

Analytical Solution of One-Dimensional Wave Equations Using the Laplace Transform Method

Mrs. Sonabai Dasarath Patil

Assistant. Professor, Dept of Mathematics, Sambhajirao Kendre Mahavidyalaya, Jalkot, Maharashtra, India

Mrs. Pooja Ramesh Mundhe

Assistant. Professor Department of Mathematics, Mahatma Basweshwar Mahavidyalaya, Latur, Maharashtra, India

Abstract

The wave equation is one of the fundamental partial differential equations used to describe wave propagation and vibrations in physical and engineering systems. In this paper, an analytical approach based on the Laplace Transform Method is presented for solving one-dimensional wave equations. The Laplace transform is applied with respect to the time variable, which converts the governing partial differential equation into an ordinary differential equation in the spatial variable. The transformed equation is solved subject to the corresponding boundary conditions, and the inverse Laplace transform is subsequently employed to obtain the solution in the physical domain. Two initial-boundary value problems are considered. The first example involves homogeneous boundary conditions and sinusoidal initial displacement, while the second example considers a non-homogeneous boundary condition. The obtained solutions are verified by direct substitution into the governing equation and by checking the corresponding initial and boundary conditions. A graphical representation is also provided to illustrate the wave behaviour. The study demonstrates that the Laplace Transform Method provides a direct and systematic analytical technique for solving linear one-dimensional wave equations with different types of boundary conditions.

Keywords: Wave equation, Laplace Transform, Inverse Laplace Transform, Partial differential equation, Homogeneous boundary condition, Non-homogeneous boundary condition, Analytical solution.

1. Introduction

Partial differential equations (PDEs) play an important role in mathematical modelling of physical phenomena such as heat transfer, diffusion, fluid flow, elasticity, vibration and wave propagation. Among the different classes of PDEs, hyperbolic equations are particularly important for describing propagation phenomena.

The classical one-dimensional wave equation is given by

where $u(x,t)$ denotes the displacement of the wave, x is the spatial variable, t is time and c represents the propagation velocity.

The wave equation has applications in the study of vibrating strings, acoustic waves, mechanical vibrations and several other physical systems. Analytical solutions can be obtained using methods such as separation of variables, Fourier series, d'Alembert's method and integral transform techniques.

The Laplace Transform Method is particularly useful for initial-value problems because the initial conditions are incorporated directly into the transformed equation. When the transform is taken with respect to time, derivatives with respect to time are converted into algebraic expressions involving the transform variable. Consequently, a partial differential equation can be reduced to an ordinary differential equation in the spatial variable.

The purpose of this work is to investigate the use of the Laplace Transform Method for solving one-dimensional wave equations with different boundary conditions. Two examples are considered. The first example is a standard problem with homogeneous boundary conditions. The second example illustrates the treatment of a non-homogeneous, time-dependent boundary condition.

The paper is organized as follows. Section 2 introduces the mathematical problem, Section 3 presents the Laplace-transform formulation, Sections 4 and 5 develop two examples, and Section 6 provides numerical calculations and a comparison of the analytical results. The physical interpretation, advantages, limitations and future scope are discussed in the subsequent sections.

The present work focuses on a direct Laplace-transform treatment rather than a numerical time-stepping scheme. This makes it possible to examine the analytical structure of the solution and identify the role of the natural frequencies of the finite interval. The first example is chosen so that the inverse transform reduces to a simple closed-form standing wave. The second example introduces a sinusoidal boundary excitation, allowing the response and resonance condition to be examined.

Integral transforms are valuable because they can convert differential operators into simpler algebraic or ordinary differential operators. In particular, the Laplace transform is naturally suited to problems in which initial conditions are prescribed at $t = 0$. Applying the transform with respect to time avoids the need to discretize the time variable and permits the initial data to enter the transformed equation directly.

For a linear uniform medium, the one-dimensional wave equation provides a fundamental model in which the constant c determines the speed at which disturbances propagate. The simplicity of this model makes it an appropriate benchmark for analytical techniques. At the same time, different initial and boundary conditions can lead to substantially different physical responses, including standing waves, travelling disturbances and resonance.

