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ANCIENT INDIAN MATHEMATICS: AN OVERVIEW

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ABSTRACT

India has had a long tradition of more than 3000 years of pursuit of Mathematical ideas, starting from the Vedic age. The Sulvasutras (which included Pythagoras theorem before Pythagoras), the Jain works, the base 10 representation (along with the use of 0), names given to powers of 10 up to 10^{53}, the works of medieval mathematicians motivated by astronomical studies, and finally the contributions of the Kerala school that came strikingly close to modern mathematics, represent the various levels of intellectual attainment. The aim of this article is to present an overview of ancient Indian mathematics together with a discussion on the sources and directions for future studies.

Key Words: Sulvasutras, Jain mathematics, Siddhanta astronomy, Bakhshali manuscript, Kerala School of Mathematics.

INTRODUCTION

Mathematics has played a significant role in the development of Indian culture for millennia. Mathematical ideas that originated in the Indian subcontinent have had a profound impact on the world. Swami Vivekananda said: 'you know how many sciences had their origin in India. Mathematics began there. You are even today counting 1, 2, 3, etc. to zero, after Sanskrit figures, and you all know that algebra also originated in India.' It is also a fitting time to review the contributions of Indian mathematicians from ancient times to the present, as in 2010, India will be hosting the International Congress of Mathematicians. This quadrennial meeting brings together mathematicians from around the world to discuss the most significant developments in the subject over the past four years and to get a sense of where the subject is heading in the next four. The idea of holding such a congress at regular intervals actually started at The Columbian Exhibition in Chicago in 1893. This exhibition had sessions to highlight the advancement of knowledge in different fields. One of these was a session on mathematics. Another, perhaps more familiar to readers of Prabuddha Bharata, was the famous Parliament of Religions in which Swami Vivekananda first made his public appearance in the West.

Following the Chicago meeting, the first International Congress of Mathematicians took place in Zurich in 1897. It was at the next meeting at Paris in 1900 that Hilbert formulated his now famous 23 Problems. Since that time, the congress has been meeting approximately every four years in different cities around the world, and in 2010, the venue will be Hyderabad, India. This is the first time in its more than hundred-year history that the congress will be held in India. This meeting could serve as an impetus and stimulus to mathematical thought in the subcontinent, provided the community is prepared for it. Preparation would largely consist in being aware of the tradition of mathematics in India, from ancient times to modern and in embracing the potential and possibility of developing this tradition to new heights in the coming millennia.

In ancient time, mathematics was mainly used in an auxiliary or applied role. Thus, mathematical methods were used to solve problems in architecture and construction (as in the public works of the Harappan civilization) in astronomy and astrology (as in the words of the Jain mathematicians) and in the construction of Vedic altars (as in the case of the Shulba Sutras of Baudhayana and his successors). By the sixth or fifth century BCE, mathematics was being studied for its own sake, as well as for its applications in other fields of knowledge.

The aim of this article is to give a brief review of a few of the outstanding innovations introduced by Indian mathematics from ancient times to modern. As we shall see, there does not seem to have been a time in Indian history when mathematics was not being developed. Recent work has unearthed many manuscripts, and what were previously regarded as dormant periods in Indian mathematics are now known to have been very active. Even a small study of this subject leaves one with a sense of wonder at the depth and breadth of ancient Indian thought.

The picture is not yet complete, and it seems that there is much work to do in the field of the history of Indian mathematics. The challenges are two-fold. First, there is the task of locating and identifying manuscripts and of translating them into a language that is more familiar to modern scholars. Second, there is the task of interpreting the significance of the work that was done.

Since much of the past work in this area has tended to adopt a Eurocentric perspective and interpretation, it is necessary to take a fresh, objective look. The time is ripe to make a major effort to develop as complete a picture as possible of Indian mathematics. Those who are interested in embarking on such an effort can find much helpful material online.

We may ask what the term Indian means in the context of this discussion. Mostly, it refers to the Indian subcontinent, but for more recent history we include also the diaspora and people whose roots can be traced to the Indian subcontinent, wherever they may be geographically located.

