Approximation of Function in the Generalized Zygmund Class by Double Euler Summability Means of Fourier Series

Santosh Kumar Sinha
Department of Mathematics, Lakhmi Chand Institute of Technology
Bilaspur, Chhattisgarh, India -495001
Email – drsksinha2024@gmail.com

Abstract

In this paper, a theorem on degree of approximation of function in the generalized Zygmund class by double Euler summability means of Fourier series has been established.

Keywords : Degree of approximation , Generalized Zygmund class , (E, q) means, $(E, 1)$ $(E, 1)$ means .

MSC : 41A24 , 41A25 , 42B05 , 42B08

1. Introduction

The degree of approximation of function belonging to different classes like $Lip\ \alpha$, $Lip\ (\alpha, r)$, $Lip(\xi(t), r)$, $W(L_r, \xi(t))$ have been studied by many researchers using different summability means (see [5], [6]). The error estimation of function in Lipschitz and Zygmund class using different means of Fourier series and conjugate Fourier series have been great interest among the researcher.The generalized Zygmund class $Z_r^{(\omega)}(r \geq 1)$ has studied by Leindler [2] Moricz [3], Moricz and Nemeth [4] etc. Recently Singh et al. [9], Mishra et al. [7], Pradhan et al. [8], Das et al. [1], find the results in Zygmund class by using different summability means . In this paper we find the degree of approximation of function in the generalized Zygmund class by (N, p_n) (E, q) means of Fourier series.To the best of our knowledge, degree of approximation of function in the generalized Zygmund class by $(E, 1)$ $(E, 1)$ product summability means of Fourier series has not been studied so far.

2. Definition

Let f be a periodic function of period 2π integrable in the sense of Lebesgue over [π, - π]. Then the Fourier series of f given by

$$f(t) \approx \frac{a_o}{2} + \sum_{n=1}^{\infty}(a_n cos\, nx + b_n sin\, nx) \qquad (2.1)$$

Zygmund class z is defined as

$$Z = \{f \in C[-\pi, \pi] \quad |f(x+t) + f(x-t) - 2f(x)| = O(|t|)\}.$$

Let $\omega: [0, 2\pi] \to R$ be an arbitrary function with $\omega(t) > 0$ for $0 < t < 2\pi$

$lim_{t \to 0}\omega(t) = \omega(0) = 0$ define

$$Z_p^w := \left\{ f \in L_p : 1 \le p \le \infty \ sup \frac{\|f(.+t) + f(.-t) - 2f(.)\|_p}{\omega(t)} < \infty \right\}$$

and $\|f\|_p^\omega := \|f\|_p + sup \frac{\|f(.+t)+f(.-t)-2f(.)\|_p}{\omega(t)}$ $p \ge 1$. (2.2)

Clearly $\|.\|_p^\omega$ is a norm on Z_p^w.

Hence the Zygmund space Z_p^w is a Banach space under the norm $\|.\|_p^\omega$.

We write through the paper

$$\emptyset_x(t) = f(x+t) - 2f(x) + f(x-t) \tag{2.3}$$

$$K_n(t) = \frac{1}{(2)^{n+k+1}\pi} \sum_{k=0}^{n} \binom{n}{k} \left\{ \sum_{v=0}^{k} \binom{k}{v} \frac{\sin\left(v+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)} \right\}. \tag{2.4}$$

3. Main Result

In this paper we prove the following theorem.

Theorem - Let f be a 2π periodic function, Lebesgue integrable in $[0, 2\pi]$ and belonging to generalized Zygmund class $Z_r^{(w)}$ $(r \ge 1)$. Then the degree of approximation of function f by $(E,1)(E,1)$ product mean of Fourier series is given

by $E_n(f) = inf \ \|t_n^{EE} - f\|_r^v = o\left(\int_{\frac{1}{n+1}}^{\pi} \frac{w(t)}{tv(t)} \, dt \right)$

where $\omega(t)$ and $v(t)$ denotes the Zygmund moduli of continuity such that $\frac{w(t)}{v(t)}$ is positive and increasing .

4 .Lemma –To prove the theorem we need the following Lemma.