The mathematical analysis of wave propagation is important because many physical systems exhibit oscillatory behaviour rather than purely diffusive behaviour. A vibrating string, for example, supports disturbances that travel through the medium with a finite propagation speed. The mathematical model must therefore account for both spatial variation and temporal acceleration.

2. Mathematical Formulation

Consider the one-dimensional homogeneous wave equation

where c is a positive constant representing the wave velocity.

The general initial conditions are

$$u(x, 0) = f(x)$$

The boundary conditions may be homogeneous or non-homogeneous. For homogeneous boundary conditions,

$$u(0, t) = 0, u(L, t) = 0$$

For non-homogeneous boundary conditions, one or both boundary values may be prescribed as non-zero functions of time.

3. Laplace Transform Method

The Laplace transform of a function $f(t)$ is defined by

The transform formulas required here are

Since the transformation is performed with respect to t ,

Therefore, the transformed wave equation becomes

$$s^2 U - s f(x) - g(x) = c^2 \frac{d^2 U}{dx^2}$$
$$\frac{d^2 U}{dx^2} - (s^2/c^2)U = \frac{-[s f(x) + g(x)]}{c^2}$$

4. Example 1: Homogeneous Boundary Conditions

Consider

with boundary conditions

$$u(0, t) = 0, u(1, t) = 0$$

and initial conditions

$$u(x, 0) = \sin(\pi x)$$

4.1 Laplace Transformation

$$s^2 U - s \sin(\pi x) = c^2 \frac{d^2 U}{dx^2}$$

$$\frac{d^2 U}{dx^2} - (s^2/c^2)U = -s \frac{\sin(\pi x)}{c^2}$$

The transformed boundary conditions are

$$U(0, s) = 0, U(1, s) = 0$$

4.2 Solution of the Transformed Equation

The complementary solution is

Since the forcing function is $\sin(\pi x)$, assume a particular solution

$$U_p = C \sin(\pi x)$$

Substitution gives

$$-\pi^2 C \sin(\pi x) - (s^2/c^2)C \sin(\pi x) = -s/c^2 \sin(\pi x)$$

$$C = s/(s^2 + \pi^2 c^2)$$

Hence

$$U_p = s \sin(\pi x)/(s^2 + \pi^2 c^2)$$

Applying the transformed boundary conditions gives $A = 0$ and $B = 0$. Therefore

$$U(x, s) = \frac{s \sin(\pi x)}{(s^2 + \pi^2 c^2)}$$

4.3 Inverse Laplace Transform

Using the standard transform pair

we obtain

$$u(x, t) = \sin(\pi x) \cos(\pi c t)$$

4.4 Verification

For the obtained solution,

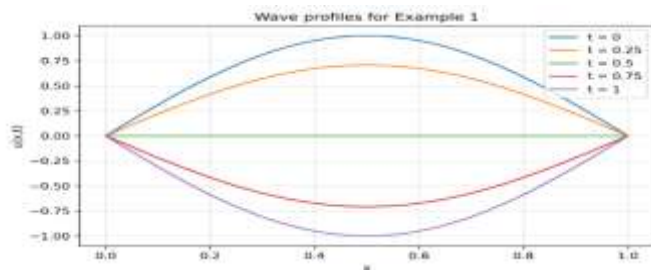


Figure 1. Wave profiles at different time levels for Example 1.

Figure 1 shows the wave profile at selected time levels. The solution remains zero at both boundaries, while the amplitude changes periodically with time. The profiles illustrate the standing-wave character of the solution.