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MATHEMATICS IN ANCIENT TIMES (3000 TO 600 BCE)

The Indus valley civilization is considered to have existed around 3000 BCE. Two of its most famous cities, Harappa and Mohenjo-Daro, provide evidence that construction of buildings followed a standardized measurement which was decimal in nature. Here, we see mathematical ideas developed for the purpose of construction. This civilization had an advanced brick-making technology (having invented the kiln). Bricks were used in the construction of buildings and embankments for flood control. The study of astronomy is considered to be even older, and there must have been mathematical theories on which it was based. Even in later times, we find that astronomy motivated considerable mathematical development, especially in the field of trigonometry.

Much has been written about the mathematical constructions that are to be found in Vedic literature. In particular, the *Shatapatha Brahmana*, which is a part of the Shukla Yajur Veda, contains detailed descriptions of the geometric construction of altars for yajnas. Here, the brick-making technology of the Indus valley civilization was put to a new use. As usual there are different interpretations of the dates of Vedic texts, and in the case of this Brahmana, the range is from 1800 to about 800 BCE. Perhaps it is even older. Supplementary to the Vedas are the Shulba Sutras. These texts are considered to date from 800 to 200 BCE. Four in numbers, they are named after their authors: *Baudhayana* (600 BCE), *Manava* (750 BCE), *Apastamba* (600 BCE), and *Katyayana* (200 BCE). The sutras contain the famous theorem commonly attributed to Pythagoras. Some scholars (such as Seidenberg) feel that this theorem as opposed to the geometric proof that the Greeks, and possibly the Chinese, were aware of.

The Shulba Sutras introduce the concept of irrational numbers, numbers that are not the ratio of two whole numbers. For example, the square root of 2 is one such number. The sutras give a way of approximating the square root of number using rational numbers through a recursive procedure which in modern language would be a 'series expansion'.

This predates, by far, the European use of Taylor series. It is interesting that the mathematics of this period seems to have been developed for solving practical geometric problems, especially the construction of religious altars. However, the study of the series expansion for certain functions already hints at the development of an algebraic perspective. In later times, we find a shift towards algebra, with simplification of algebraic formulate and summation of series acting as catalysts for mathematical discovery.

JAIN MATHEMATICS (600 BCE TO 500 CE)

This is a topic that scholars have started studying only recently. Knowledge of this period of mathematical history is still fragmentary, and it is a fertile area for future scholarly studies. Just as Vedic philosophy and theology stimulated the development of certain aspects of mathematics, so too did the rise of Jainism. Jain cosmology led to ideas of the infinite. This in turn, led to the development of the notion of orders of infinity as a mathematical concept. By orders of infinity, we mean a theory by which one set could be deemed to be 'more infinite' than another. In modern language, this corresponds to the notion of cardinality. For a finite set, its cardinality is the number of elements it contains. However, we need a more sophisticated notion to measure the size of an infinite set. In Europe, it was not until Cantors work in the nineteenth century that a proper concept of cardinality was established.

Besides the investigations into infinity, this period saw developments in several other fields such as number theory, geometry, computing, with fractions and combinatorics. In particular, the recursion formula for binomial coefficients and the 'Pascal's triangle' were already known in this period. As mentioned in the previous section, astronomy had been studied in India since ancient times. This subject is often confused with astrology. Swami Vivekananda has speculated that astrology came to India from the Greeks and that astronomy was borrowed by the Greeks from India. Indirect evidence for this is provided by a text by Yavaneshvara (c. 200 CE) which popularized a Greek astrology text dating back to 120 BCE.

The period 600 CE coincides with the rise and dominance of Buddhism. In the *Lalitavistara*, a biography of the Buddha which may have been written around the first century CE, there is an incident about Gautama being asked to state the name of large powers of 10 starting with 10. He is able to give names to numbers up to 10 (tallaksana). The very fact that such large numbers had names suggests that the mathematicians of the day were comfortable thinking about very large numbers. It is hard to imagine calculating with such numbers without some form of place value system.

BRAHMI NUMERALS, THE PLACE-VALUE SYSTEM AND ZERO

No account of Indian mathematics would be complete without a discussion of Indian numerals, the place-value system, and the concept of zero. The numerals that we use even today can be traced to the Brahmi numerals that seem to have made their appearance in 300 BCE. But Brahmi numerals were not part of a place value system. They evolved into the Gupta numerals around 400 CE and subsequently into the Devnagari numerals, which developed slowly between 600 and 1000 CE.