Lemma 4(a) - For $0 \le t \le \frac{\pi}{n+1}$ we have $\sin nt = n \ sint$

$$|K_n(t)| = o(n) \tag{4.1}$$

Proof - For $0 \le t \le \frac{\pi}{n+1}$ and $\sin nt = n \ sint$ then

$$|\,K_n(t)| = \left| \frac{1}{(2)^{n+k+1}\pi} \sum_{k=0}^{n} \binom{n}{k} \left\{ \sum_{v=0}^{k} \binom{k}{v} \frac{\sin\left(v+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)} \right\} \right|$$

$$\le \frac{1}{(2)^{n+k+1}\pi} \left| \sum_{k=o}^{n} \binom{n}{k} \left\{ \sum_{v=0}^{k} \binom{k}{v} \frac{(2v+1)\sin\left(\frac{t}{2}\right)}{\sin\left(\frac{t}{2}\right)} \right\} \right|$$

$$\le \frac{1}{(2)^{n+k+1}\pi} \left| \sum_{k=o}^{n} \binom{n}{k} (2k+1) \left\{ \sum_{v=0}^{k} \binom{k}{v} \right\} \right|$$

$$\le \frac{1}{(2)^{n+1}\pi} \left| \sum_{k=o}^{n} \binom{n}{k} (2k+1) \right|$$

$$= \frac{(2n+1)}{2\pi}$$

$$= o(n)$$

Lemma 4(b) - For $\frac{\pi}{n+1} \leq t \leq \pi$, $sin\frac{t}{2} \geq \frac{t}{\pi}$ and $sin\, nt \leq 1$ we have

$$| K_n(t)| = o\left(\frac{1}{t}\right) \tag{4.2}$$

Proof - For $\frac{\pi}{n} \leq t \leq \pi$, $sin\frac{t}{2} \geq \frac{t}{\pi}$ and $sin\, nt \leq 1$

$$| K_n(t)| = \left|\frac{1}{(2)^{n+k+1}\pi} \sum_{k=0}^{n}\binom{n}{k}\left\{\sum_{v=0}^{k} \binom{k}{v} \frac{\sin\,(v+\frac{1}{2})t}{\sin\,(\frac{t}{2})}\right\}\right|$$

$$\leq \frac{1}{(2)^{n+k+1}\pi}\left|\sum_{k=o}^{n}\binom{n}{k}\left\{\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\frac{\pi}{t}\right\}\right|$$

$$\leq \frac{1}{(2)^{n+k+1}t}\left|\sum_{k=o}^{n}\binom{n}{k}\left\{\sum_{v=0}^{k}\binom{k}{v}\right\}\right|$$

$$\leq \frac{1}{(2)^{n+1}t}\left|\sum_{k=o}^{n}\binom{n}{k}\right|$$

$$= o\left(\frac{1}{t}\right)$$

Lemma 4(c) – Let $f \in Z_p^{(w)}$ then for $0 < t \leq \pi$

(i) $\|\phi(.,t)\|_p = o(w(t))$

(ii) $\|\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)\|_p = \begin{cases} o(w(t) \\ o(w(y) \end{cases}$

(iii) If $\omega(t)$ and $v(t)$ are defined as in theorem then

$$\|\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)\|_p = \left\{v(y)\frac{\omega(t)}{v(t)}\right.$$

where $\phi(x,t) = f(x+t) + f(x-t) - 2f(x)$.

5. Proof of Theorem 3

Let $s_n(x)$ denotes the partial sum of fourier series given in (2.1) then we have

$$s_n(x) - f(x) = \frac{1}{2\pi}\int_0^{\pi} \emptyset(t)\frac{\sin\,(n+\frac{1}{2})t}{sin\frac{t}{2}}dt \tag{5.1}$$

The (E, 1) transform E_n^1 of s_n is given by $E_n^1 - f(x) = \frac{1}{\pi(2)^{n+1}}\int_0^{\pi} \emptyset(t)\left\{\sum_{k=0}^{n}\binom{n}{k}\frac{sin\left(k+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\right\}dt.$ (5.2)