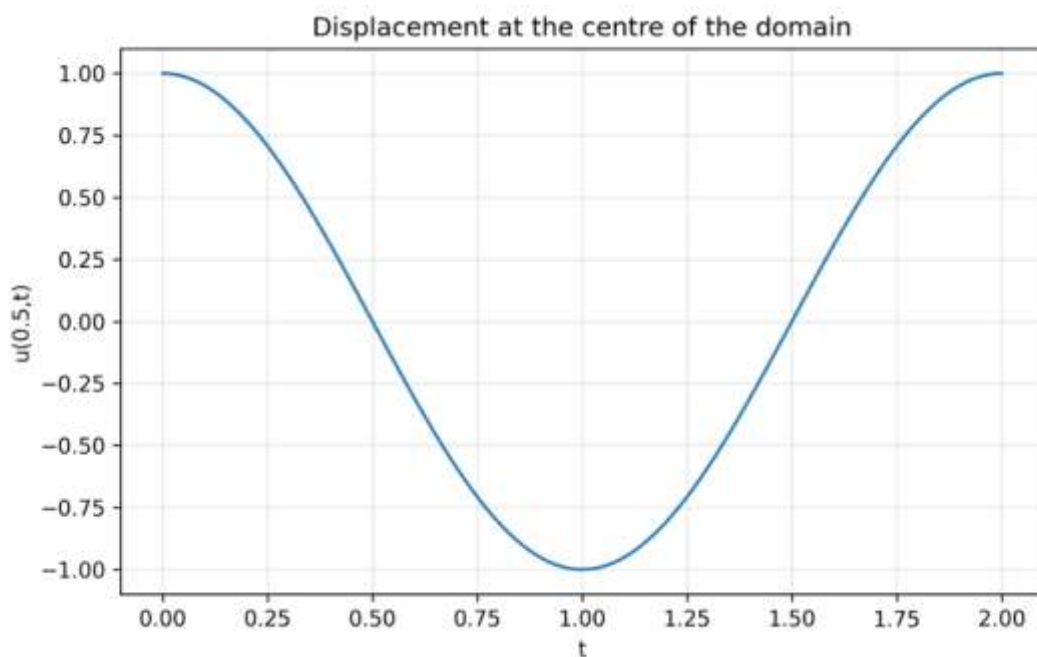


Figure 2. Displacement at the centre of the domain as a function of time for Example 1.

At $x = 0.5$, the solution reduces to $u(0.5,t) = \cos(\pi ct)$ for $c = 1$. Figure 2 shows the periodic oscillation of the centre point between positive and negative displacement.

5. Example 2: Non-Homogeneous Boundary Condition

Consider

with

$$u(0,t) = 0$$

$$u(1,t) = \sin(\omega t)$$

and zero initial conditions

$$u(x,0) = 0$$

Here ω is the frequency of the prescribed boundary motion. This represents a string with one fixed end and one end subjected to a sinusoidal displacement.

5.1 Laplace Transformation

Taking the Laplace transform with respect to t gives

$$s^2 U = c^2 \frac{d^2 U}{dx^2}$$

$$\frac{d^2 U}{dx^2} - (s^2/c^2)U = 0$$

The general solution is

5.2 Transformed Boundary Conditions

From $u(0, t) = 0$, we obtain $U(0, s) = 0$, so $A = 0$. For the second boundary condition,

Therefore

Thus, the transformed solution is

5.3 Representation of the Inverse Transform

For the non-homogeneous boundary condition, introduce

$$u(x, t) = x \sin(\omega t) + v(x, t)$$

The first term satisfies the prescribed boundary values. The remaining function v satisfies homogeneous boundary conditions. Substitution gives

$$v(0, t) = v(1, t) = 0$$

The function x can be expanded in a Fourier sine series

For the non-resonant case $\omega^2 \neq n^2 \pi^2 c^2$, the solution can be written as

where

$$A_n = 2(-1)^{n+1} \omega^2 / [n\pi(\omega^2 - n^2 \pi^2 c^2)]$$

This representation is equivalent to the inverse-transform solution expressed through the natural modes of the finite interval.

5.4 Resonance

The coefficient A_n contains the factor $\omega^2 - n^2 \pi^2 c^2$ in the denominator. Therefore, resonance occurs when

$$\omega = n\pi c$$

for some positive integer n . At resonance, the non-resonant expression must be replaced by its limiting form. Physically, this corresponds to excitation at a natural frequency of the system and may produce a substantially larger response.