By 600 CE, a place-value decimal system was well in use in India. This means that when a number is written down, each symbol that is used has an absolute value, but also a value relative to its position. For example, the numbers 1 and 5 have a value on their own, but also have a value relative to their position in the number 15. The importance of a place-value system need hardly be emphasized. It would suffice to cite an often-quoted remark by La-place: 'It is India that gave us the ingenious method of expressing all numbers by means of ten symbols, each symbol receiving a value of position as well as an absolute value; a profound and important idea which appears so simple to us now that we ignore its true merit. But its very simplicity and the great ease which it has lent to computations put our arithmetic in the first rank of useful inventions; and we shall appreciate the grandeur of the achievement the more when we remember that it escaped the genius of Archimedes and Apollonius, two of the greatest men produced by antiquity.

A place-value system of numerals was apparently known in other cultures; for example, the Babylonians used a sexagesimal place-value system as early as 1700 BCE, but the Indian system was the first decimal system. Moreover, until 400 BCE, THE Babylonian system had an inherent ambiguity as there was no symbol for zero. Thus it was not a complete place-value system in the way we think of it today.

The elevation of zero to the same status as other numbers involved difficulties that many brilliant mathematicians struggled with. The main problem was that the rules of arithmetic had to be formulated so as to include zero. While addition, subtraction, and multiplication with zero were mastered, division was a more subtle question. Today, we know that division by zero is not well-defined and so has to be excluded from the rules of arithmetic. But this understanding did not come all at once, and took the combined efforts of many minds. It is interesting to note that it was not until the seventeenth century that zero was being used in Europe, and the path of mathematics from India to Europe is the subject of much historical research.

THE CLASSICAL ERA OF INDIAN MATHEMATICS (500 TO 1200 CE)

The most famous names of Indian mathematics belong to what is known as the classical era. This includes Aryabhata I (500 CE) Brahmagupta (700 CE), Bhaskara I (900 CE), Mahavira (900 CE), Aryabhatta II (1000 CE) and Bhaskarachrya or Bhaskara II (1200 CE).

During this period, two centers of mathematical research emerged, one at Kusumapura near Pataliputra and the other at Ujjain. Aryabhata I was the dominant figure at Kusumapura and may even have been the founder of the local school. His fundamental work, the *Aryabhatiya,* set the agenda for research in mathematics and astronomy in India for many centuries.

One of Aryabhata's discoveries was a method for solving linear equations of the form
$ax + by = c$. Here a, b, and c are whole numbers, and we seeking values of x and y in whole numbers satisfying the above equation. For example if $a = 5$, $b = 2$, and $c = 8$ then $x = 8$ and $y = -16$ is a solution. In fact, there are infinitely many solutions:

$$x = 8 - 2m$$
$$y = 5m - 16$$

where m is any whole number, as can easily be verified. Aryabhata devised a general method for solving such equations, and he called it the *kuttaka (*or pulverizer) method. He called it the pulverizer because it proceeded by a series of steps, each of which required the solution of a similar problem, but with smaller numbers. Thus, a, b, and c were pulverized into smaller numbers.

The Euclidean algorithm, which occurs in the Elements of Euclid, gives a method to compute the greatest common divisor of two numbers by a sequence of reductions to smaller numbers. As far as I am aware Euclid does not suggest that this method can be used to solve linear equations of the above sort. Today, it is known that if the algorithm in Euclid is applied in reverse order then in fact it will yield Aryabhata's method. Unfortunately the mathematical literature still refers to this as the extended Euclidean algorithm, mainly out of ignorance of Aryabhata's work.

It should be noted that Aryabhata's studied the above linear equations because of his interest in astronomy. In modern times, these equations are of interest in computational number theory and are of fundamental importance in cryptography.

Amongst other important contributions of Aryabhata is his approximation of *Pie* to four decimal places (3.14146). By comparison the Greeks were using the weaker approximation 3.1429. Also of importance is Aryabhata's work on trigonometry, including his tables of values of the sine function as well as algebraic formulate for computing the sine of multiples of an angle.

The other major centre of mathematical learning during this period was Ujjain, which was home to Varahamihira, Brahmagupta and Bhaskaracharya. The text *Brahma-sphuta-siddhanta* by Brahmagupta, published in 628 CE, dealt with arithmetic involving zero and negative numbers. As with Aryabhata, Brahmagupta was an astronomer, and much of his work was motivated by problems that arose in astronomy. He gave the famous formula for a solution to the quadratic equation

It is not clear whether Brahmagupta gave just this solution or both solutions to this equation. Brahmagupta also studied quadratic equation in two variables and sought solutions in whole numbers. Such equations were studied only much later in Europe. We shall discuss this topic in more detail in the next section. This period closes with Bhaskaracharya (1200 CE). In his fundamental work on arithmetic (titled *Lilavati*) he refined the kuttaka method of Aryabhata and Brahmagupta. The Lilavati is impressive for its originality and diversity of topics.

