

Analysis of Fractional Gas Dynamic Equations using Atangana-Baleanu Derivative Operator

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Abstract- This paper presents the approximate analytical solutions to solve the nonlinear homogeneous and non-homogeneous gas dynamics equations with Atangana-Baleanu fractional derivative operator. The validity of the solutions is verified by providing the existence and uniqueness criteria of solution for used technique. To illustrate the ability and efficiency of the method, absolute error, tables and graphical presentations of numerical solution for specific cases are obtained.

Keywords: Atangana-Baleanu operator; Fractional gas Dynamic equations; Iterative Laplace transform method.

1. Introduction

Gas dynamic equations are mathematical representations defined as the physical laws of energy conservation, mass conservation, momentum conservation, etc. Nonlinear fractional-order gas dynamics equations are applied in shock fronts, unusual factions, and connection discontinuities. Gas dynamics is a study in the field of fluid dynamics that studies gas motion and its effect on physical constructions based on the concept of fluid mechanics and fluid dynamics. In 1981, Steger and Warming [23] addressed that the conservation-law form of the inviscid gas dynamic equation possesses a remarkable property by virtue of which the nonlinear flux vectors are homogeneous functions. The science emerges from research of gas flows involve choked flows in nozzles and pipes, gas fuel streams in a rocket engine, aerodynamic heating on atmospheric reentry cars, and shock waves around aircraft [8].

During the last decade, fractional calculus has found applications in numerous seemingly diverse fields of science and engineering. Fractional differential equations are increasingly used to model problems in fluid mechanics, acoustics, biology, electromagnetism, diffusion, signal processing, and many other physical processes [14, 24, 13, 12].

A major importance is given to the fractional calculus due to its very important role in various fields over the integral calculus to form the various day-today models and to study these models. To find the better approximation, convergent numerical solution of a nonlinear partial and fractional partial differential equation numerous methods are established in past years by several scholars and scientists. There are so many recent approaches which solve analytically or numerically as finite difference method, differential transform method, homotopy analysis method, homotopy analysis transform method, and several many iterative approaches [2, 9, 21, 1, 10, 18]. Recently, Biazar and Eslami [5] and Das and Kumar [7] have applied to obtain the solutions of the homogenous and non-homogenous time-fractional gas dynamics equation. Rasulov et al. [16] used the finite difference method to find the solution of nonlinear equations of gas dynamic problems for discontinuous functions. In [15], Prakash and Kumar investigated numerical method for time-fractional gas dynamic equations. Solutions of fractional-Order gas dynamics equation by effective techniques is obtained in [11] by Iqbal et.al. Mohamed [3] used new iterative method for fractional gas dynamics and coupled burger's equations.

These fractional differential equation involves several fractional differential operators like Riemann-Liouville operator [19], Caputo operator [20], Caputo-Fabrizio operator [17, 22], etc. However these operators possesses a power law kernel, exponential kernel and has singularity. Therefore, they have some limitations in

modeling physical problems. To overcome this difficulty, in recent times Atangana and Baleanu have proposed a reliable operator having nonlocal and nonsingular kernel in the form of Mittag-Leffler function known as Atangana-Baleanu operator [4].

The goal of this study is to apply iterative Laplace transform to find approximate solutions of time fractional homogeneous and non-homogeneous gas dynamic equations having Atangana-Baleanu operator. These equations are given below as follows:

$$\frac{ABC_{\partial^{\beta}u}}{\partial t^{\beta}} + u \frac{\partial u}{\partial x} - (1-u) \log \lambda = 0, \quad \lambda > 0, \quad 0 < \beta \leq 1 \quad (1.1)$$

with initial condition $u(x, 0) = \lambda^{-x}$ and

$$\frac{ABC_{\partial^{\beta}u}}{\partial t^{\beta}} + u \frac{\partial u}{\partial x} + (1+t)^2 u^2 = x^2, \quad 0 < \beta \leq 1 \quad (1.2)$$

with initial condition $u(x, 0) = x$

The paper is organized in following order. In Section 2, basic results of fractional calculus is presented. Working rule for iterative Laplace transform method is given in Section 3. Section 4, presents existence and uniqueness criteria for solutions of time fractional homogeneous and non-homogeneous gas dynamic equations. In section 5, numerical simulation, tables and plots for the obtained solutions are obtained. Finally, we give our conclusions in Section 6.

2. Preliminaries

Definition 2.1 The left and right Atangana-Baleanu fractional derivative for a given function of order β in Caputo sense are defined as

$${}^{ABC}D_t^{\beta} f(t) = \frac{B(\beta)}{1-\beta} \int_a^t \frac{df(\tau)}{d\tau} E_{\beta} \left[-\frac{\beta}{(1-\beta)} (t-\tau)^{\beta} \right] d\tau, \quad (left)$$

$${}^{ABC}D_t^{\beta} f(t) = -\frac{B(\beta)}{1-\beta} \int_t^b \frac{df(\tau)}{d\tau} E_{\beta} \left[-\frac{\beta}{(1-\beta)} (t-\tau)^{\beta} \right] d\tau, \quad (right)$$

where $B(\beta) = (1-\beta) + \frac{\beta}{\Gamma(\beta)}$ is normalization function and $E_{\beta}(\cdot)$ is the Mittag-Leffler function.

Definition 2.2 Atangana-Baleanu fractional integral of order β is defined as

$${}^{AB}I_t^{\beta} (f(t)) = \frac{1-\beta}{B(\beta)} f(t) + \frac{\beta}{B(\beta)\Gamma(\beta)} \int_a^t f(\tau) (t-\tau)^{(\beta-1)} d\tau$$

if $f(t)$ is a constant, integral will be resulted to zero.

