations, and accessing key shape information like vertices, edges, areas, volumes, and more.

2) Common Tools

The Neotrie interface features a table where users can access a variety of classic drawing tools. These include a paintbrush and colour palette, a stamp for copying and pasting, an eraser for removing entire shapes, tools for rotation, reflection, and midpoint calculation, an angle transporter, a ruler, and a tool for drawing parallel lines. (Figure 2b).

Fig. 2. (a) Virtual hands action menu. (b) Basic actions and tools.

Neotrie offers tools like the perpendicular tool, creating lines perpendicular to others or planes, and the intersection tool, indicating where lines meet or their closest points of intersection. It also displays points where planes intersect or lines meet planes.

Moreover, users can visualize object traces, shown as curves for vertices or ruled surfaces for edges. It also includes a labelling tool enabling users to add information to the elements of shapes, enhancing clarity and understanding (as illustrated in Figure 3).

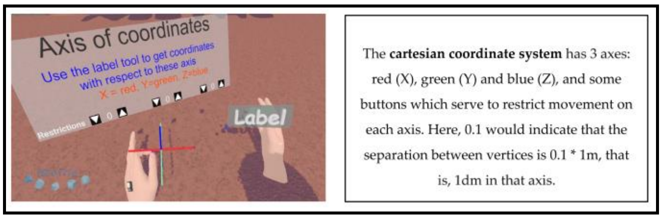

Fig. 3. The labeling tool and the 3D axis system.

3) Results

In teaching experiments, Neotrie was observed to reduce students' sole reliance on mathematical language for building and sustaining shared mathematical understanding. Its immersive environment encourages interactive learning, integrating actions and gestures with discourse. It effectively channels students' initial enthusiasm into genuine interest and effort, boosting engagement and improving geometrical competence, even among those with lower math skills.

B. The AR Application Geo+

The Geo+ app enables small groups of students to examine and understand the structure of geometric solid shapes, using multimedia resources to gather and discuss essential information. The app necessitates the utilization of smartphones and tablets, alongside a specific target image serving as an initial reference point. Its primary learning objectives include:

- Recognizing solid figures
- Drawing and translating representations of different 3D solids in 2D figures
- Recognizing the significant features (corners and edges) and distinctive properties

Initially, Geo+ presents a main menu where users can choose the geometric solid figures they want to explore. Navigation among these figures is facilitated through the introduction of buttons, which are categorized into two groups (Figure 4):

- On the left side are the rotating solids, for example, the sphere, the cylinder and the cone ("Sfera'', ''Cilindro'' and ''Cono'' respectively in Italian)
- On the right side are the polyhedral solids, for example, the cube, the parallelepiped and the pyramid (''Cubo'', ''Parallelepipedo'' and ''Piramide'' respectively in Italian)

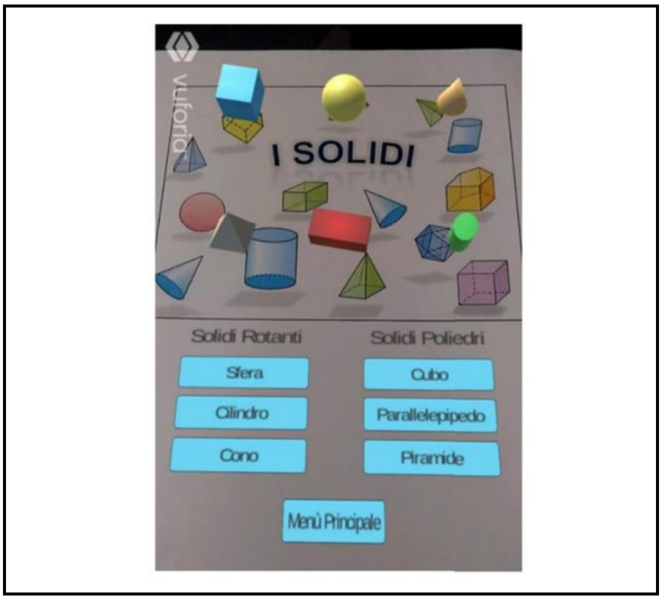

Fig. 4. The menu to select the solid figure. On the left are the rotating solids and on the right are the polyhedral solids

Once a solid is selected, Geo+ presents the augmented object for exploration. In this view (Figure 5), users have the ability to zoom in and out, rotate the object, and pause the rotation to observe detailed features of the solid. This functionality is crucial as it enables users to count the faces, vertices, and edges, and to identify the shape of each face and polygon. Additionally, a video is provided to illustrate how the solid is constructed (Figure 6).

