

Analytical Investigation of Fractional Partial Differential Equations for Complex Dynamical Systems

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Abstract

Fractional partial differential equations (FPDEs) provide a useful mathematical framework for describing systems whose present behavior depends on their previous history. Such memory and non-local effects are encountered in anomalous diffusion, porous media, viscoelasticity, biological transport, and heat transfer. Classical partial differential equations use integer-order derivatives and may therefore be inadequate for processes in which the evolution is not purely local in time or space. In this paper, a mathematical investigation of a class of time-fractional partial differential equations is presented. The Caputo fractional derivative is employed for the temporal derivative, and its relationship with the classical first-order derivative is discussed. A one-dimensional time-fractional diffusion equation is formulated and solved using the method of separation of variables and Fourier series. The resulting solution is expressed in terms of the Mittag–Leffler function, which replaces the exponential function appearing in the corresponding classical diffusion problem. A theorem concerning the representation and uniqueness of the solution is established and proved using eigenfunction expansion. Four worked examples are included to demonstrate fractional differentiation, fractional relaxation, a single-mode fractional diffusion problem, and a problem with a general initial distribution. The analysis shows that the fractional order significantly influences the temporal evolution of the system. In particular, the classical diffusion equation is recovered when the fractional order is one, whereas lower fractional orders introduce memory-dependent behavior. The results demonstrate the potential of FPDEs for modelling complex dynamical systems where conventional integer-order models do not adequately describe observed behavior.

Keywords: Fractional partial differential equations, fractional calculus, Caputo derivative, Mittag–Leffler function, fractional diffusion, anomalous diffusion, mathematical modeling.

1. Introduction

Partial differential equations are among the most important tools used to model physical and natural phenomena. Heat conduction, diffusion, fluid flow, elasticity, electromagnetic fields, population dynamics, and chemical reactions can all be described using suitable partial differential equations. In many classical models, the derivatives appearing in the governing equations are of integer order. Although these models are successful for a wide range of applications, some real systems display memory, hereditary effects, or anomalous transport.

A system possessing memory does not respond solely according to its present state. Its current behavior can depend on what happened during an earlier period. This feature is difficult to express using an ordinary first-order derivative because the derivative at a particular time is essentially a local quantity. Fractional calculus offers a natural extension by allowing derivatives of non-integer order.

Fractional derivatives have been investigated for applications involving viscoelastic materials, diffusion in heterogeneous media, biological systems, porous structures, and relaxation processes. A fractional derivative introduces an additional parameter, generally denoted by α , which can be used to characterize the degree of memory or anomalous behavior.

The classical diffusion equation is

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2},$$

where $D > 0$ is the diffusion coefficient. A time-fractional version is

$${}^C D_t^\alpha u = D \frac{\partial^2 u}{\partial x^2}, \quad 0 < \alpha \leq 1.$$

When $\alpha = 1$, the classical equation is recovered. For $0 < \alpha < 1$, the time derivative incorporates the history of the solution.

The purpose of this paper is to investigate this fractional diffusion model analytically and to illustrate its properties through four worked examples.

2. Basic Concepts of Fractional Calculus

2.1 Fractional Integral

For an integrable function $f(t)$, its fractional integral of order $\alpha > 0$ is defined by

where $\Gamma(\alpha)$ denotes the Gamma function.

The Gamma function is

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt.$$

For positive integers,

$$\Gamma(n) = (n-1)!.$$

Thus fractional integration extends the familiar concept of repeated integration to arbitrary positive orders.

2.2 Caputo Fractional Derivative

For $0 < \alpha < 1$, the Caputo derivative is defined by

$${}^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha} d\tau.$$

The definition clearly shows the memory property because the derivative at time t involves $f'(\tau)$ over the complete interval $0 < \tau < t$.

For $n-1 < \alpha < n$,

$${}^C D_t^\alpha u f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau.$$

One reason for using the Caputo derivative in physical models is that conventional initial conditions such as $f(0)$ can be used directly.

3. Mittag–Leffler Function

The Mittag–Leffler function is fundamental in fractional differential equations. It is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}.$$

For $\alpha = 1$,

$$E_1(z) = e^z.$$

Consequently,

$$E_1(-\lambda t) = e^{-\lambda t}.$$

The Mittag–Leffler function therefore plays a role in fractional equations similar to that of the exponential function in classical differential equations.

An important property is

$${}^C D_t^\alpha E_\alpha(\lambda t^\alpha) = \lambda E_\alpha(\lambda t^\alpha)$$

This property is used to construct solutions of fractional evolution equations.

4. Mathematical Model

Consider the time-fractional diffusion equation

$${}^C D_t^\alpha u(x, t) = D u_{xx}(x, t), \quad 0 < x < L, \quad t > 0, \quad (1)$$

with boundary conditions

$$u(0, t) = u(L, t) = 0, \quad (2)$$

and initial condition

$$u(x, 0) = u_0(x). \quad (3)$$

Here $u(x, t)$ represents the quantity being transported, D is a positive diffusion coefficient, and $0 < \alpha \leq 1$.