The (E, 1) , (E, 1) transform of $s_n(x)$ is given by

$$t_n^{EE}(f) - f(x) = \frac{1}{\pi(2)^{n+1}}\sum_{k=0}^{n}\binom{n}{k}\int_0^{\pi} \emptyset(t)\left\{\frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}\frac{sin\left(v+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\right\}dt \tag{5.3}$$

$$= \int_0^{\pi} \emptyset(t)\ k_n(t). \tag{5.4}$$

Let $l_n(x) = t_n^{NE} - f(x) = \int_0^{\pi} \emptyset(x,t)\ k_n(t)dt$ then

$$l_n(x+y) + l_n(x-y) - 2\, l_n(x) = \int_0^{\pi} [\phi(x+y,t) + \phi(x-y,t) - 2\phi(x,t)]\ k_n(t)\, dt.$$

Using the generalized Minkowaski's inequality we get

$$\|\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)\|_p = \left\{\frac{1}{2\pi}\int_0^{2\pi} |l_n(x+y) + l_n(x-y) - 2\, l_n(x)|^p\, dx\right\}^{\frac{1}{p}}$$

$$= \left\{\frac{1}{2\pi}\int_0^{2\pi} \left|\int_0^{\pi} [\phi(x+y,t) + \phi(x-y,t) - 2\phi(x,t)]\ k_n(t)\, dt\right|^p dx\right\}^{\frac{1}{p}}$$

$$\leq \int_0^{\pi} \left\{\frac{1}{2\pi}\int_0^{2\pi} |[\phi(x+y,t) + \phi(x-y,t) - 2\phi(x,t)]\ k_n(t)|^p\, dx\right\}^{\frac{1}{p}} dt$$

$$= \int_0^{\pi} (|\ k_n(t)|^p)^{\frac{1}{p}} \left\{\frac{1}{2\pi}\int_0^{2\pi} |[\phi(x+y,t) + \phi(x-y,t) - 2\phi(x,t)]|^p\, dx\right\}^{\frac{1}{p}} dt$$

$$= \int_0^{\pi} \|\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)\|_p |\ k_n(t)|\, dt$$

$$= \int_0^{\frac{1}{n+1}} \|\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)\|_p |\ k_n(t)|\, dt + \int_{\frac{1}{n+1}}^{\pi} \|\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)\|_p |\ k_n(t)|\, dt$$

$$= I_1 + I_2.\ (say) \tag{5.5}$$

Using Lemma 4(a) and 4(c) and the monotonically of $\frac{\omega(t)}{v(t)}$ with respect to t, we have

$$I_1 = \int_0^{\frac{1}{n+1}} \|\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)\|_p |\ k_n(t)|\, dt$$

$$= \int_0^{\frac{1}{n+1}} o\left(v(y)\frac{\omega(t)}{v(t)}\right) o(n)\, dt$$

$$= o\left(nv(y)\int_0^{\frac{1}{n+1}} \frac{\omega(t)}{v(t)}\, dt\right).$$

Using second mean value theorem of integral, we have

$$I_1 \leq o\left(nv(y)\int_0^{\frac{1}{n+1}} \frac{\omega(t)}{v(t)}\, dt\right)$$

$$= o\left(\frac{n}{n+1}\ v(y)\ \frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}\right)$$

$$= o\left(v(y)\ \frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}\right). \tag{5.6}$$

For I_2 using lemma 4(b) and 4(c) , we have

$$I_2 = \int_{\frac{1}{n+1}}^{\pi} \|\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)\|_p | \, k_n(t)| \, dt$$

$$= o\left(\int_{\frac{1}{n+1}}^{\pi} \left(v(y)\frac{\omega(t)}{v(t)}\right) \frac{1}{t} \, dt\right)$$

$$= o\left(v(y) \int_{\frac{1}{n+1}}^{\pi} \left(\frac{\omega(t)}{t\, v(t)}\right) \, dt\right). \qquad (5.7)$$