3. Fundamental idea of iterative Laplace transform method

Here, we demonstrate the fundamental solution procedure of the proposed algorithm by Daftardar-Gejji and Jafari [6] by considering the arbitrary fractional order differential equation with Atangana Baleanu operator to

$${}^{ABC}D_t^{\beta} u(x,t) + \mathcal{R}u(x,t) + \mathcal{N}u(x,t) = f(x,t) \quad (3.1)$$

$$\text{with initial condition } u(x, t) = \psi(x, t) \quad (3.2)$$

Where $\mathcal{R}$ and $\mathcal{N}$ are given linear and non-linear operator respectively and $f(x, t)$ denotes source term. Applying Laplace transform on (3.1) we obtain

$$L\{u(x, t)\} - \frac{1}{s} u(x, 0) + \frac{1}{B(\beta)} (1 - \beta + \frac{\beta}{s^\beta}) (L\{\mathcal{R}u(x, t)\} + L\{\mathcal{N}u(x, t)\} - L\{f(x, t)\}) = 0 \quad (3.3)$$

After regrouping terms we obtain

$$L\{u(x, t)\} = \frac{1}{s} \psi(x, t) - \frac{1}{B(\beta)} (1 - \beta + \frac{\beta}{s^\beta}) (L\{\mathcal{R}u(x, t)\} + L\{\mathcal{N}u(x, t)\} - L\{f(x, t)\}) \quad (3.4)$$

Further, we apply inverse Laplace transform on (3.4) then we obtain

$$u(x, t) = \psi(x, t) + L^{-1} \left[\frac{-1}{B(\beta)} (1 - \beta + \frac{\beta}{s^\beta}) (L\{\mathcal{R}u(x, t)\} + L\{\mathcal{N}u(x, t)\} - L\{f(x, t)\}) \right] \quad (3.5)$$

Next, we use iterative method [6] to obtain series solution as below given by,

$$u(x, t) = \sum_{j=0}^{\infty} u_j(x, t) \quad (3.6)$$

As $\mathcal{R}$ is linear,

$$\mathcal{R} \left(\sum_{j=0}^{\infty} u_j(x, t) \right) = \sum_{j=0}^{\infty} \mathcal{R}(u_j(x, t)) \quad (3.7)$$

The non linear operator $\mathcal{N}$ is decomposed as

$$\mathcal{N} \left(\sum_{j=0}^{\infty} u_j(x, t) \right) = \mathcal{N}(u_0(x, t)) + \sum_{j=1}^{\infty} \left\{ \mathcal{N} \left(\sum_{i=0}^j u_i(x, t) \right) - \mathcal{N} \left(\sum_{i=0}^{j-1} u_i(x, t) \right) \right\} \quad (3.8)$$

$$= \sum_{i=0}^{\infty} \mathcal{P}_i \quad (3.9)$$

where $\mathcal{P}_0 = \mathcal{N}(u_0(x, t))$ and $\mathcal{P}_i = \mathcal{N}(\sum_{i=0}^j u_i(x, t)) - \mathcal{N}(\sum_{i=0}^{j-1} u_i(x, t))$, $i \geq 1$

By using of (3.6), (3.7) and (3.8), the equation (3.1) is transforms to

$$\begin{aligned} \sum_{j=0}^{\infty} u_j(x, t) &= \psi(x, t) \\ &+ L^{-1} \left[\frac{-1}{B(\beta)} (1 - \beta + \frac{\beta}{s^\beta}) (L \{ \sum_{j=0}^{\infty} \mathcal{R}(u_j(x, t)) \} \right. \\ &\quad \left. + L \{ \mathcal{N}(u_0(x, t)) + \sum_{j=1}^{\infty} \left\{ \mathcal{N} \left(\sum_{i=0}^j u_i(x, t) \right) - \mathcal{N} \left(\sum_{i=0}^{j-1} u_i(x, t) \right) \right\} - L\{f(x, t)\} \} \right] \quad (3.10) \end{aligned}$$

Further, consider the recurrence relation as given below

$$u_0(x, t) = \psi(x, t)$$

$$u_1(x, t) = L^{-1} \left\{ \frac{-1}{B(\beta)} \left(1 - \beta + \frac{\beta}{s^\beta} \right) (L\{\mathcal{R}(u_0(x, t))\} + L\{\mathcal{N}(u_0(x, t))\}) \right\} \quad (3.11)$$

⋮

$$u_{n+1}(x, t) = L^{-1} \left\{ \frac{-1}{B(\beta)} \left(1 - \beta + \frac{\beta}{s^\beta} \right) (L\{\mathcal{R}(u_n(x, t))\} + L\{\mathcal{N}(\sum_{i=0}^j u_i(x, t)) - \mathcal{N}(\sum_{i=0}^{j-1} u_i(x, t))\}) \right\} \quad (3.12)$$

The n-term approximate solution is given by

$$u = u_0 + u_1 + u_2 + \cdots + u_{n-1} \quad (3.13)$$

4. Existence and Uniqueness

4.1 Existence of solutions for the homogeneous fractional gas dynamics equation

In this section, we have obtained existence and uniqueness criteria of the solution for the gas dynamics equation with the help of fixed-point theorem. Let us consider the time fractional gas dynamics equation (1.1) using Atangana-Baleanu fractional derivative operator as below

$${}^{ABC}D^\beta u(x, t) = \mathcal{H}(x, t, u(x, t)) = -u \frac{\partial u}{\partial x} + (1-u) \log \lambda \quad (4.1)$$

where ${}^{ABC}D^\beta$ represents the fractional operator of type Atangana-Baleanu-Caputo (ABC) having fractional order β ; where $0 < \beta < 1$, subject to initial conditions $u_0(x, t) = u(x, t)$

The equation (4.1) can be converted to the Volterra-type integral equation by using the ABC fractional integral which is written by referring definition 2.2 as follows:

$$u(x, t) - u(x, 0) = \frac{1-\beta}{M(\beta)} \mathcal{H}(x, t, u) + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_a^t \mathcal{H}(x, t, u)(t-\tau)^{(\beta-1)} d\tau \quad (4.2)$$

Theorem 4.1: The kernels $\mathcal{H}(x, t, u)$ given in Eq. (4.1) satisfy the Lipschitz condition and contraction if the following condition holds

$$0 \leq \frac{\mu}{2} (\varphi_1 + \varphi_2) + \log \lambda (1 + \varphi_1 + \varphi_2) < 1$$

Proof: We have $\mathcal{H}(x, t, u(x, t)) = -u \frac{\partial u}{\partial x} + (1-u) \log \lambda$.