Assume

$$u(x, t) = X(x)T(t).$$

Substitution into (1) gives

$$X^C D_t^\alpha T = DT X''.$$

Therefore,

$$\frac{{}^C D_t^\alpha T}{DT} = \frac{X''}{X} = -\lambda.$$

The spatial problem is

$$X'' + \lambda X = 0,$$

with

$$X(0) = X(L) = 0.$$

Hence,

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

and

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right).$$

The corresponding temporal equation is

$${}^C D_t^\alpha T_n = -D\lambda_n T_n.$$

Its solution is

$$T_n(t) = E_\alpha(-D\lambda_n t^\alpha).$$

Therefore,

$$u(x, t) = \sum_{n=1}^{\infty} A_n E_\alpha \left[-D \left(\frac{n\pi}{L} \right)^2 t^\alpha \right] \sin \left(\frac{n\pi x}{L} \right) \quad (4)$$

where

$$A_n = \frac{2}{L} \int_0^L u_0(x) \sin \left(\frac{n\pi x}{L} \right) dx. \quad (5)$$

5. Theorem and Proof

Theorem

Let $0 < \alpha \leq 1$, $D > 0$, and $u_0 \in L^2(0, L)$. Then the initial-boundary value problem

$${}^C D_t^\alpha u = D u_{xx}$$

with

$$u(0, t) = u(L, t) = 0, \quad u(x, 0) = u_0(x)$$

has the eigenfunction representation

$$u(x, t) = \sum_{n=1}^{\infty} A_n E_\alpha(-D \lambda_n t^\alpha) \sin\left(\frac{n\pi x}{L}\right),$$

where

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

and A_n is given by (5). The solution is unique in the corresponding mild-solution class.

Proof

The eigenvalue problem

$$X'' + \lambda X = 0, \quad X(0) = X(L) = 0$$

has eigenfunctions

$$\sin\left(\frac{n\pi x}{L}\right)$$

and eigenvalues

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2.$$

These eigenfunctions form a complete orthogonal system in $L^2(0, L)$. Therefore,

$$u_0(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right).$$

Representing $u(x, t)$ using the same basis gives

$$u(x, t) = \sum_{n=1}^{\infty} T_n(t) \sin\left(\frac{n\pi x}{L}\right).$$

Substitution into the governing equation produces

$${}^C_{D_t^\alpha} T_n = -D\lambda_n T_n.$$

With $T_n(0) = A_n$, the Mittag–Leffler solution is

$$T_n(t) = A_n E_\alpha(-D\lambda_n t^\alpha).$$

Consequently,

$$u(x, t) = \sum_{n=1}^{\infty} A_n E_\alpha(-D\lambda_n t^\alpha) \sin\left(\frac{n\pi x}{L}\right).$$

For uniqueness, suppose u_1 and u_2 satisfy the same problem. Their difference

$w = u_1 - u_2$ satisfies

$${}^C_{D_t^\alpha} w = Dw_{xx}$$

with zero initial and boundary conditions. Expansion in the eigenfunctions leads to

$${}^C_{D_t^\alpha} W_n = -D\lambda_n W_n, \quad W_n(0) = 0.$$

The unique scalar solution is $W_n(t) = 0$ for every n . Hence $w = 0$, and therefore

$$u_1 = u_2.$$

Thus the solution is unique.

6. Worked Examples

Example 1: Caputo Derivative of t^2

Find

$${}^C_{D_t^\alpha} (t^2), \quad 0 < \alpha < 1.$$

Using

$${}^C_{D_t^\alpha} t^\beta = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} t^{\beta - \alpha},$$

with $\beta = 2$,

$${}^C_{D_t^\alpha} t^2 = \frac{\Gamma(3)}{\Gamma(3 - \alpha)} t^{2 - \alpha}.$$

Since $\Gamma(3) = 2$,

$$\boxed{{}^C_{D_t^\alpha} t^2 = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}}.$$

For $\alpha = 1$,

$${}^C D_t^1 t^2 = 2t,$$

which agrees with ordinary calculus.

Example 2: Fractional Relaxation Equation

Consider

$${}^C D_t^\alpha y = -\lambda y, y(0) = y_0$$

The solution is

$$y(t) = y_0 E_\alpha(-\lambda t^\alpha).$$

Using the series definition,

$$y(t) = y_0 \left[1 - \frac{\lambda t^\alpha}{\Gamma(\alpha + 1)} + \frac{\lambda^2 t^{2\alpha}}{\Gamma(2\alpha + 1)} - \dots \right].$$

When $\alpha = 1$,

$$y(t) = y_0 e^{-\lambda t}.$$

Hence the fractional model generalizes the familiar exponential relaxation law.