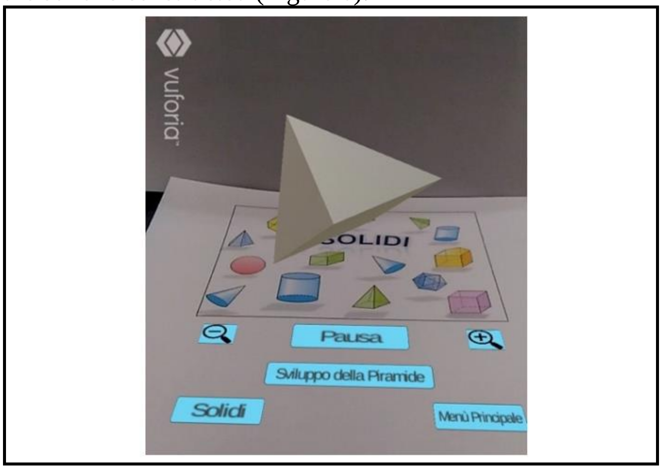

Fig. 5. Augmented object visualization

The video lasts approximately three minutes and shows real objects with the same shape as the selected solid.

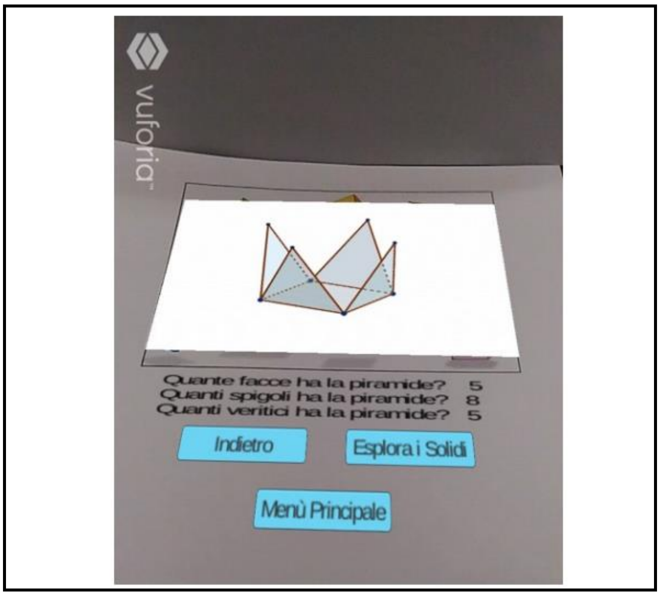

Fig. 6. The video-lesson of the development of the solid

A field study was conducted on an advanced interactive prototype of Geo+ to assess its effectiveness in learning, engagement, and perceived workload within the school environment. The data collected indicated that students showed an improvement in their understanding of geometric solids after engaging with Geo+.

Furthermore, following both pre-test and post-test assessments, it was verified that integrating augmented reality technology in educational settings was beneficial in terms of enhancing student learning outcomes. Specifically, the AR application served as a supplementary tool for reinforcing concepts that students had already mastered. Notably, it effectively assisted students in grasping concepts related to solid dimensions, vertices, edges, and faces.

C. *Construct3D- A Dynamic Geometry Construction Tool*

Construct3D is a cutting-edge tool for dynamic geometry construction in three dimensions, specifically created to address the need for innovative user interfaces in mathematics and geometry education.

The configuration utilized for Construct3D accommodates two users who wear stereoscopic see-through Head-Mounted Displays (HMDs), enabling them to share a virtual space. Interaction with the system is facilitated through pen and pad props (Figure 7). Both users have access to the same virtual objects, as well as each other's pens and menu systems, allowing for collaborative assistance if needed.

Fig. 7. Immersive multi-user setup

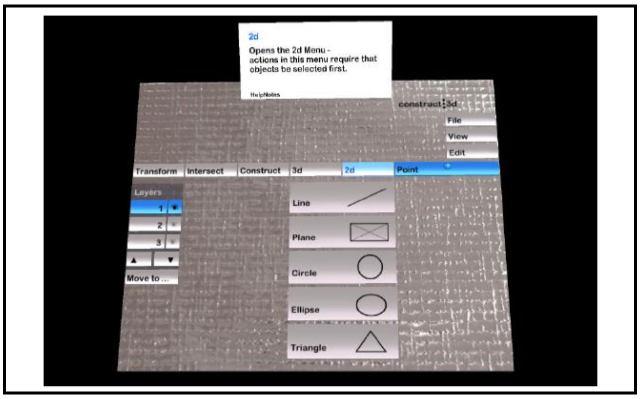

Fig. 8. The menu system

The current version of Construct3D provides tools for building 3D points and a range of geometric shapes. It offers functions for performing both planar and spatial operations on these shapes, as well as tools for measurements and organizing elements into layers. Construct3D supports the creation and manipulation of various fundamental object types, including points (which can be positioned freely or fixed on curves and surfaces), lines, planes, circles, ellipses, cuboids, spheres, cylinders, cones, and more.

