A New faster Iteration Process to fixed points of mean-non expansive Mappings in Banach Space

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Abstract. Let X be a Banach space and F a subset of X. In this paper, we introduce a new iterative scheme to approximate fixed point of mean non-expansive mappings. We first prove that proposed iteration process is faster than all of Mann, Ishikawa, Picard, Agarwal, Noor and Thakur processes for contractive mappings. We also show that some weak and strong convergence theorems for mean non-expansive mappings. Using example presented in mean non-expansive mappings in Banach space. We compare the convergence behavior of the new iterative process with other iterative processes.

Keywords. Mean non-expansive mappings, Iterative Process, Uniformly Convex Banach Space, Convergence Theorem.

1.INTRODUCTION

In this paper, we denote by E the set of positive integers and by R the set of real numbers. Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $F(T) = \{Tx=x: x\epsilon C\}$ is denote the set of fixed point of T. Let $T:C\to C$ be mapping then T is said to be,

(a) Contraction- If $\|Tx-Ty\|\leq k\|x-y\|$, for all $x,y\epsilon C$ and $k\epsilon[0,1)$, where k is called contraction constant.
(b) Non expansive- If k=1, i.e. $\|Tx-Ty\|\leq\|x-y\|$, for all $x,y\epsilon C$
(c) Quasi non-expansive- If $\|Tx-p\|\leq\|x-p\|$, for all $x\epsilon C$ and $p\epsilon F(T)$

In 1920, S.Banach[1] proved most important result n complete Metric space, it state that if X is a compete Metric space and $T:X\to X$ is contraction mapping, then T has a unique fixed point.

In 1965, Browder[2], Gohde[3] and Kirk[4] independently prove that every non expansive mapping of a closed convex and bounded subset of uniformly convex Banach space has a fixed point.

Recently, Akutsah et al. [8] introduced the following iteration scheme in the frame work of Banach space. Let $(X,\|.\|)$ be a Banach space and C be a non empty closed convex subset of X.

Let $\{x_n\}$ in iterative manner is given below,

$$z_n=(1-\beta_n)x_n+\beta_nTx_n$$
$$y_n=Tz_n$$
$$x_{n+1}=T((1-\alpha_n)y_n+\alpha_nTy_n) \ , \ n\geq1, \quad (1.1)$$

$$x_{n+1}=Tz_n$$
$$z_n=T((1-\beta_n)x_n+\beta_n Tx_n$$
$$y_n=T((1-\alpha_n)z_n+\alpha_n Tz_n \quad (1.2)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence in(0,1).

2.PRELIMINARIES

Defintion 2.1 - Let C be a non empty subset of Banach space X and T:C→C be mapping then T is said to be,

(a) Mean non-expansive[5], if there exists $\alpha,\beta\geq 0$ with $\alpha+\beta\leq 1$, such that
$$\|Tx-Ty\|\leq\alpha\|x-y\|+\beta\|x-Ty\|, \text{ for all } x,y\epsilon C$$
(b) Satisfy condition(C) Suzuki type[6], if
$$\frac{1}{2}\|Tx-x\|\leq\|x-y\|=>\|Tx-Ty\|\leq\|x-y\|, \text{ for all } x,y\epsilon C$$
(c) Satisfy condition(C_λ), if
$$\lambda\|Tx-x\|\leq\|x-y\|=>\|Tx-Ty\|\leq\|x-y\|, \text{ for all } x,y\epsilon C$$

Let C be a nonempty subset of a Banach space X,a mapping T:C→C is said to satisfy condition(C) if,

$$\frac{1}{2}\|Tx-x\|\leq\|x-y\|=>\|Tx-Ty\|\leq\|x-y\|$$

For all x,yϵC. it is obvious that every condition (C) mapping with a fixed point is quasi non-expansive mapping.

Recently, S-S Zhang in[5], introduced the class of mean non-expansive mappings in Banach spaces and obtained a fixed point theorems for such mappings. Let C be a non empty subset of a Banach space X, a mapping T:C→C is said to be mean non-expansive if for a given real number $\alpha<1$,

$$\|Tx-Ty\|\leq\alpha\|x-y\|+\beta\|x-Ty\| \text{ , for all } x,y\epsilon C.$$

There exists iteration processes which is often used to approximation fixed points of non-expansive mappings.