Example 3: Single-Mode Fractional Diffusion

Consider

$${}^C D_t^\alpha u = D u_{xx}, \quad 0 < x < \pi,$$

with

$$u(0, t) = u(\pi, t) = 0$$

and

$$u(x, 0) = \sin x.$$

Let

$$u(x, t) = T(t) \sin x.$$

Since

$$u_{xx} = -T(t) \sin x,$$

we obtain

$${}^C D_t^\alpha T = -DT.$$

Because $T(0) = 1$,

$$T(t) = E_\alpha(-Dt^\alpha).$$

Therefore,

$$u(x, t) = E_\alpha(-Dt^\alpha) \sin x.$$

For $\alpha = 1$,

$$u(x, t) = e^{-Dt} \sin x.$$

The fractional order therefore changes the temporal decay while preserving the spatial mode.

Example 4: General Initial Distribution

Consider

$${}^C D_t^\alpha u = Du_{xx}, \quad 0 < x < \pi,$$

with zero boundary conditions and

$$u(x, 0) = x(\pi - x).$$

Expand the initial condition as

$$x(\pi - x) = \sum_{n=1}^{\infty} A_n \sin(nx),$$

where

$$A_n = \frac{2}{\pi} \int_0^\pi x(\pi - x) \sin(nx) dx.$$

Evaluation gives

$$A_n = \frac{4}{\pi n^3} [1 - (-1)^n].$$

Thus even terms vanish and, for odd n ,

$$A_n = \frac{8}{\pi n^3}.$$

Hence

$$u(x, t) = \frac{8}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{E_{\alpha}(-Dn^2 t^{\alpha})}{n^3} \sin(nx).$$

The first three non-zero terms are

$$u(x, t) \approx \frac{8}{\pi} \left[E_{\alpha}(-Dt^{\alpha}) \sin x + \frac{E_{\alpha}(-9Dt^{\alpha})}{27} \sin 3x + \frac{E_{\alpha}(-25Dt^{\alpha})}{125} \sin 5x \right].$$

This example demonstrates how an arbitrary initial profile can be decomposed into eigenmodes, each of which evolves according to a fractional relaxation law.

7. Applications

Fractional PDEs have potential applications in:

- **Anomalous diffusion:** transport through heterogeneous materials.
- **Porous media:** contaminant and fluid transport.
- **Viscoelasticity:** materials whose response depends on loading history.
- **Biological systems:** transport in complex biological structures.
- **Heat transfer:** thermal systems exhibiting non-classical relaxation.
- **Financial mathematics:** models involving long-range dependence and memory.

The suitability of a fractional model should, however, be established by comparing its predictions with experimental or observational data.

8. Conclusion

A mathematical investigation of a time-fractional diffusion equation has been presented. The Caputo derivative was used to introduce temporal memory, while the Mittag-Leffler function provided the natural counterpart of the exponential function found in classical differential equations.

Using separation of variables and Fourier expansion, an analytical solution of the fractional diffusion problem was obtained. A theorem establishing the solution representation and uniqueness was proved using the eigenfunction method. Four solved examples demonstrated the principal mathematical ideas and illustrated the transition from classical to fractional behavior.

The results show that

$$\alpha = 1$$

recovers the classical diffusion model, whereas

$$0 < \alpha < 1$$

allows the model to incorporate memory-dependent temporal behavior. This additional flexibility makes fractional PDEs useful for systems where classical models fail to represent observed dynamics adequately.

The present work provides an analytical foundation for further investigation. Future research can focus on numerical approximation, variable-order fractional derivatives, multidimensional models, fractional reaction-diffusion equations, and parameter estimation using experimental data.

References

1. Podlubny, I. (1999). *Fractional Differential Equations*. Academic Press.
2. Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). *Theory and Applications of Fractional Differential Equations*. Elsevier.
3. Diethelm, K. (2010). *The Analysis of Fractional Differential Equations*. Springer.
4. Mainardi, F. (2010). *Fractional Calculus and Waves in Linear Viscoelasticity*. Imperial College Press.
5. Samko, S. G., Kilbas, A. A., & Marichev, O. I. (1993). *Fractional Integrals and Derivatives: Theory and Applications*. Gordon and Breach.
6. Oldham, K. B., & Spanier, J. (1974). *The Fractional Calculus*. Academic Press.
7. Metzler, R., & Klafter, J. (2000). The random walk's guide to anomalous diffusion: A fractional dynamics approach. *Physics Reports*, 339(1), 1–77.
8. Gorenflo, R., Mainardi, F., Moretti, D., Pagnini, G., & Paradisi, P. (2002). Discrete random walk models for space-time fractional diffusion. *Chemical Physics*, 284, 521–544.
9. Luchko, Y. (2009). Some uniqueness and existence results for the initial-boundary-value problems for the generalized time-fractional diffusion equation. *Computers & Mathematics with Applications*, 59, 1766–1772.
10. Sakamoto, K., & Yamamoto, M. (2011). Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems. *Journal of Mathematical Analysis and Applications*, 382, 426–447.