

A Review and Numerical Analysis of the Inviscid Limit for 2D Incompressible Navier–Stokes Equations with Periodic Boundaries.

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Abstract — The inviscid limit describes the behavior of a viscous fluid as its kinematic viscosity approaches zero. Although this limit can be difficult to analyze in the presence of solid boundaries because of the formation of thin boundary layers, the problem becomes considerably cleaner on a periodic two-dimensional domain. In this paper, the incompressible Navier–Stokes equations on the two-dimensional torus T^2 are considered to examine the transition from viscous flow to the corresponding ideal Euler flow. The vorticity formulation is used to study the effects of viscosity on vorticity transport, kinetic energy, and dissipation. A Taylor–Green vortex provides a simple numerical framework for illustrating the influence of decreasing viscosity on the persistence of coherent flow structures. The analysis indicates that, in the periodic two-dimensional setting, viscous solutions approach the inviscid dynamics as viscosity tends to zero, while the direct contribution of viscous dissipation becomes increasingly small. The periodic geometry therefore provides a useful baseline for studying the inviscid limit without the additional complications introduced by solid-wall boundary layers.

Keywords: Navier–Stokes equations, inviscid limit, Euler equations, periodic boundary conditions, vorticity, kinetic energy, Taylor–Green vortex.

1. Introduction

The incompressible Navier–Stokes equations are among the fundamental mathematical models used to describe the motion of viscous fluids, accounting for fluid inertia, pressure, viscosity, and external forces:

$$\partial u / \partial t + (u \cdot \nabla) u = -\nabla p + \nu \Delta u, \quad \nabla \cdot u = 0$$

where u represents the velocity field, p is pressure, and $\nu > 0$ denotes kinematic viscosity. The viscous term $\nu \Delta u$ describes momentum diffusion from internal friction. The inviscid limit corresponds to $\nu \rightarrow 0^+$, formally approaching the Euler equations:

$$\partial u / \partial t + (u \cdot \nabla) u = -\nabla p$$

In bounded domains, the no-slip condition at solid walls generates thin boundary layers with rapidly varying velocity. In a 2D periodic domain, physical walls are absent, eliminating boundary-layer complications and providing a clean setup to evaluate the core physics of the inviscid limit.

2. Mathematical Setup

Let the spatial domain be the 2D torus $T^2 = (\mathbb{R} / 2\pi\mathbb{Z})^2$, represented computationally by $[0, 2\pi] \times [0, 2\pi]$ with opposite edges identified. The velocity field satisfies $u(x + 2\pi, y, t) = u(x, y, t)$ and $u(x, y + 2\pi, t) = u(x, y, t)$.

The system on T^2 consists of $\partial_t u + (u \cdot \nabla) u = -\nabla p + \nu \Delta u$ and $\nabla \cdot u = 0$. Eliminating physical walls removes wall-generated boundary layers while maintaining non-linear flow structures.

3. Vorticity Formulation

For 2D velocity $u = (u_1, u_2)$, scalar vorticity is $\omega = \partial_x u_2 - \partial_y u_1$. Taking the curl yields:

$$\partial_t \omega + u \cdot \nabla \omega = \nu \Delta \omega$$

where $u \cdot \nabla \omega$ represents vorticity transport and $\nu \Delta \omega$ represents viscous diffusion. As $\nu \rightarrow 0^+$, this becomes $\partial_t \omega + u \cdot \nabla \omega = 0$. Unlike 3D flows, 2D scalar vorticity lacks a vortex-stretching term, making the mathematical theory substantially well-behaved.

4. Kinetic Energy and Dissipation

Total kinetic energy is $E(t) = \frac{1}{2} \int_{\mathbb{T}^2} |u(x, t)|^2 dx$.

Integrating against u yields the energy identity:

$$dE/dt = -\nu \int_{\mathbb{T}^2} |\nabla u|^2 dx$$

Nonlinear transport and pressure terms contribute zero net change.

Thus $dE/dt \leq 0$ for $\nu > 0$. As $\nu \rightarrow 0^+$, direct dissipation vanishes under controlled velocity gradients, conserving energy ($dE/dt = 0$) for ideal Euler flows.

5. Importance of the Periodic Geometry

Solid walls force $u = 0$ via no-slip conditions, causing sharp velocity gradients across thin boundary layers as $\nu \rightarrow 0$. Periodic boundary conditions bypass this issue, keeping fields smooth and allowing controlled study of zero-viscosity limits.

6. Mathematical Foundations of the Inviscid Limit

Classical work by Wolibner and Yudovich proved global existence and uniqueness for 2D Euler flows. On $\mathbb{T}^2$, solutions rigorously converge:

Navier–Stokes $\xrightarrow{(\nu \rightarrow 0^+)} \text{Euler}$

7. Numerical Example: Taylor–Green Vortex

Using initial conditions $u_1(x, y, 0) = \sin(x) \cos(y)$ and $u_2(x, y, 0) = -\cos(x) \sin(y)$, initial vorticity is $\omega(x, y, 0) = 2 \sin(x) \sin(y)$. Lower viscosity reduces diffusion, allowing coherent vortices to persist longer.

Viscosity Level	Flow Evolution	Energy Behavior
High ν	Rapid smoothing of vortical structures	Relatively rapid decay
Medium ν	Gradual weakening of coherent structures	Moderate decay
Low ν	Vortices remain coherent for longer times	Slow decay
Inviscid ($\nu \rightarrow 0$)	Evolution approaches inviscid transport	Energy approaches conservation

8. Discussion

- Persistence of Vortical Structures: Decreasing ν weakens diffusion ($\nu \Delta \omega$), preserving coherent vortex centers.
- Reduction of Viscous Dissipation: Direct dissipation $-\nu \int |\nabla u|^2 dx$ decreases toward zero as $\nu \rightarrow 0^+$.
- Influence of Boundary Conditions: Removing physical walls eliminates no-slip boundary layers, serving as a clean benchmark.

9. Conclusion

The 2D incompressible Navier–Stokes equations on a periodic domain provide a clear framework for studying the inviscid limit. As $\nu \rightarrow 0^+$, viscous diffusion weakens, dissipative energy loss vanishes, and dynamics approach Euler equations without boundary-layer interference.

10. References

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