Consider u_1 and u_2 be two functions then we get the following

$$\begin{aligned}
 & \| \mathcal{H}(x, t, u_1(x, t)) - \mathcal{H}(x, t, u_2(x, t)) \| \\
 &= \| -u_1(x, t) \frac{\partial u_1(x, t)}{\partial x} + u_2(x, t) \frac{\partial u_2(x, t)}{\partial x} + \log \lambda (u_1(x, t) - u_2(x, t)) \\
 &\quad + \log \lambda (-u_1^2(x, t) + u_2^2(x, t)) \| \\
 &\leq \frac{1}{2} \| \frac{\partial}{\partial x} [u_1^2(x, t) - u_2^2(x, t)] \| + \log \lambda \| u_1(x, t) - u_2(x, t) \| \\
 &\leq \frac{\mu}{2} \| \frac{\partial}{\partial x} [u_1^2(x, t) - u_2^2(x, t)] \| + \log \lambda \| u_1(x, t) - u_2(x, t) \| \\
 &\quad + \log \lambda (\varphi_1 + \varphi_2) \| u_1(x, t) - u_2(x, t) \| \\
 &\leq \frac{\mu}{2} (\varphi_1 + \varphi_2) + \log \lambda (1 + \varphi_1 + \varphi_2) \| u_1(x, t) - u_2(x, t) \| \\
 &\leq \Delta \| u_1(x, t) - u_2(x, t) \|.
 \end{aligned}$$

where $\mu = \frac{\partial}{\partial x}$ is the differential operator. u_1 and u_2 are bounded functions, therefore

we get $\|u_1\| \leq \varphi_1, \|u_2\| \leq \varphi_2$.

Moreover, $\Delta = (\frac{\mu}{2}(\varphi_1 + \varphi_2) + \log \lambda (1 + \varphi_1 + \varphi_2))$. Hence we get

$$\| \mathcal{H}(x, t, u_1(x, t)) - \mathcal{H}(x, t, u_2(x, t)) \| \leq \Delta \| u_1(x, t) - u_2(x, t) \| \quad (4.3)$$

As a result, the Lipschitz condition is obtained for $\mathcal{H}$. Furthermore, we see that if

$0 \leq (\frac{\mu}{2}(\varphi_1 + \varphi_2) + \log \lambda (1 + \varphi_1 + \varphi_2)) < 1$ then it implies the contraction.

The equation (4.2) is defined as follows

$$u_n(x, t) = \frac{1 - \beta}{M(\beta)} \mathcal{H}(x, t, u_{n-1}) + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_a^t \mathcal{H}(x, t, u_{n-1})(t - \tau)^{(\beta-1)} d\tau. \quad (4.4)$$

Next, the following expression gives the difference between the successive iterative terms

$$\begin{aligned}
 \pi_n(x, t) &= u_n(x, t) - u_{n-1}(x, t) \\
 &= \frac{1 - \beta}{M(\beta)} (\mathcal{H}(x, t, u_{n-1}) - \mathcal{H}(x, t, u_{n-2})) \\
 &\quad + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_0^t (\mathcal{H}(x, t, u_{n-1}) - \mathcal{H}(x, t, u_{n-2}))(t - \tau)^{(\beta-1)} d\tau.
 \end{aligned} \quad (4.5)$$

Note that

$$u_n(x, t) = \sum_{i=1}^n \pi_i(x, t) \quad (4.6)$$

Next, using equation (4.4) and (4.5), we get

$$\| \pi_n(x, t) \| \leq \frac{1 - \beta}{M(\beta)} \Delta \| \pi_n(x, t) \| + \frac{\beta}{M(\beta)\Gamma(\beta)} \Delta \int_0^t \| \pi_n(x, \tau) \| d\tau. \quad (4.7)$$

This gives the required result of the theorem.

Theorem 4.2: The solution for the (4.1) will exist and unique under the condition that we can find t_0 satisfying

$$\frac{1-\beta}{M(\beta)}\Delta + \frac{\beta t_0}{M(\beta)\Gamma(\beta)}\Delta < 1.$$

Proof: We assume that bounded function $u(x, t)$ satisfies the Lipschitz condition. Using equations (4.5) and (4.7), we get the following equation

$$\|\pi_n(x, t)\| \leq \|u_n(x, 0)\| \left[\frac{1-\beta}{M(\beta)}\Delta + \frac{\beta}{M(\beta)\Gamma(\beta)}\Delta \right]^n \quad (4.8)$$

This shows that the solution is smooth, moreover existence is proved for the obtained solution. Further, we show that the equation (4.8) is the solution of the equation (4.1). For this, we consider

$$u(x, t) - u(x, 0) = u_n(x, t) - \omega_n(x, 0). \quad (4.9)$$

where $\omega_n(x, t)$ are reminder terms of series solution. Then, we must show that these terms approach to zero at infinity, that is, $\|\omega_n(x, t)\| \rightarrow 0$.

$$\begin{aligned} \|\omega_n(x, t)\| &= \left\| \frac{1-\beta}{M(\beta)}(\mathcal{H}(x, t, u) - \mathcal{H}(x, t, u_{n-1})) \right. \\ &\quad \left. + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_a^t (\mathcal{H}(x, \tau, u) - \mathcal{H}(x, \tau, u_{n-1}))(t - \tau)^{(\beta-1)} d\tau \right\| \\ &\leq \frac{1-\beta}{M(\beta)} \|\mathcal{H}(x, t, u) - \mathcal{H}(x, t, u_{n-1})\| \\ &\quad + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_a^t \|\mathcal{H}(x, \tau, u) - \mathcal{H}(x, \tau, u_{n-1})\| (t - \tau)^{(\beta-1)} d\tau \\ &\leq \frac{1-\beta}{M(\beta)} \Delta \|u - u_{n-1}\| + \frac{\beta}{M(\beta)\Gamma(\beta)} \Delta \|u - u_{n-1}\| t. \end{aligned} \quad (4.10)$$

Hence, Continuing this way recursively at t_0 we get

$$\|\omega_n(x, t)\| = \left(\frac{1-\beta}{M(\beta)}\Delta + \frac{\beta t_0}{M(\beta)\Gamma(\beta)}\Delta \right)^{n+1} \Delta^{n+1} P. \quad (4.11)$$

where $P = \|u - u_{n-1}\|$. After taking limit of both sides as n tends to infinity, we get $\|\omega_n(x, t)\| \rightarrow 0$

Next, it is necessity to demonstrate uniqueness for the solution of the proposed problem.