Until recently, it was a popularly held view that there was no original Indian mathematics before Bhaskaracharya. However, the above discussion shows that his work was the culmination of a series of distinguished mathematicians who came before him. Also, after Bhaskaracharya, there seems to have been a gap of two hundred years before the next recorded work. Perhaps this is another time period about which more research is needed.

THE SOLUTION OF PELL'S EQUATION

In Brahmagupta's work, Pell's equation had already made an appearance. This is the equation that for a given whole number D, asks for whole numbers x and y satisfying the equation

Xsquare – Dysquare = I.

In modern times, it arises in the study of units of quadratic fields and is a topic in the field of algebraic number theory. If D is a whole square (such as 1, 4, 9 and so on), the equation is easy to solve, as it factors into the product

$$(x- my)(x + my) = 1$$

where D = m square. This implies that each factor is + 1 or – 1 and the values of x and y can be determined from that. However, if D is not a square, then it is not even clear that there is a solution. Moreover, if there is a solution it is not clear how one can determine all solutions. For example consider the case D=2. Here, x = 3 and y=2 gives a solution. But if D=61, then even the smallest solutions are huge.

Brahmagupta discovered a method, which he called *samasa*, by which; given two solutions of the equation a third solution could be found. That is, he discovered a composition law on the set of solutions. Brahmagupta's lemma was known one thousand years before it was rediscovered in Europe by Fermat, Legendre, and others. This method appears now in most standard text books and courses in number theory. The name of the equation is a historical accident. The Swiss mathematician Leonhard Euler mistakenly assumed that the English mathematician John Pell was the first to formulate the equation, and began referring to it by this name.

The work of Bhaskaracharya gives an algorithmic approach which he called the *cakrawala* (cyclic) method to finding all solutions of this equation. The method depends on computing the continued fraction expansion of the square root of D and using the convergents to give values of x and y. Again, this method can be found in most modern books on number theory, though the contributions of Bhaskaracharya do not seem to be well-known.

MATHEMATICS IN SOUTH INDIA

We described above the centres at Kusumapara and Ujjain. Both of these cities are in North India. There was also a flourishing tradition of mathematics in South India which we shall discuss in brief in this section. Mahavira is a mathematician belonging to the ninth century who was most likely from modern day Karnataka. He studied the problem of cubic and quartic equations and solved them for some families of equations. His work had a significant impact on the development of mathematics in South India. His book *Ganita– sara–sangraha* amplifies the work of Brahmagulpta and provides a very useful reference for the state of mathematics in his day. It is not clear what other works he may have published; further research into the extent of his contributions would probably be very fruitful.

Another notable mathematician of South India was Madhava from Kerala. Madhava belongs to the fourteenth century. He discovered series expansions for some trigonometric functions such as the sine, cosine and arctangent that were not known in Europe until after Newton. In modern terminology, these expansions are the Taylor series of the functions in question.

Madhava gave an approximation to Pie of 3.14159265359, which goes far beyond the four decimal places computed by Aryabhata. Madhava deduced his approximation from an infinite series expansion for Pie by 4 that became known in Europe only several centuries after Madhava (due to the work of Leibniz).

Madhava's work with series expansions suggests that he either discovered elements of the differential calculus or nearly did so. This is worth further analysis. In a work in 1835, Charles Whish suggested that the Kerala School had laid the foundation for a complete system of fluxions. The theory of fluxions is the name given by Newton to what we today call the differential calculus. On the other hand, some scholars have been very dismissive of the contributions of the Kerala School, claiming that it never progressed beyond a few series expansions. In particular, the theory was not developed into a powerful tool as was done by Newton. We note that it was around 1498 that Vasco da Gama arrived in Kerala and the Portuguese occupation began. Judging by evidence at other sites, it is not likely that the Portuguese were interested in either encouraging or preserving the sciences of the region. No doubt, more research is needed to discover where the truth lies.