From (5.5) , (5.6) and (5.7), we get

$$\|l_n(.+y) + l_n(.-y) - 2l_n(.)\|_p = o\left(v(y) \frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)})\right) + o\left(v(y) \int_{\frac{1}{n+1}}^{\pi} \left(\frac{\omega(t)}{t\, v(t)}\right) \, dt\right)$$

$$\sup_{y \neq 0} \frac{\|l_n(.+y)+l_n(.-y)-2l_n(.)\|_p}{v(y)} = o\left(\frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}\right) + o\left(\int_{\frac{1}{n+1}}^{\pi} \left(\frac{\omega(t)}{t\, v(t)}\right) \right). \qquad (5.8)$$

Again using Lemma we have

$$\|l_n(.)\|_p \leq \left(\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi} .\right) \|\phi(.,t)\| |K_n(t)| \, dt$$

$$= o\left(n \int_0^{\frac{1}{n+1}} \omega(t) dt\right) + o\left(\int_{\frac{1}{n+1}}^{\pi} \frac{\omega(t)}{t} \, dt\right)$$

$$= o\left(\frac{n}{n+1}\, \omega\left(\frac{1}{n+1}\right)\right) + o\left(\int_{\frac{1}{n+1}}^{\pi} \frac{\omega(t)}{t} \, dt\right)$$

$$= o\left(\omega\left(\frac{1}{n+1}\right)\right) + o\left(\int_{\frac{1}{n+1}}^{\pi} \frac{\omega(t)}{t} \, dt\right). \qquad (5.9)$$

From (5.8) and (5.9), we obtain

$$\|l_n(.)\|_p^v = \|l_n(.)\|_p + \sup_{y \neq 0} \frac{\|l_n(.+y)+l_n(.-y)-2l_n(.)\|_p}{v(y)}$$

$$= o\left(\omega\left(\frac{1}{n+1}\right)\right) + o\left(\int_{\frac{1}{n+1}}^{\pi} \frac{\omega(t)}{t} \, dt\right) + o\left(\frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}\right) + o\left(\int_{\frac{1}{n+1}}^{\pi} \left(\frac{\omega(t)}{t\, v(t)}\right) \right)$$

$$= \sum_{i=1}^{4} J_i \ .$$

Now we write J_1 in terms of J_3 *and* J_2 , J_3 *in* term of J_4.

In view of the monotonicity of v(t) we have

$$\omega(t) = \left(\frac{\omega(t)}{v(t)}\right), \quad v(t) \leq v(\pi)\left(\frac{\omega(t)}{v(t)}\right) = o\left(\frac{\omega(t)}{v(t)}\right) \quad for\ 0 < t \leq \pi$$

therefore we can write

$$J_1 = o(\,J_3) \,.$$

Again using monotonicity of v(t)

$$J_2 = \int_{\frac{1}{n+1}}^{\pi} \frac{\omega(t)}{t}\, dt = \int_{\frac{1}{n+1}}^{\pi} \left(\frac{\omega(t)}{t\, v(t)}\right) dt \le v(\pi) \int_{\frac{1}{n+1}}^{\pi} \left(\frac{\omega(t)}{t\, v(t)}\right) dt = o(J_4). \qquad (5.10)$$

Using the fact $\frac{\omega(t)}{v(t)}$ is positive and non decreasing , we have

$$J_4 = \int_{\frac{1}{n+1}}^{\pi} \left(\frac{\omega(t)}{t\, v(t)}\right) dt = \frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)} \int_{\frac{1}{n+1}}^{\pi} \left(\frac{1}{t}\right) dt \ge \frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}$$

therefore we can write

$$J_3 = o(J_4)\ .$$

so we have

$$\|l_n(.)\|_p^v = o(J_4) = o\left(\int_{\frac{1}{n+1}}^{\pi} \left(\frac{\omega(t)}{t\, v(t)}\right) dt\right).$$

Hence

$$E_n(f) = \inf\ \|l_n(.)\|_p^v = o\left(\int_{\frac{1}{n+1}}^{\pi} \left(\frac{\omega(t)}{t\, v(t)}\right) dt\right)$$

This completes the proof.

7. Conclusion.

In this study, different types of results on the degree of approximation of periodic function belonging to the Lipschitz classes and Zygmund classes of function are reviewed.The established theorem in this paper is on degree of approximation of function in the generalized Zygmund class by $(E, 1)\,(E, 1)$ summability means of Fourier series. The result can be extended for other functions belonging to weighted Zygmund class.

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