Let C be non empty subset of Banach space X and T:C→C be mean non-expansive with $F(T)\neq\varphi$, then

(a) T is a quasi non-expansive.
(b) F(T) is closed.

Definition 2.2 Let C be a non empty subset of a uniformly convex Banach space X. A sequence $\{x_n\}$ in X is said to be monotone with respect to subset C, if $\|x_{n+1}-z\|\leq\|x_n-z\|$, for all zϵC, $n\geq 1$.

Preposition 2.3 Let C be a non empty subset of a uniformly convex Banach space X. Suppose that $\{x_n\}$ is Fejer monotone sequence with respect to C. Then the following holds:

(a) Sequence $\{x_n\}$ is bounded.
(b) For every xϵC, $\|x_n-x\|$ converges.

Preposition 2.4 Let C be a nonempty subset of Banach space X and $T:C\to C$ be mean non-expansive mapping $F(T)\neq\varphi$, Then T is quasi non-expansive mappings.

Lemma 2.5 Let X be uniformly convex Banach space and $\{t_n\}$ be a sequence in $[\delta,1-\delta]$ for some $0\leq t_n\leq q<1$ for all $n\epsilon N$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of X such that $\lim_{n\to\infty} sup\|x_n\|\leq r$ and $\lim_{n\to\infty} sup\|y_n\|\leq r$ and $\lim_{n\to\infty} \|t_nx_n+(1-t_n)y_n\|=r$ holds for some $r\geq 0$. Then $\lim_{n\to\infty} \|x_n-y_n\|=0$

Definition 2.6 [18] Let $\{t_n\}$ be any arbitrary sequence in K. An iteration procedure $x_{n+1}=f(T,x_n)$ converging to fixed point z is said to be T stable or stable with respect to T if for

$\epsilon_n=\|t_{n+1}-f(T,t_n)\|$, for all $n\epsilon N$. we have $\lim_{n\to\infty} a_n=0$ if and only if $\lim_{n\to\infty} t_n=z$.

Lemma 2.7 Let $\{\Psi_n\}$ and $\{\xi_n\}$ be non-negative real sequences satisfying the following inequality:

$$\Psi_{n+1}<(1-\varphi_n)\Psi_n+\xi_n,$$

where $\varphi_n\epsilon(0,1)$ for all $n\epsilon N$, $\sum_{n=0}^{\infty} \phi n=\infty$ and $\lim_{n\to\infty} \frac{\xi n}{\phi n}=0$,then $\lim_{n\to\infty} \Psi n=0$.

3.Main Results

Lemma 3.1 Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $T:C\to C$ be a mean non-expansive mapping . Then F(T) is closed.

Proof. Let $\{x_n\}$ be a sequence in F(T) such that $x_n\to x$ as $n\to\infty$. Since T is mean non-expansive mapping. We have,

$$\begin{aligned}
0\leq\|x-Tx\|&=\|x-x_n+x_n-Tx\|\\
&\leq\|x-x_n\|+\|x_n-Tx\|\\
&=\|x-x_n\|+\|Tx_n-Tx\|\\
&\leq\|x-x_n\|+a\|x_n-x\|+b\|x_n-Tx\|\\
&\leq\|(1+a)\|x-x_n\|+b\|x_n-Tx\|\\
&\leq(1+a)\|x-x_n\|+b\|x_n-x\|+b\|x-Tx\|\\
\|x-Tx\|-b\|x-Tx\|&\leq(1+a)\|x-x_n\|+b\|x_n-x\|\\
(1-b)\|x-Tx\|&\leq(1+a+b)\|x-x_n\|\to 0 \text{ as } n\to\infty
\end{aligned}$$

This shows that Tx=x, i.e. $x\epsilon F(T)$.
Hence F(T) is closed.

Lemma 3.2 Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $T:C\to C$ be a mean non-expansive mapping such that $F(T)\neq\varphi$. Let $\{x_n\}$ be a sequence in C defined by (1.1)&(1.2).Then $\lim_{n\to\infty} \|x_n-z\|$ exists for all $z\epsilon F(T)$.