Madhava spawned a school of mathematics in Kerala, and among his followers may be noted Nilakantha and Jyesthadeva. It is due to the writings of these mathematicians that we know about the work of Machala, as all of Madhava's own writings seem to be lost.

MATHEMATICS IN THE MODERN AGE

In more recent times there have been many important discoveries made by mathematicians of Indian origin. We shall mention the work of three of them: Srinivasa Ramanujan, Harish-Chandra, and Manjul Bhargava.

Ramanujan (1887- 1920) is perhaps the most famous of modern Indian mathematicians. Though he produced significant and beautiful results in many aspects of number theory, his most lasting discovery may be the arithmetic theory of modular forms. In an important paper published in 1916, he initiated the study of the Pie function. The values of this function are the Fourier coefficients of the unique normalized cusp form of weight 12 for the modular group SL2 (Z). Ramanujan proved some properties of the function and conjectured many more. As a result of his work, the modern arithmetic theory of modular forms, which occupies a central place in number theory and algebraic geometry, was developed by Hecke.

Harish-Chandra (1923- 83) is perhaps the least known Indian mathematician outside of mathematical circles. He began his career as a physicist, working under Dirac. In his thesis, he worked on the representation theory of the group SL2 (C). This work convinced him that he was really a mathematician, and he spent the remainder of his academic life working on the representation theory of semi-simple groups. For most of that period, he was a professor at the Institute for Advanced Study in Princeton, New Jersey. His *Collected Papers* published in four volumes contain more than 2,000 pages. His style is known as meticulous and thorough and his published work tends to treat the most general case at the very outset. This is in contrast to many other mathematicians, whose published work tends to evolve through special cases. Interestingly, the work of Harish-Chandra formed the basis of Langlands's theory of automorphic forms, which are a vast generalization of the modular forms considered by Ramanujan.

Manjul Bhargava (b. 1974) discovered a composition law for ternary quadratic forms. In our discussion of Pell's equation, we indicated that Brahmagupta discovered a composition law for the solutions. Identifying a set of importance and discovering an algebraic structure such as a composition law is an important theme in mathematics. Karl Gauss, one of the greatest mathematicians of all time, showed that binary quadratic forms, that is, functions of the form

axsquare + bxy + cysquare

where a, b, and c are integers, have such a structure. More precisely, the set of primitive SLsquare (Z) orbits of binary quadratic forms of given discriminant D has the structure of an abelian group. After this fundamental work of Gauss, there had been no progress for several centuries on discovering such structures in other classes of forms.

REFERENCES

- History of Hindu Mathematics-B.B. Datta & A.N. Singh
- Geometry According to Sulba Sutra- Dr. R.P. Kulkarni
- Geometry in Ancient and Medieval India – Dr. T.A.S. Amma
- Vedic Mathematics in School- J.T. Glover.
- Aryabhatiya of Aryabhatta-Edited by.- Walter Eugene Clark
- Surya-Siddhanta – Edited by- Phanindra Lal Ganguli
- Lilavati- Edited by- Krishnaji Shankar Patwardhan
- Buddhism and Science – Buddhadasa P. Kirthisinghe
- India's Contribution to the West – Dr. P. Priyadarshi
- What Happened in History – Gordon Childe
- The Story of Mathematics – Richard Mankiewicz
- Joy of Mathematics – Theoni Pappas
- Mathematics for the Non Mathematicians – Morris Kline
- Mathematics for Millions – Lancelot Hogben
- God Created the Integers – Stephen Hawkins
- Universal History of Numbers – Georges Ifrah
- History of Mathematics – D.E. Smith
- The World of Mathematics – James R Newman
- K Ramasubramanian and M S Sriram (Ed.), Tantrasangraha of Nilakantha Somayaji. With a foreword by B V Subbarayappa, Culture and History of Mathematics, Vol.6, Hindustan Book Agency, New Delhi, 2011.
- S G Dani, Geometry in Sulvasutras, in: Studies in the history of Indian mathematics, pp.9–37, Cult. Hist. Math. 5, Hindustan Book Agency, New Delhi, 2010.

- George G Joseph, Crest of the Peacock, Non-European roots of mathematics, Third edition, Princeton University Press, Princeton, NJ, 2011.
- K Ramasubramanian and M D Srinivas, Development of Calculus in India, in: Studies in the history of Indian mathematics, pp.201–286, Cult. Hist. Math. 5, Hindustan Book Agency, New Delhi, 2010.