Some of the implemented geometric operations in Construct3D include Boolean operations (such as union, difference, and intersection) on 3D objects, finding intersections between various types of 2D and 3D objects which result in intersection points and curves treated as primary objects, planar slicing of objects, rotational sweeps, determining surface normals, tangential planes, and tangents, among other functionalities.

Additionally, it encourages and facilitates exploratory behavior in dynamic geometry without imposing any limitations. This means that all geometric entities can be continuously modified by the user, and dependent entities maintain their geometric relationships. For instance, adjusting the position of a point on a sphere will automatically change the sphere's radius accordingly.

1) User Interface Design

Construct3D employs a menu system integrated with a handheld pen and panel interface known as the Personal Interaction Panel (PIP). This panel simplifies the incorporation of conventional 2D interface components like buttons, sliders, and dials, alongside 3D interaction widgets. Passive haptic feedback from physical props assists users in navigating the panel, while overlaid graphics enable the props to function as versatile tools. The pen serves dual purposes: operating on the panel and directly manipulating the 3D scene.

The menu in Construct3D is structured based on experts' knowledge and experience, employing logical grouping of functionality into five submenus accessible via tabs (Figure 8). Certain essential functions are always visible, ensuring easy access for users at all times.

2) Transperancy

Construct3D employs transparency for geometric primitives, allowing users to visualize the interiors of objects. Direct manipulation of points inside other objects is facilitated only if those points are visible. The current transparency mode is enhanced with a colour scheme and six lights in the scene, which create visually appealing spotlights (see Figure 9). Furthermore, the midpoint of the sphere is distinctly visible.

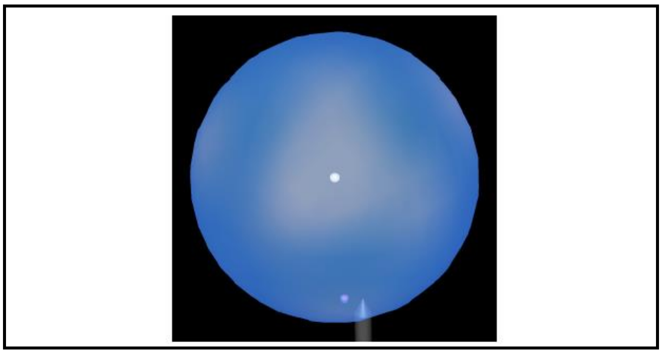

Fig. 9. Rendering of a transparent sphere in the rendering mode

Complex objects are initially drawn in an opaque manner in Construct3D, giving users the ability to individually switch them to wireframe mode if desired. This functionality enables users to see inside or behind these objects more effectively. For instance, the rotational sweep surface is visible both in its normal opaque mode and in wireframe mode, providing enhanced clarity (Figure 10).

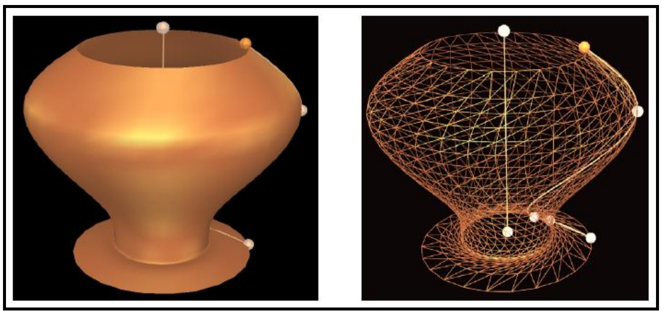

Fig. 10. Left: The lower point of the axis and points on the BSpline curve are hidden behind and inside the surface of revolution.
Right: In wireframe mode all points are visible and easily accessible.

3) *Results*

The most critical priorities for an educational application, aligning with pedagogic theories like constructivism, are that the application (1) is user-friendly and requires minimal learning time, (2) promotes exploration of new functions by learners, and (3) offers consistent usage and design that facilitates effective memorization of learned concepts. These aspects are highly valued by students, as demonstrated by their positive ratings, indicating that Construct3D software enhances their understanding of geometry more effectively through AR/VR. Almost all students expressed that they could imagine using the current version of Construct3D in high school or university education.

VI. DRAWBACKS OF AR/VR IN MATHEMATICS LEARNING

It has also been noted that AR/VR technologies could pose certain drawbacks for students involved in mathematics education, as indicated in Table 1. Only a restricted number of studies highlighted potential drawbacks to AR (29%, n = 17) and VR technologies (7%, n = 4) in mathematics education, primarily encompassing pedagogical, socio-emotional, and cognitive issues.