Suppose, $u^*(x, t)$ be the another solution, then we get

$$\begin{aligned} u(x, t) - u^*(x, t) &= \frac{1-\beta}{M(\beta)} (\mathcal{H}(x, t, u) - \mathcal{H}(x, t, u^*)) \\ &\quad + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_a^t (\mathcal{H}(x, \tau, u) - \mathcal{H}(x, \tau, u^*))(t - \tau)^{(\beta-1)} d\tau \end{aligned} \quad (4.12)$$

On applying norms on both side of above equation we get

$$\begin{aligned} \|u(x, t) - u^*(x, t)\| &= \left\| \frac{1-\beta}{M(\beta)} (\mathcal{H}(x, t, u) - \mathcal{H}(x, t, u^*)) \right. \\ &\quad \left. + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_a^t (\mathcal{H}(x, \tau, u) - \mathcal{H}(x, \tau, u^*))(t - \tau)^{(\beta-1)} d\tau \right\| \\ &\leq \frac{1-\beta}{M(\beta)} \Delta \|u(x, t) - u^*(x, t)\| + \frac{\beta}{M(\beta)\Gamma(\beta)} \Delta t \|u(x, t) - u^*(x, t)\|. \end{aligned} \quad (4.13)$$

After simplification we get

$$\|u(x, t) - u^*(x, t)\| \left(1 - \frac{1 - \beta}{M(\beta)} \Delta + \frac{\beta t}{M(\beta)\Gamma(\beta)} \Delta\right) \leq 0. \quad (4.14)$$

From the above inequality, we get that if

$$\left(1 - \frac{1 - \beta}{M(\beta)} \Delta + \frac{\beta t}{M(\beta)\Gamma(\beta)} \Delta\right) \geq 0 \quad (4.15)$$

then $u(x, t) - u^*(x, t) = 0$.

Therefore, (4.15) is required condition for uniqueness.

4 Existence and Uniqueness

4.1 Existence of solutions for the homogeneous fractional gas dynamics equation

In this section, we have obtained existence and uniqueness criteria of the solution for the gas dynamics equation with the help of fixed-point theorem. Let us consider the time fractional gas dynamics equation (1.1) using Atangana-Baleanu fractional derivative operator as below

$${}^{ABC}_{D^\beta} u(x, t) = \mathcal{T}(x, t, u(x, t)) = -u \frac{\partial u}{\partial x} - (1+t)^2 u^2 + x^2 \quad (4.16)$$

where ${}^{ABC}_{D^\beta}$ represents the fractional operator of type Atangana-Baleanu-Caputo (ABC) having fractional order β ; where $0 < \beta < 1$, subject to initial condition $u_0(x, t) = u(x, 0)$. The equation (4.16) can be converted to the Volterra-type integral equation by using the ABC fractional integral as follows:

$$u(x, t) - u(x, 0) = \frac{1 - \beta}{M(\beta)} \mathcal{T}(x, t, u) + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_a^t \mathcal{T}(x, \tau, u) (t - \tau)^{(\beta-1)} d\tau \quad (4.17)$$

Theorem 4.3 The kernel $\mathcal{T}(x, t, u)$ given in Eq. (4.16) satisfy the Lipschitz condition and contraction if the following condition holds

$$0 \leq \left(\frac{\nu}{2}(\varphi_3 + \varphi_4) + (1+t)^2(\varphi_3 + \varphi_4)\right) < 1$$

Proof: Let $\mathcal{T}(x, t, u(x, t)) = -u \frac{\partial u}{\partial x} - (1+t)^2 u^2 + x^2$.

Consider u_1 and u_2 be two functions, then we obtain the following

$$\begin{aligned} & \|\mathcal{T}(x, t, u_1(x, t)) - \mathcal{T}(x, t, u_2(x, t))\| \\ &= \left\| -u_1(x, t) \frac{\partial u_1(x, t)}{\partial x} + u_2(x, t) \frac{\partial u_2(x, t)}{\partial x} - (1+t)^2 u_1^2(x, t) + (1+t)^2 u_2^2(x, t) \right\| \\ &\leq \frac{1}{2} \left\| \frac{\partial}{\partial x} [u_1^2(x, t) - u_2^2(x, t)] \right\| + (1+t)^2 \|u_1^2(x, t) - u_2^2(x, t)\| \\ &\leq \frac{\nu}{2} \| [u_1^2(x, t) - u_2^2(x, t)] \| + (1+t)^2 \|u_1^2(x, t) - u_2^2(x, t)\| \\ &\leq \left(\frac{\nu}{2}(\varphi_3 + \varphi_4) + (1+t)^2(\varphi_3 + \varphi_4)\right) \|u_1(x, t) - u_2(x, t)\| \\ &\leq \Theta \|u_1(x, t) - u_2(x, t)\|. \end{aligned}$$

where $\nu = \frac{\partial}{\partial x}$ is the differential operator. Since, u_1 and u_2 are bounded functions, therefore we get $\|u_1\| \leq \varphi_3, \|u_2\| \leq \varphi_4$.

Also, $\Theta = \left(\frac{\nu}{2}(\varphi_3 + \varphi_4) + (1+t)^2(\varphi_3 + \varphi_4)\right)$. Therefore we have

$$\|\mathcal{T}(x, t, u_1(x, t)) - \mathcal{T}(x, t, u_2(x, t))\| \leq \Theta \|u_1(x, t) - u_2(x, t)\| \quad (4.18)$$

This shows that, the Lipschitz condition is obtained for $\mathcal{T}$. Moreover, we see that if

$$0 \leq \binom{2}{2}(\varphi_3 + \varphi_4) + (1 + t)^2(\varphi_3 + \varphi_4) < 1 \text{ then it implies the contraction.}$$

The recursive form equation (4.17) is defined as follows

$$u_n(x, t) = \frac{1 - \beta}{M(\beta)} \mathcal{T}(x, t, u_{n-1}) + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_a^t \mathcal{T}(x, \tau, u_{n-1})(t - \tau)^{(\beta-1)} d\tau. \quad (4.19)$$

Next, we get the difference between the successive iterative terms in the form of following expression

$$\sigma_n(x, t) = u_n(x, t) - u_{n-1}(x, t)$$

$$\begin{aligned} &= \frac{1 - \beta}{M(\beta)} (\mathcal{T}(x, t, u_{n-1}) - \mathcal{T}(x, t, u_{n-2})) \\ &+ \frac{\beta}{M(\beta)\Gamma(\beta)} \int_0^t (\mathcal{T}(x, \tau, u_{n-1}) - \mathcal{T}(x, \tau, u_{n-2}))(t - \tau)^{(\beta-1)} d\tau. \end{aligned} \quad (4.20)$$

Note that

$$u_n(x, t) = \sum_{i=1}^n \sigma_i(x, t) \quad (4.21)$$

By, using equation (4.19) and then using norm of equation (4.20), we get

$$\|\sigma_n(x, t)\| \leq \frac{1 - \beta}{M(\beta)} \Theta \|\sigma_{n-1}(x, t)\| + \frac{\beta}{M(\beta)\Gamma(\beta)} \Theta \int_0^t \|\sigma_n(x, \tau)\| d\tau. \quad (4.22)$$

This completes the proof of the theorem.