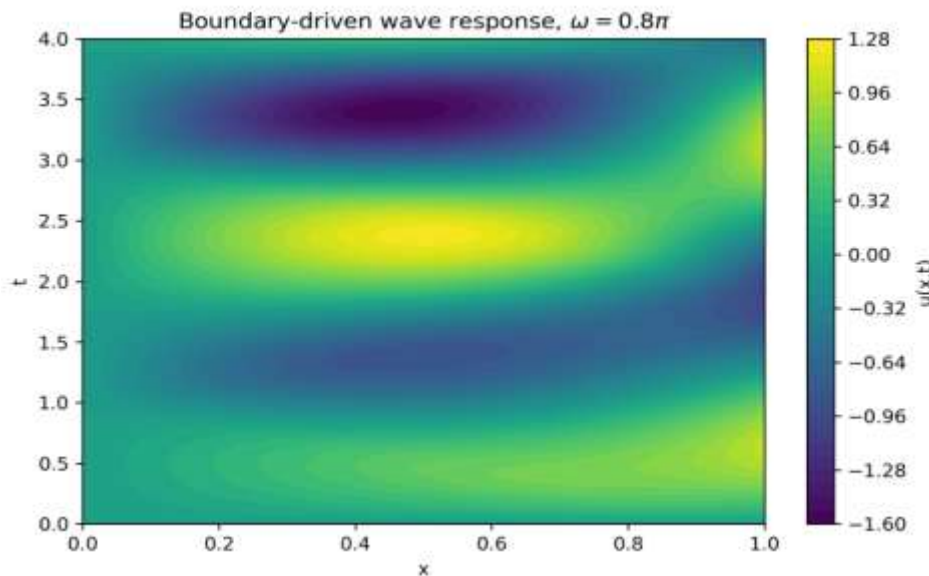


Figure 3. Boundary-driven wave response for the non-homogeneous boundary condition with $\omega = 0.8\pi$ and $c = 1$.

Figure 3 illustrates the boundary-driven response for a non-resonant excitation. The disturbance generated by the moving boundary produces an oscillatory response throughout the spatial domain. The selected value $\omega = 0.8\pi$ is below the first natural frequency πc , avoiding the resonance singularity in the displayed non-resonant representation.

6. Numerical Illustration

For Example 1, let $c = 1$. Then

$$u(x,t) = \sin(\pi x) \cos(\pi t)$$

At the centre $x = 0.5$,

$$u(0.5,t) = \cos(\pi t)$$

t	u(0.5,t)
0	1.000000
0.10	0.951057
0.25	0.707107
0.50	0.000000
0.75	-0.707107
1.00	-1.000000

The table confirms the periodic oscillation of the centre displacement. The value changes from +1 to -1 over one time unit when $c = 1$.

6.1 Detailed Calculation for Example 1

For $c = 1, x = 0.5$ and $t = 0.25$, the exact analytical solution is evaluated as follows.

$$\begin{aligned} u(0.5,0.25) &= \sin(\pi/2) \cos(\pi/4) \\ &= 1 \times 0.70710678 = 0.70710678 \end{aligned}$$

Thus, the displacement at the centre of the domain at $t = 0.25$ is approximately 0.707107.

The same calculation at selected times gives the following values:

Because the solution is obtained analytically by the Laplace Transform Method, the numerical values generated from the closed-form expression agree with the exact solution to the displayed precision. The absolute error is therefore zero when the same analytical expression is used for evaluation.

6.2 Detailed Calculation for Example 2

For illustration, take $c = 1$ and $\omega = 0.8\pi$. The first three modal coefficients are calculated from

$$A_1 = -1.13176848$$

$$A_2 = 0.06063045$$

$$A_3 = -0.01624548$$

For $n = 1$, the denominator is $\omega^2 - \pi^2 = (0.8\pi)^2 - \pi^2 = -0.36\pi^2$, so the first coefficient is

$$A_1 = -0.71111111$$

The denominator is non-zero because $\omega = 0.8\pi$ is not equal to $\pi, 2\pi, 3\pi, \dots$. Hence the selected example is non-resonant.

7. Comparison of the Two Examples

Feature	Example 1	Example 2
Initial displacement	$\sin(\pi x)$	0
Initial velocity	0	0
Boundary at $x = 0$	0	0
Boundary at $x = 1$	0	$\sin(\omega t)$
Boundary type	Homogeneous	Non-homogeneous
Response	Standing wave	Boundary-driven wave
Method	Laplace Transform	Laplace Transform

Example 1 produces a simple standing-wave solution because both ends are fixed. Example 2 represents a boundary-driven system in which one endpoint oscillates in time. The two examples demonstrate that the Laplace Transform Method can handle different boundary-condition structures.