The most commonly mentioned downside of AR/VR technologies involved technical issues such as inadequate infrastructure, insufficient devices and software, and technological glitches like internet connectivity issues and audio-visual problems (AR: 15%, n = 9; VR: 7%, n = 4). The accessibility of AR/VR applications for all students might be constrained by cost-related factors (AR: 7%, n = 4; VR: 5%, n = 3). Several studies (7%, n = 4) have pointed out that AR applications can be time-consuming, with users facing long waiting times due to inadequate devices, which led to boredom among some students (3%, n = 2). Another concern is the lack of prior experience with AR among students and the need for professional support to use AR apps effectively (5%, n = 3). Additionally, there are reports indicating potential health issues related to prolonged screen time and the small screen sizes of mobile devices used for AR apps (2%, n = 1), and that they may increase cognitive load by requiring users to process information from both real and virtual environments simultaneously (2%, n = 1). Furthermore, social interaction and communication can be limited, especially in single-user modes (2%, n = 1).

Category	Sub-Category	AR		VR	
		n	%	*n*	%
Pedagogical outcomes	Technological glitches, technical deficiencies	9	15	4	7
	Cost	4	7	3	5
	Time-consuming	4	7	-	-
	Lack of user knowledge/experience in using AR tools	3	5	-	-
	Health problems	1	2	-	-
Socio-emotional outcomes	Being bored	2	3	-	-
	Lack of interaction and communication	1	2	1	2
Cognitive outcomes	Cognitive load	1	2	-	-

Table 1. Drawbacks of AR/VR technology in mathematics education

These noted drawbacks might hinder the effectiveness of AR/VR technologies for mathematics learning. However, it's worth noting that these drawbacks were reported by only a small subset of students in the study samples. It is noteworthy that the progression of digital technologies presents a promising pathway to potentially overcoming these obstacles in the near future. Overall, to improve mathematics learning through the effective use of AR/VR, it's crucial to address their potential drawbacks and ensure accessibility for all students.

VII. CHALLENGES OF MIXED REALITY IN MATHEMATICS

A. *Hardware Costs*

Current AR/VR hardware is expensive mainly because of no existing mass market for AR/VR solutions. It is unrealistic that an average high school can afford an immersive setup, which is most favoured by teachers and students. Tracking position and orientation of multiple users and all their devices accurately is the most expensive part of an immersive system. While prices are still high compared to the available budget of schools, but movement can be seen in the rigid motion tracking market recently.

B. *Support of Limited Number of Users*

Most of the applications support only a very limited number of users. This narrows down the possibilities of usage for teachers. Integration into regular lessons becomes difficult if only a very few students are able to use advanced technology. Pedagogical concepts are needed to integrate high end technology in a meaningful way for all students to benefit. Another option is to use VR technology only in special courses where few students attend, for example, in courses for high or low achievers.

C. *Technical Complexity*

Even if the hardware was cheap, it has been reviewed that teachers who were thrilled by the possibilities confirmed they would not use a VR setup in their school – given they had a spare room to set it up – because of its technical complexity. A technical setup consisting of multiple components is prone to errors and requires maintenance. Maintaining a standard school PC lab, with computers, monitors, mice, and keyboards, demands ongoing attention and effort. Introducing even basic additional technology increases complexity. Implementing a tracking system, interaction devices, extra displays, wireless communication, and more would necessitate additional maintenance personnel dedicating several hours each week to upkeep.

Besides the aforementioned points, the need for sufficient fluidity and immersion within the VR environment for effective learning to occur adds to the list of problems faced. The absence of these qualities negatively impacts learning outcomes and the user experience. Moreover, while VR has the potential to enhance mathematical literacy, further research and improvement are necessary, particularly in the interpret dimension of mathematical literacy skills. Finally, the use of virtual manipulatives in mathematics education, although promising, lacks sufficient study, especially for students with mathematics difficulties or learning disabilities.

VIII. CONCLUSION

The different methods are designed for teaching geometry at different stages of education, combining didactic values and fun. Each method aims at focusing different aspects of geometry, in the end providing an immersive way of solving difficult and comprehensive problems. While adapting traditional content to a new medium is one challenge, in some cases VR technology enables the teaching of completely new content, that one may face difficulty learning it the traditional way. Enhancing the learning experience hinges on effectively utilizing the strengths of available new technologies. Since many of the technological problems have been solved in previous years and VR technology is becoming mature, the presented review paper idea give hints at how VR can be utilized for special target groups such as dynamic geometry. Despite barriers to implementing VR in math education currently, its allure to students is undeniable and should not be disregarded.

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