Proof. Let $z\epsilon F(T)$,

$$\begin{aligned}
\|y_n-z\|&=\|Tz_n-z\|\\
&\leq\|Tz_n-Tz\|\\
&\leq a\|z_n-z\|+b\|z_n-Tz\|\\
&=(a+b)\|z_n-z\|\\
&\leq\|z_n-z\|
\end{aligned}$$

$$\|z_n-z\|=\|(1-\beta_n)x_n+\beta_nTx_n-z\|$$
$$\leq(1-\beta_n)\|x_n-z\|+\beta_n\|Tx_n-z\|$$
$$\leq(1-\beta_n)\|x_n-z\|+\beta_n(a\|x_n-z\|+b\|x_n-Tz\|)$$
$$=(1-\beta_n)\|x_n-z\|+\beta_n(a+b)\|x_n-z\|$$
$$\leq\|x_n-z\|$$

And

$$\|X_{n+1}-z\|=\|T((1-\alpha_n)yn+\alpha_nTy_n)-z\|$$
$$\leq a\|(1-\alpha_ny_n)+\alpha_nTyn-z\|+b\|(1-\alpha_n)y_n+\alpha_nTy_n-Tz\|$$
$$\leq a(1-\alpha_n)\|y_n-z\|+a\alpha_n\|Ty_n-z\|+b(1-\alpha_n)\|y_n-Tz\|+b\alpha_n\|Ty_n-Tz\|$$
$$=(a+b)(1-\alpha_n)\|y_n-z\|+(a+b)\alpha_n\|Ty_n-Tz\|$$
$$\leq(1-\alpha_n)\|y_n-z\|+\alpha_n(a\|y_n-z\|+b\|y_n-Tz\|)$$
$$\leq\|y_n-z\|$$
$$\leq\|z_n-z\|\leq\|x_n-z\|$$

This shows that $\{\|x_n-z\|\}$ is bounded and non-decreasing for all $z\epsilon F(T)$.Thus $\{x_n\}$ is bounded and $\{\|x_n-z\|\}$ converges. i.e. $\lim_{n\to\infty}\|x_n-z\|$ exists.

Lemma 3.3 Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $T:C\to C$ be a mean non-expansive such that $F(T)\neq\varphi$, let $\{x_n\}$ be a sequence in C defined by (1.1) & (1.2) and $\{\gamma_n\}$ be a sequence in$[\delta,1-\delta]$ for some $\delta\epsilon(0,1)$.Then $\lim_{n\to\infty}\|x_n-Tx_n\|=0$.

Proof. From Lemma 3.2

$\lim_{n\to\infty}\|x_n-z\|$ exists for all $z\epsilon F(T)$, so suppose that

$\lim_{n\to\infty}\|x_n-z\|=q$, where $q\geq0$. If $q=0$, then

$$\|x_n-Tx_n\|=\|x_n-z+z-Tx_n\|$$
$$\leq\|x_n-z\|+\|z-Tx_n\|$$
$$\leq\|x_n-z\|+a\|x_n-z\|+b\|x_n-Tz\|$$
$$\leq(1+a+b)\|x_n-z\|\to0 \text{ as } n\to\infty$$

Let $q>0$, since $\lim_{n\to\infty}\|x_n-z\|=q\Rightarrow\lim_{n\to\infty}sup\|x_n-z\|\leq q$, also

$$\|Tx_n-z\|\leq(a+b)\|x_n-z\|$$
$$\lim_{n\to\infty}sup\|Tx_n-z\|\leq\lim_{n\to\infty}sup\|x_n-z\|$$
$$\lim_{n\to\infty}sup\|Tx_n-z\|\leq q$$

Let$\{\gamma_n\}$ be a sequence in$[\delta,1-\delta]$ for some $\delta\epsilon(0,1)$, then we have

$$\lim_{n\to\infty}sup\|(1-\gamma_n)(x_n-z)+\gamma_n(Tx_n-z)\|\leq(1-\gamma_n)\lim_{n\to\infty}sup\|x_n-z\|+\gamma_n\lim_{n\to\infty}sup\|Tx_n-z\|$$
$$\lim_{n\to\infty}sup\|(1-\gamma_n)(x_n-z)+\gamma_n(Tx_n-z)\|\leq q$$

So, from lemma 2.5, we have $\lim_{n\to\infty}\|(x_n-z)-(Tx_n-z)\|=0$

i.e. $\lim_{n\to\infty}\|x_n-Tx_n\|=0$.

Theorem 3.4 Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $T:C\to C$ be a mean non-expansive mapping such that $F(T)\neq\varphi$. Let$\{x_n\}$ be a sequence in C defined by (1.1)

& (1.2).Then the sequence $\{x_n\}$ converges strongly to a fixed point of T iff $\lim_{n\to\infty} d(x_n,F(T))=0$, where $d(x_n,F(T))=\inf\{\|x_n-z\|: y\epsilon F(T)\}$.