Theorem 4.4: The solution for the (4.16) will exist and unique under the condition that we can find t_0 satisfying

$$\frac{1 - \beta}{M(\beta)} \Theta + \frac{\beta t_0}{M(\beta)\Gamma(\beta)} \Theta < 1.$$

Proof: First we consider that bounded function $u(x, t)$ satisfies the Lipschitz condition. Using equations (4.20) and (4.22), we get the following equation

$$\|\sigma_n(x, t)\| \leq \|u_n(x, 0)\| \left[\frac{1 - \beta}{M(\beta)} \Theta + \frac{\beta}{M(\beta)\Gamma(\beta)} \Theta \right]^n \quad (4.23)$$

Therefore, the solution is smooth, moreover existence is proved for the obtained solution. Next, we show that the equation (4.23) is the solution for the equation (4.16). For this, we consider

$$u(x, t) - u(x, 0) = u_n(x, t) - \xi_n(x, 0). \quad (4.24)$$

where $\xi_n(x, t)$ are reminder terms of series solution. Then, we must show that these terms approach to zero at infinity, that is, $\|\xi_n(x, t)\| \rightarrow 0$.

$$\begin{aligned} \|\xi_n(x, t)\| &= \left\| \frac{1 - \beta}{M(\beta)} (\mathcal{T}(x, t, u) - \mathcal{T}(x, t, u_{n-1})) \right. \\ &\quad \left. + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_a^t (\mathcal{T}(x, \tau, u) - \mathcal{T}(x, \tau, u_{n-1}))(t - \tau)^{(\beta-1)} d\tau \right\| \end{aligned}$$

$$\begin{aligned}
 & \leq \frac{1-\beta}{M(\beta)} \|(\mathcal{T}(x, t, u) - \mathcal{T}(x, t, u_{n-1}))\| \\
 & \quad + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_a^t \|(\mathcal{T}(x, \tau, u) - \mathcal{T}(x, \tau, u_{n-1}))\| (t-\tau)^{(\beta-1)} d\tau \\
 & \leq \frac{1-\beta}{M(\beta)} \Theta \|u - u_{n-1}\| + \frac{\beta}{M(\beta)\Gamma(\beta)} \Theta \|u - u_{n-1}\| t.
 \end{aligned} \tag{4.25}$$

Further, we continue recursively at t_0 we get:

$$\|\xi_n(x, t)\| = \left(\frac{1-\beta}{M(\beta)} \Theta + \frac{\beta t_0}{M(\beta)\Gamma(\beta)} \right)^{n+1} \Theta^{n+1} N. \tag{4.26}$$

where $N = \|u - u_{n-1}\|$. After taking limit of both sides as n tends to infinity, we get $\|\xi_n(x, t)\| \rightarrow 0$

Next, it is necessity to demonstrate uniqueness for the solution of the proposed problem.

Suppose, $u^*(x, t)$ be the another solution, then we get

$$\begin{aligned}
 u(x, t) - u^*(x, t) &= \frac{1-\beta}{M(\beta)} (\mathcal{T}(x, t, u) - \mathcal{T}(x, t, u^*)) \\
 & \quad + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_a^t (\mathcal{T}(x, \tau, u) - \mathcal{T}(x, \tau, u^*)) (t-\tau)^{(\beta-1)} d\tau
 \end{aligned} \tag{4.27}$$

On applying norms on both side of above equation we get

$$\begin{aligned}
 \|u(x, t) - u^*(x, t)\| &= \left\| \frac{1-\beta}{M(\beta)} (\mathcal{T}(x, t, u) - \mathcal{T}(x, t, u^*)) \right. \\
 & \quad \left. + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_a^t (\mathcal{T}(x, \tau, u) - \mathcal{T}(x, \tau, u^*)) (t-\tau)^{(\beta-1)} d\tau \right\| \\
 &\leq \frac{1-\beta}{M(\beta)} \Theta \|u(x, t) - u^*(x, t)\| + \frac{\beta}{M(\beta)\Gamma(\beta)} \Theta t \|u(x, t) - u^*(x, t)\|.
 \end{aligned} \tag{4.28}$$

After simplification we get

$$\|u(x, t) - u^*(x, t)\| \left(1 - \frac{1-\beta}{M(\beta)} \Theta + \frac{\beta t}{M(\beta)\Gamma(\beta)} \Theta \right) \leq 0. \tag{4.29}$$

From the above inequality, we get that if

$$\left(1 - \frac{1-\beta}{M(\beta)} \Theta + \frac{\beta t}{M(\beta)\Gamma(\beta)} \Theta \right) \geq 0 \tag{4.30}$$

then $u(x, t) - u^*(x, t) = 0$.

Therefore, (4.30) is required condition for uniqueness.

5 Numerical Simulation

In this section, we demonstrate the efficiency of iterative Laplace transform method by applying it on the time fractional homogeneous and non-homogeneous gas dynamics equation. Computations are done with the help of Mathematica software.

5.1 Approximate solution for time fractional homogeneous gas dynamics equation

Consider the time fractional gas dynamics equation (1.1) with initial condition $u(x, 0) = \lambda^{-x}$. The exact solution to (1.1) is given in [7] as

$$u(x, t) = \lambda^{t-x} \tag{5.1}$$

Applying Laplace transform on both side of (1.1) we get

$$L\{u(x, t)\} - \frac{1}{s} u(x, 0) + \frac{1}{M(\beta)} (1 - \beta + \frac{1}{s^\beta}) L \{u \frac{\partial u}{\partial x} - (1 - u) \log \lambda\} = 0 \quad (5.2)$$

Rearranging terms we get

$$L\{u(x, t)\} = \frac{1}{s} (\lambda^{-x}) - \frac{1}{M(\beta)} (1 - \beta + \frac{1}{s^\beta}) L \{u \frac{\partial u}{\partial x} - (1 - u) \log \lambda\} \quad (5.3)$$

