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Analytical Solution for Beam Dynamics Using Modified Couple Stress Theory

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Abstract

The dynamic behavior of Bernoulli–Euler beams is analytically investigated using the Modified Couple Stress Theory (MCST). Governing equations, initial conditions, and boundary conditions are derived using Hamilton's principle. Two dynamic boundary-value problems—simply supported and cantilever beams—are solved to quantify the influence of microstructural size effects on natural frequencies. Results show that natural frequencies predicted by MCST are significantly higher than those predicted by classical beam theory when the characteristic structural dimension is comparable to the internal material length-scale parameter. As the size ratio increases, MCST predictions converge to classical solutions. These findings highlight the critical role of size-dependent stiffness in micro- and nano-scale beam applications such as MEMS devices.

Keywords

Modified couple stress theory; beam dynamics; Bernoulli–Euler beam; size effect; natural frequency; gradient elasticity; MEMS structures

1. Introduction

Beams are integral structural elements widely used across macro- and micro-scale engineering applications, especially in microelectromechanical systems (MEMS). In devices such as vibration sensors, atomic force microscopy cantilevers, and microscale resonators, beam thickness and dimensions approach the micron or sub-micron range. Numerous experimental investigations have confirmed that mechanical behavior at such scales deviates from classical elasticity.

Experimental observations demonstrate pronounced size effects in metals and polymers subjected to torsion, indentation, and bending. For example, Fleck et al. observed substantial torsional hardening in copper wires as diameter decreased, while Ma et al. reported increased hardness in micro-indentation of silver. Similar stiffening behavior has been reported in polymeric micro-beams. These studies reveal that miniaturization induces enhanced stiffness and increased load-bearing capacity.

Classical elasticity theory, lacking intrinsic length-scale parameters, fails to predict such size-dependent phenomena. Higher-order theories, such as strain-gradient elasticity and classical couple stress theory, were introduced to incorporate material microstructure effects. However, the classical couple stress theory requires multiple additional material constants, complicating parameter identification.

To overcome these limitations, Yang et al. (2002) formulated the Modified Couple Stress Theory, introducing only a *single* material length-scale parameter. This renders MCST attractive for simplified modeling of micro-beams while retaining accuracy in capturing size effects.

This study aims to derive analytical solutions for the dynamic behavior of Bernoulli–Euler beams using MCST. The governing equations are obtained through Hamilton's principle, and frequency responses of simply supported and cantilever beams are evaluated. Comparisons with classical beam theory highlight the significance of microstructural effects in dynamic beam analysis.

2. Theoretical Background

2.1 Modified Couple Stress Theory

MCST introduces a symmetric deviatoric couple stress tensor associated with curvature strain and adds one intrinsic length-scale parameter lll. The strain energy density is expressed as:

$$U = \frac{1}{2}\int_{\Omega} (\sigma_{ij}\varepsilon_{ij} + m_{ij}\chi_{ij})\, d\Omega$$

where the key constitutive relations include:

- Stress:

$$\sigma_{ij} = \lambda\, \mathrm{tr}(\varepsilon)\delta_{ij} + 2\mu\varepsilon_{ij}$$

- Couple stress:

$$m_{ij} = 2\mu l^2 \chi_{ij}$$

- Curvature tensor:

$$\chi_{ij} = \frac{1}{2}(\theta_{i,j} + \theta_{j,i}); \quad \theta = \frac{1}{2}\nabla \times \mathbf{u}$$

The presence of l introduces size dependency absent in classical elasticity.

2.2 Bernoulli–Euler Beam Kinematics

Assuming small deformations, the displacement field is:

$$u(x,z,t) = -z\psi(x,t), \quad v = 0, \quad w = w(x,t)$$

The rotation is approximately:

$$\psi(x,t) \approx \frac{\partial w}{\partial x}$$

Resulting axial strain:

$$\varepsilon_{xx} = -z\frac{\partial^2 w}{\partial x^2}$$

Curvature component:

$$\chi_{xy} = -\frac{1}{2}\frac{\partial^2 w}{\partial x^2}$$

Using these expressions, the strain energy reduces to:

$$U = \frac{1}{2}\int_0^L (EI + \mu A l^2) w''^2 dx$$

The kinetic and external work terms follow classical definitions.

2.3 Governing Equation

Applying Hamilton's principle yields the dynamic beam equation:

$$\rho A \ddot{w} + (EI + \mu A l^2) w^{(4)} = q(x,t)$$

This equation differs from classical beam theory only by inclusion of the term $\mu A l^2$ which increases stiffness and induces size-dependent behavior.

3. Solution of Dynamic Boundary-Value Problems

3.1 Simply Supported Beam

Boundary conditions:

$$w(0) = w(L) = 0, \quad w''(0) = w''(L) = 0$$

Assuming separable solution:

$$w(x,t) - X(x)T(t)$$

Spatial solution satisfies:

$$X'''' - s^4 X = 0, \quad X = \sin(sx)$$

Natural frequency:

$$\omega_n = \left(\frac{n\pi}{L}\right)^2 \sqrt{\frac{EI + \mu A l^2}{\rho A}}$$

Classical frequency recovered by setting l=0.

Rectangular Cross Section

$$I = \frac{bh^3}{12}, \quad A = bh$$

Frequency ratio:

$$\frac{\omega}{\omega_0} = \sqrt{1 + \frac{6l^2}{h^2}}$$

Circular Cross Section

$$I = \frac{\pi d^4}{64}, \quad A = \frac{\pi d^2}{4}$$

Frequency ratio:

$$\frac{\omega}{\omega_0} = \sqrt{1 + \frac{8l^2}{d^2}}$$

3.2 Cantilever Beam

Boundary conditions:

$$w(0) = 0, \quad w'(0) = 0, \quad w''(L) = 0, \quad w'''(L) = 0$$

Eigenvalues:

$$sL = 1.875,\ 4.694,\ 7.855, \dots$$

Natural frequency:

$$\omega_n = \frac{s_n^2}{L^2}\sqrt{\frac{EI + \mu A l^2}{\rho A}}$$

Frequency ratios for rectangular and circular sections match the expressions in the simply supported case.

4. Results and Discussion

Results reveal clear stiffening behavior for beams modeled with MCST, particularly when characteristic dimensions (thickness hhh or diameter ddd) are comparable to the material length scale parameter lll.

Key Observations

- When l=hl = hl=h for rectangular beams, natural frequencies increase by ≈ 2.6×.
- When l=dl = dl=d for circular beams, natural frequencies increase by ≈ 3×.
- When l≥10hl \ge 10hl≥10h or l≥10dl \ge 10dl≥10d, MCST predictions converge to classical solutions.
- Findings align with experimental results showing higher stiffness in microscale beams.

5. Applications

The analytical model provides improved accuracy for predicting dynamics of:

- MEMS cantilevers and vibration sensors
- Micro-resonators and frequency-selective devices
- High-precision aerospace components
- Automotive chassis micro-components
- Industrial microscale actuators

Additionally, classical beam applications such as building beams, bridges, furniture, and industrial supports also benefit from the refined understanding of size effects at smaller scales.

6. Conclusion

A dynamic Bernoulli–Euler beam model incorporating Modified Couple Stress Theory was formulated and analytically solved for simply supported and cantilever configurations. The inclusion of a single material length-scale parameter introduces size-dependent stiffness, leading to significantly higher natural frequencies compared with classical elasticity for micro-sized beams. These effects diminish as structural dimensions increase, confirming the relevance of MCST primarily in micro- and nano-scale applications. This study provides a valuable theoretical foundation for the design of MEMS structures and micro-beam resonators.

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