Proof. Let $\{x_n\}$ converges to z a fixed point of T. Then $\lim_{n\to\infty} d(x_n,z)=0$ and $0\le d(x_n,F(T))\le d(x_n,z)$, it follows that $\lim_{n\to\infty} d(x_n,F(T))=0$.Therefore $\lim_{n\to\infty} d(x_n,F(T))=0$,

Conversaly, suppose that $\lim_{n\to\infty} d(x_n,F(T))=0$, for $\epsilon>0$, there exists $n_o\epsilon N$ such that for all $n\ge n_o$.

$$d(x_n,F(T))<\frac{\epsilon}{4}$$

By lemma (3.2 & 3.3), that $\lim_{n\to\infty} \|x_n-z\|$ exists and that $\lim_{n\to\infty} d(x_n,F(T))$ exists for all $z\epsilon F(T)$.

By our hypothesis,

$$\lim_{n\to\infty} d(x_n,F(T))=0.$$

Suppose $\{x_{no}\}$ is any arbitrary subsequence of $\{x_n\}$ & $\{r_k\}$ is a sequence in F(T) such that for all $n\epsilon N$,

$$\|x_{no}-r_k\|<\frac{\epsilon}{2}$$

It follows that $\|x_{n+m}-r_k\|\le\|x_n-r_k\|<\frac{\epsilon}{2}$, for $n,m\ge n_o$, we have

$$\begin{aligned}\|x_{n+m}-x_n\|&=\|x_{n+m}-r_k+r_k-x_n\|\\&\le\|x_{n+m}-r_k\|+\|r_k-x_n\|\\&=2\|x_{no}-r_k\|\\&<\epsilon\end{aligned}$$

It follows that $\{x_n\}$ is a Cauchy sequence in C. Since C is closed subset of uniformly convex Banach space X. So there exists a point say $x\epsilon C$ such that

$\|x_n-x\|\to0$ as $n\to\infty$. By our assumption $\lim_{n\to\infty} d(x_n,F(T))=0$,

it gives that,

$$d(x,F(T))=0 \Rightarrow x\epsilon F(T).$$

Theorem 3.5 Let C be a nonempty compact convex subset of a uniformly convex Banach space X and $T:C\to C$ be a mean non-expansive mapping such that $a+b<1$ and $F(T)\neq\varphi$. Let $\{x_n\}$ be a sequence in C defined by (1.1) & (1.2). Then the sequence $\{x_n\}$ is T stable(T a function on X).

Proof. Let $\{s_n\}\subseteq X$ be an arbitrary sequence in C and suppose that the sequence $\{x_n\}$ generated by (1.1) & (1.2) is $x_{n+1}=f(T,x_n)$ converges to a point $z\epsilon F(T)$ and $\epsilon_n=\|s_{n+1}-f(T,s_n)\|$. It is sufficient to prove that $\lim_{n\to\infty} \epsilon_n=0$ iff $\lim_{n\to\infty} s_n=z$.

Let $\lim_{n\to\infty} \epsilon_n=0$,

$$\begin{aligned}\|y_n-z\|&\le(a+b)\|z_n-z\|\\&\le\|Tx_n-z\|\\&\le\|x_n-z\|\end{aligned}$$

And

$$\begin{aligned}\|s_{n+1}-z\|&\le\|s_{n+1}-f(T,s_n)\|+\|f(T,s_n)-z\|\\&=\epsilon_n+\|f(T,s_n)-z\|\\&\le\epsilon_n+(a+b)(1-\alpha_n+\alpha_n(a+b))(a+b)(1-\beta_n+\beta_n(a+b))\|s_n-z\|\\&=\epsilon_n+(a+b)^2[(1-\alpha_n)+\alpha_n(a+b)][(1-\beta_n)+\beta_n(a+b)]\|s_n-z\|\\&=\epsilon_n+(a+b)^2[1-((\alpha_n+\beta_n)(1-(a+b)-\alpha_n\beta_n(1-2(a+b))-(a+b)^2]\|s_n-z\|,\end{aligned}$$