Further, the inverse Laplace transform on (5.3), yields

$$u(x, t) = \lambda^{-x} - L^{-1} \{ \frac{1}{M(\beta)} (1 - \beta + \frac{1}{s^\beta}) L \{u \frac{\partial u}{\partial x} - (1 - u) \log \lambda\} \} \quad (5.4)$$

The series solution obtained by the method is given by,

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) \quad (5.5)$$

The nonlinear term $u \frac{\partial u}{\partial x}$ is written as $u \frac{\partial u_n}{\partial x} = \sum_{n=0}^{\infty} \mathbb{A}_n(x, t)$; where as $\mathbb{A}_n$ is further decomposed as follows

$$\mathbb{A}_n = \sum_{i=0}^n u_i \cdot \frac{\partial}{\partial x} (\sum_{i=0}^n u_i) - \sum_{i=0}^{n-1} u_i \cdot \frac{\partial}{\partial x} (\sum_{i=0}^{n-1} u_i)$$

By using $u_0(x, t) = \lambda^{-x}$, we get the recursive formula as follows

$$u_n(x, t) = u_0(x, t) - L^{-1} \{ \frac{1}{M(\beta)} (1 - \beta + \frac{1}{s^\beta}) L \{u_n \frac{\partial u_n}{\partial x} - (1 - u_n) \log \lambda\} \} \quad (5.6)$$

The n-terms approximate solution is given as

$$u(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + \dots + u_{n-1}(x, t) \quad (5.7)$$

Therefore, using (5.6) the first three terms of approximate solution of (1.1) are obtained as follows

$$u_0 = \lambda^{-x}$$

$$u_1 = - \frac{t \Gamma(\beta) \log(\lambda) \lambda^{-x} (-\beta + \frac{\beta t^\beta}{\Gamma(\beta + 2)} + 1)}{\beta + \beta(-\Gamma(\beta)) + \Gamma(\beta)}$$

$$u_2 = - \frac{\beta t^2 \Gamma(\beta) \log^2(\lambda) \lambda^{-x}}{(-\beta + \frac{\beta}{\Gamma(\beta)} + 1) ((\beta - 1) \Gamma(\beta) - \beta)} - \frac{t^2 \Gamma(\beta) \log^2(\lambda) \lambda^{-x}}{2 (-\beta + \frac{\beta}{\Gamma(\beta)} + 1) ((\beta - 1) \Gamma(\beta) - \beta)} \\ - \frac{2 \beta t^{\beta+2} \Gamma(\beta) \log^2(\lambda) \lambda^{-x}}{(-\beta + \frac{\beta}{\Gamma(\beta)} + 1) ((\beta - 1) \Gamma(\beta) - \beta) \Gamma(\beta + 3)} - \frac{\beta^2 t^2 \Gamma(\beta) \log^2(\lambda) \lambda^{-x}}{2 (-\beta + \frac{\beta}{\Gamma(\beta)} + 1) ((\beta - 1) \Gamma(\beta) - \beta)} \\ - \frac{2 \beta t^{2\beta+2} \Gamma(\beta) \log^2(\lambda) \lambda^{-x}}{(-\beta + \frac{\beta}{\Gamma(\beta)} + 1) ((\beta - 1) \Gamma(\beta) - \beta) \Gamma(2\beta + 3)} + \dots$$

$$u_3 = - \frac{\beta t^3 \Gamma(\beta)^2 \log^3(\lambda) \lambda^{-x}}{2 (-\beta + \frac{\beta}{\Gamma(\beta)} + 1) (\beta + \beta(-\Gamma(\beta)) + \Gamma(\beta))} - \frac{t^3 \Gamma(\beta)^2 \log^3(\lambda) \lambda^{-x}}{6 (-\beta + \frac{\beta}{\Gamma(\beta)} + 1) (\beta + \beta(-\Gamma(\beta)) + \Gamma(\beta))} \\ + \frac{\beta^3 t^3 \Gamma(\beta)^2 \log^3(\lambda) \lambda^{-x}}{6 (-\beta + \frac{\beta}{\Gamma(\beta)} + 1) (\beta + \beta(-\Gamma(\beta)) + \Gamma(\beta))} - \frac{\beta^2 t^3 \Gamma(\beta)^2 \log^3(\lambda) \lambda^{-x}}{2 (-\beta + \frac{\beta}{\Gamma(\beta)} + 1) (\beta + \beta(-\Gamma(\beta)) + \Gamma(\beta))} \\ - \frac{\beta^3 t^{3\beta+3} \Gamma(\beta)^2 \log^3(\lambda) \lambda^{-x}}{(-\beta + \frac{\beta}{\Gamma(\beta)} + 1) (\beta + \beta(-\Gamma(\beta)) + \Gamma(\beta)) \Gamma(3\beta + 4)} + \dots$$

Continuing in the same way, remaining terms of the iteration formula (3.11) can be calculated.

t	x	$\beta = 0.6$	$\beta = 0.8$	$\beta = 1$	Absolute error $ u_{exact} - u_{approx} $ for $\beta = 1$
0.03	0	0.997213	0.998720	0.999926	0.005052
	1	0.845096	0.846373	0.847395	0.004282
	2	0.766183	0.717265	0.718131	0.003628
0.06	0	0.994109	0.997202	0.999702	0.010278
	1	0.842465	0.845086	0.847205	0.008710
	2	0.713954	0.716175	0.717970	0.007382
0.09	0	0.990782	0.995483	0.999330	0.015678
	1	0.839645	0.843630	0.846890	0.013286
	2	0.711564	0.714941	0.717703	0.012597

Table1: Comparison of absolute error between the exact solution with three term approximations obtained by ILTM of (1.1) for $\beta=1$ and numerical results for $\beta=0.6, 0.8$.