8. Physical Interpretation

The first example represents a freely oscillating standing wave with both endpoints fixed. The initial displacement has a sinusoidal shape and the zero initial velocity means that the system is initially released from rest.

The second example represents a boundary-driven wave. One end remains fixed while the other end oscillates sinusoidally. Consequently, waves are generated by the moving boundary and propagate through the domain. The resonance condition identified in Example 2 is important because excitation close to a natural frequency can produce a large response.

9. Advantages of the Laplace Transform Method

Initial conditions are incorporated directly into the transformed equation.

Time derivatives are converted into algebraic expressions.

The PDE is reduced to an ordinary differential equation in the spatial variable.

No spatial discretization is required for the analytical problems considered here.

No iterative numerical algorithm is required for the simple linear problems.

The method provides analytical solutions or analytical series representations.

Both homogeneous and non-homogeneous boundary conditions can be treated.

The transformed solution gives useful information about resonance and natural frequencies.

10. Limitations

The Laplace Transform Method is particularly effective for linear equations, but obtaining an explicit inverse transform can become difficult for complicated problems. For nonlinear wave equations, variable coefficients, complicated geometries or strongly non-homogeneous forcing terms, the transformed equation may not have a simple analytical inverse.

In such cases, the Laplace transform may need to be combined with Fourier series, eigenfunction expansions or numerical inversion techniques.

11. Future Scope

Damped wave equations.

Forced wave equations.

Variable wave-speed problems.

Nonlinear wave equations.

Two-dimensional wave equations.

Wave equations with mixed boundary conditions.

Resonance and frequency-response analysis.

Comparison of Laplace-transform solutions with finite-difference and finite-element methods.

12. Conclusion

In this paper, the Laplace Transform Method has been applied to one-dimensional wave equations with different boundary conditions. The transformation was performed with respect to the time variable, converting the governing partial differential equation into an ordinary differential equation in the spatial variable.

Two examples were considered. The first example involved homogeneous boundary conditions and sinusoidal initial displacement. The resulting solution was obtained in closed form as

$$u(x,t) = \sin(\pi x) \cos(\pi ct)$$

and was verified by direct substitution into the governing equation and by checking the initial and boundary conditions.

The second example involved a non-homogeneous sinusoidal boundary condition. The Laplace transform was used to obtain the transformed solution, and the inverse solution was represented through the natural modes of the finite interval. The analysis also identified the resonance condition associated with the natural frequencies of the system.

The results demonstrate that the Laplace Transform Method provides a direct and systematic analytical framework for solving one-dimensional wave equations. The method is particularly useful for linear initial-boundary value problems and provides clear mathematical and physical interpretation of wave propagation and resonance.

References

1. Strauss, W. A., Partial Differential Equations: An Introduction, 2nd ed., Wiley, New York, 2008.
2. Haberman, R., Applied Partial Differential Equations with Fourier Series and Boundary Value Problems, 5th ed., Pearson, Boston, 2013.
3. Evans, L. C., Partial Differential Equations, 2nd ed., American Mathematical Society, Providence, 2010.

4. Debnath, L. and Bhatta, D., Integral Transforms and Their Applications, 3rd ed., Chapman & Hall/CRC, Boca Raton, 2015.
5. Churchill, R. V. and Brown, J. W., Fourier Series and Boundary Value Problems, 8th ed., McGraw-Hill, New York, 2012.
6. Kreyszig, E., Advanced Engineering Mathematics, 10th ed., Wiley, New York, 2011.
7. O'Neil, P. V., Advanced Engineering Mathematics, Cengage Learning, Boston, 2017.
8. John, F. B. and Haberman, R., Partial Differential Equations: An Introduction, 4th ed., Pearson, 2013.
9. Zauderer, E., Partial Differential Equations of Applied Mathematics, 3rd ed., Wiley, Hoboken, 2006.
10. Logan, J. D., Applied Partial Differential Equations, 3rd ed., Springer, New York, 2015.
11. Strauss, W. A., Wave Equations with Applications to the Vibrating String and Other Continuum Models, relevant chapters in Partial Differential Equations: An Introduction, Wiley.
12. Polyanin, A. D. and Zaitsev, V. F., Handbook of Exact Solutions for Ordinary Differential Equations, 2nd ed., Chapman & Hall/CRC, 2003.