Let $\varphi_n=(\alpha_n+\beta_n)(1-(a+b)-\alpha_n\beta_n(1-2(a+b))-(a+b)^2$

Since $\{\alpha_n\}\{\beta_n\}\subset(0,1)$ and $a+b<1$, we conclude that $\alpha_n\beta_n\epsilon(0,1)$, $\alpha_n+\beta_n\epsilon(0,1)$.
Therefore, $\varphi_n\epsilon(0,1)$
By our assumption $\lim_{n\to\infty}\epsilon_n=0$, we have $\lim_{n\to\infty}\frac{\epsilon n}{\phi n}=0$, so from lemma 2.7,
We have $\lim_{n\to\infty} s_n=z$.
Conversely suppose that $\lim_{n\to\infty} s_n=z$, since

$$\epsilon_n=\|s_{n+1}-f(T,s_n)\|$$
$$\leq\|s_{n+1}-z\|+\|z-f(T,s_n)\|$$
$$\leq\|s_{n+1}-z\|+(a+b)^2[1-((\alpha_n+\beta_n)(1-(a+b))-\alpha_n\beta_n(1-2(a+b))-(a+b)^2]\|s_n-z\|\to 0 \text{ as } \lim_{n\to\infty}$$

It follows that $\lim_{n\to\infty}\epsilon_n=0$.This show that the iteration scheme(1.1) & (1.2) is stable(function on X) with respect to T.

4.NUMERICAL EXAMPLE

Example 4.1 Let X be a real uniformly convex Banach space and C=[0,1] be a non empty closed subset of X. Let T:C→C be a mapping defined by

$$Tx=\begin{cases}0, 0\leq x<\frac{1}{2}\\ \frac{x}{8}, \frac{1}{2}\leq x\leq 1\end{cases}$$

Proof- Since T is not continuous at $x=\frac{1}{2}$, hence T is not a non-expansive mapping.

We claim that mean non-expansive mapping.

Case1-Suppose that $0\leq x,y<\frac{1}{2}$.Then $\|Tx-Ty\|=0$.For $a=b=\frac{1}{3}$, we have $\|Tx-Ty\|=0\leq a\|x-y\|+b\|x-Ty\|$.

Case2-Suppose that $\frac{1}{2}\leq x,y\leq 1$.Then $\|Tx-Ty\|=\frac{1}{8}\|x-y\|$. Also

$$a\|x-y\|+b\|x-Ty\|=\frac{1}{3}\|x-y\|+\frac{1}{3}\|x-\frac{y}{8}\|$$
$$\geq\frac{1}{3}\|x-y\|$$
$$\geq\frac{1}{8}\|x-y\| = \|Tx-Ty\|.$$

Case3-Let $0\leq x<\frac{1}{2},\frac{1}{2}\leq y\leq 1$.Then $\|Tx-Ty\| = \frac{1}{8}\|y\|$, and

$$a\|x-y\|+b\|x-Ty\|= \frac{1}{3}\|x-y\|+\frac{1}{3}\|x-\frac{y}{8}\|$$
$$\geq\frac{1}{3}\|x-y-(x-\frac{y}{8})\|$$
$$=\frac{7}{24}\|y\|\geq\frac{1}{8}\|y\|=\|Tx-Ty\|.$$

Case4-Let $\frac{1}{2}\leq x\leq 1, 0<y<\frac{1}{2}$, then $\|Tx-Ty\|=\frac{1}{8}\|x\|$, and

$$a\|x-y\|+b\|x-Ty\|=\frac{1}{3}\|x-y\|+\frac{1}{3}\|x\|$$
$$\geq\frac{1}{3}\|x\|\geq\frac{1}{8}\|x\|=\|Tx-Ty\|.$$

Thus we conclude that T is mean non-expansive mapping.
Comparison between convergence of (1.1) & (1.2) with Akutasah[8], Abbas[9],Thakur[14] iteration scheme to the fixed point x=0 of T in example[4.1].

Iteration	Akutsah	Thakur	Abbas
0	0.50000000	0.50000000	0.50000000
1	0.00000000	-0.01953125	0.00781250
2	0.00000000	0.00029564	-0.00002480
3	0.00000000	-0.00000257	0.00000011
4	0.00000000	0.00000001	0.00000000
5	0.00000000	0.00000000	0.00000000
6	0.00000000	0.00000000	0.00000000

5.CONCLUSION

We conclude that via numerical example our iterative process is faster than the well known iteration appeared in the literature and our results obtained in this paper improve.

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