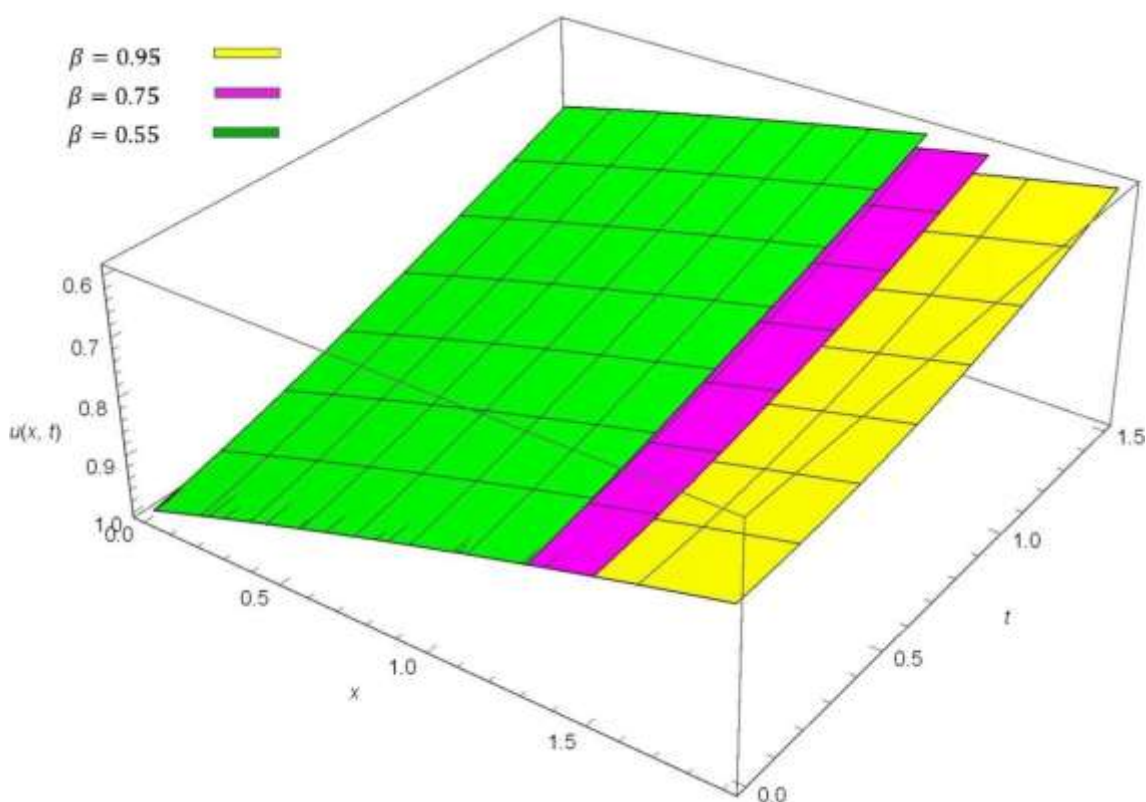


Figure 1: Approx. solution of Eq. (1.1), for $\beta = 0.95, 0.75, 0.65$

5.2. Approximate solution for time fractional non-homogeneous gas dynamics equation

In this section, we study the time fractional non-homogeneous gas dynamics equation (1.2) with initial condition $u(x, 0) = x$.

The exact solution for the classical non-homogeneous gas dynamics equation is given by [7]

$$u(x, t) = \frac{x}{x + t} \quad (5.8)$$

Applying Laplace transform on both side of (1.2) we get,

$$L\{u(x, t)\} - \frac{1}{s}u(x, 0) + \frac{1}{M(\beta)}(1 - \beta + \frac{1}{s^\beta})L\{u \frac{\partial u}{\partial x} + (1 + t)^2u^2 - x^2\} = 0 \quad (5.9)$$

Rearranging terms we get,

$$L\{u(x, t)\} = \frac{x}{s} - \frac{1}{M(\beta)}(1 - \beta + \frac{1}{s^\beta})L\{u \frac{\partial u}{\partial x} + (1 + t)^2u^2 - x^2\} \quad (5.10)$$

Further, the inverse Laplace transform on (5.10), yields

$$u(x, t) = x - L^{-1}\{\frac{1}{M(\beta)}(1 - \beta + \frac{1}{s^\beta})L\{u \frac{\partial u}{\partial x} + (1 + t)^2u^2 - x^2\}\} \quad (5.11)$$

The series solution obtained by the method is given by,

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) \quad (5.12)$$

The nonlinear term $u \frac{\partial u}{\partial x}$ is written as $u \frac{\partial u}{\partial x} = \sum_{n=0}^{\infty} \mathbb{B}_n(x, t)$; where as $\mathbb{B}_n$ is further decomposed as follows

$$\mathbb{B}_n = \sum_{i=0}^n u_i \cdot \frac{\partial}{\partial x} (\sum_{i=0}^n u_i) - \sum_{i=0}^{n-1} u_i \cdot \frac{\partial}{\partial x} (\sum_{i=0}^{n-1} u_i)$$

By using $u_0(x, t) = x$, we get the recursive formula as follows

$$u_n(x, t) = u_0(x, t) + L^{-1}\{\frac{1}{M(\beta)}(1 - \beta + \frac{1}{s^\beta})L\{u_n \frac{\partial u_n}{\partial x} + (1 + t)^2u_n^2 - x^2\}\} \quad (5.13)$$

The n-terms approximate solution is given as

$$u(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + \dots + u_{n-1}(x, t) \quad (5.14)$$

Therefore, using (5.13) the first three terms of approximate solution of (1.2) are obtained as follows

$$u_0 = x$$

$$u_1 = \frac{t^3x\Gamma(\beta)}{3(\beta + \beta(-\Gamma(\beta)) + \Gamma(\beta))} + \frac{t^2x^2\Gamma(\beta)}{\beta + \beta(-\Gamma(\beta)) + \Gamma(\beta)} + \frac{tx\Gamma(\beta)}{\beta + \beta(-\Gamma(\beta)) + \Gamma(\beta)} - \frac{\beta t^3x\Gamma(\beta)}{3(\beta + \beta(-\Gamma(\beta)) + \Gamma(\beta))} - \frac{\beta t^2x^2\Gamma(\beta)}{\beta + \beta(-\Gamma(\beta)) + \Gamma(\beta)} - \frac{\beta tx\Gamma(\beta)}{\beta + \beta(-\Gamma(\beta)) + \Gamma(\beta)} + \frac{2\beta x^2\Gamma(\beta)t^{\beta+2}}{(\beta + \beta(-\Gamma(\beta)) + \Gamma(\beta))\Gamma(\beta + 3)} + \frac{\beta x\Gamma(\beta)t^{\beta+1}}{(\beta + \beta(-\Gamma(\beta)) + \Gamma(\beta))\Gamma(\beta + 2)}$$

$$u_2 = \frac{1}{-\beta + \frac{\beta}{\Gamma(\beta)} + 1} \frac{1}{h} \frac{2t^5x}{15(-\beta + \frac{\beta}{\Gamma(\beta)} + 1)} + \frac{t^3x}{3(-\beta + \frac{\beta}{\Gamma(\beta)} + 1)} + \beta t x^2 - t^2 x + \frac{t^7x}{63(-\beta + \frac{\beta}{\Gamma(\beta)} + 1)^2} + \frac{t^3x^2}{3(-\beta + \frac{\beta}{\Gamma(\beta)} + 1)^2} + \frac{8\beta^3x^3\Gamma(2\beta + 8)t^{\beta+2(\beta+4)}}{(-\beta + \frac{\beta}{\Gamma(\beta)} + 1)^2\Gamma(\beta + 3)\Gamma(\beta + 4)} + \frac{8\beta^3x^2\Gamma(2\beta + 8)t^{\beta+2(\beta+4)}}{(-\beta + \frac{\beta}{\Gamma(\beta)} + 1)^2\Gamma(\beta + 4)^2}$$

$$u_3 = \frac{1}{h^{945}} \frac{t^{12}x}{(-\beta + \frac{\beta}{\Gamma(\beta)} + 1)^6} + \frac{134t^{10}x}{4725(-\beta + \frac{\beta}{\Gamma(\beta)} + 1)^6} + \frac{4t^8x}{45(-\beta + \frac{\beta}{\Gamma(\beta)} + 1)^6} + \frac{t^6x}{9(-\beta + \frac{\beta}{\Gamma(\beta)} + 1)^6} \\ + \frac{t^{14}x}{3969(-\beta + \frac{\beta}{\Gamma(\beta)} + 1)^6} + \frac{t^6x^2}{3(-\beta + \frac{\beta}{\Gamma(\beta)} + 1)^6} \\ + \frac{384\beta^6x^5\Gamma(2\beta + 8)^2t^{\beta+4}(\beta+4)}{(-\beta + \frac{\beta}{\Gamma(\beta)} + 1)^6\Gamma(\beta + 3)^2\Gamma(\beta + 4)^2\Gamma(2(\beta + 4) + 1)\Gamma(\beta + 2(\beta + 4) + 1)} + \dots$$

t	x	$\beta = 0.6$	$\beta = 0.8$	$\beta = 1$	Absolute error $ u_{exact} - u_{approx} $ for $\beta = 1$
0.4	0.2	0.285745	0.243016	0.21468	0.00118654
	0.6	0.862571	0.713500	0.631011	0.00310109
	1	1.46107	1.16689	1.03016	0.00031587
0.6	0.2	0.392077	0.296655	0.237210	0.0012790
	0.6	1.33626	0.91455	0.692986	0.00192986
	1	2.57989	1.58823	1.12576	0.000500755
0.8	0.2	0.581501	0.390823	0.276887	0.0076886
	0.6	2.35671	1.34482	0.825964	0.00428571
	1	5.29352	2.62657	1.37925	0.000823695

Table 2: Comparison of absolute error between the exact solution with four term approximations obtained by ILTM of (1.2) for $\beta = 1$ and solution for of $\beta = 0.6, 0.8$

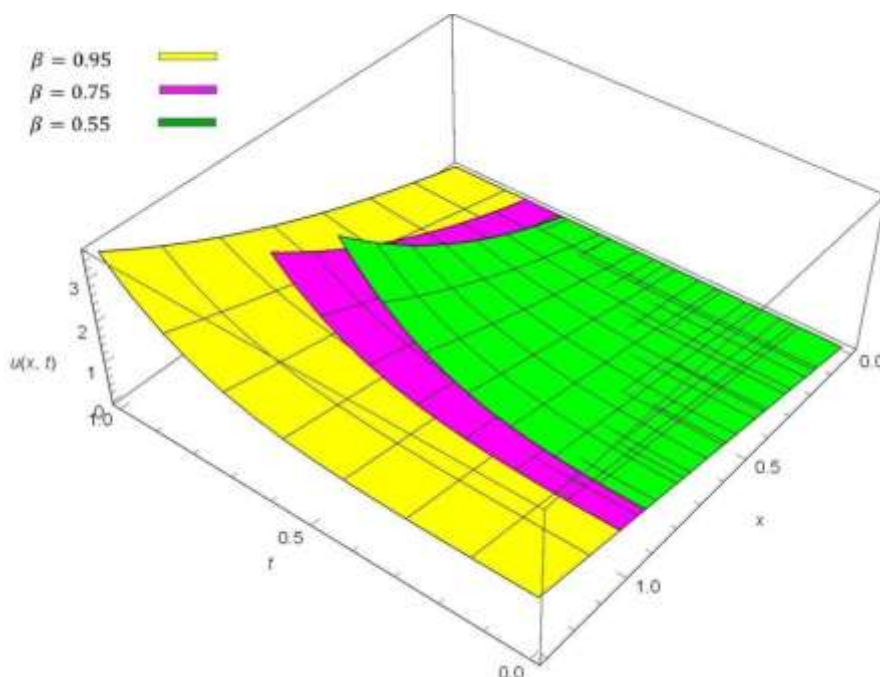


Figure 2: Approx. soln of Eq. (1.2), for $\beta = 0.95, 0.75, 0.55$

In tables 1 and 2 the computational results are obtained to get approximate solution of equations (1.1) and (1.2) respectively. This technique provides accurate numerical solutions even if lower order approximations are used. The accuracy for the time fractional nonlinear homogeneous and non-homogeneous gas dynamics equations is demonstrated by the absolute errors of Eq. (1.1) and (1.2) respectively with their exact solutions.

Also fig.1 shows surfaces for approximate solution of Eq. (1.1) and exact solution of homogeneous fractional gas dynamics equations and fig. 2 shows surfaces for approximate solution of Eq. (1.2) and exact solution of non-homogeneous fractional gas dynamics equations. Also, from these surfaces it can be observed that approximate solution and exact solution are of both the equations are very near to each other.

6 Conclusions

In this work, iterative Laplace transform method has been successfully applied for numerical solution of time-fractional homogeneous and non-homogeneous gas dynamic equations. The method gives more realistic series solutions that converge very rapidly in physical problems. The theoretical analysis of solutions is carried out by providing the existence and uniqueness criteria of solution for used technique. It is apparently seen from comparison of absolute error that the proposed method is very efficient and powerful numerical method to find the approximate solution. Also, it is important to point out that even if the problem is nonlinear this method does not require linearization or perturbation. Therefore, ILTM is easy to implement and more convenient than other methods.

Acknowledgement

The authors are thankful to the referees for their valuable comments.

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