International Journal of Scientific Research in Engineering and Management (IJSREM)
Volume: 08 Issue: 04 | April - 2024 SJIF Rating: 8.448 ISSN: 2582-3930

Characterization of μ**- R_1 Spaces

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ABSTRACT: In this paper we introduce $\mu^{**}R_1$- spaces and we study some characterization of $\mu^{**}R_1$- spaces. We analyse the relation between μ**-closed sets with already existing closed sets.

KEYWORDS: μ**-closed sets, μ**-open sets, μ**-closure, μ**-R_o spaces.

I INTRODUCTION

Levine[7] introduced generalized closed sets (briefly g-closed sets) in topological spaces and studied their basic properties. R. Devi[4] introduced and studied μ* -closed sets. Veerakumar [10] introduced g*-closed sets in topological spaces and studied their properties. Pauline Mary Helan [8] introduced and studied g** -closed sets in topological spaces. The aim of this paper is to introduce a $\mu^{**}R_1$- spaces and we investigate some characterization of $\mu^{**}R_1$- spaces.

II PRELIMINARIES

Definition 2.1 A subset A of a topological space (X,τ) is called

(i) a semi-open set if A ⊆ cl(int(A)) and a semi-closed set if int(cl(A)) ⊆A,
(ii) a preopen set if A ⊆ int(cl(A)) and a preclosed set if cl(int(A)) ⊆ A,
(iii) an α- open set if A ⊆ int(cl(int(A))) and an α -closed set if cl(int(cl(A))) ⊆ A,
(iv) a semi-preopen set if A⊆ cl(int(cl(A))) and a semi-preclosed set if int(cl(int(A))) ⊆A
(v) a regular open set if A=int(cl(A))and a regular closed set if cl(int(A))=A.

The semi-closure (resp.preclosure , semi-preclosure) of a subset A of a space (X,τ) is the intersection of all semi-closed(resp. preclosed , α-closed, semi-preclosed) sets that contain A and is denoted by scl(A) (resp.pcl(A), Acl(A), spcl(A)).

Definition 2.2 A subset A of a space (X,τ) is called

(i) a generalized closed (briefly g-closed) set[10] if cl(A)⊆U whenever A⊆U and U is open in (X,τ); the compliment of a g-closed set is called a g-open set,
(ii) a semi-generalized closed (briefly sg-closed) set[2] if scl(A)⊆U whenever A⊆U and U is semi-open in(X,τ); the compliment of sg-closed set is called a sg-open set,
(iii) a generalized semi-closed (briefly gs-closed) set if scl(A)⊆U whenever A⊆U and U is open in (X,τ)

(iv) an α-generalized closed (briefly αg-closed) set[3] if αcl(A)⊆U whenever A⊆U and U is α-open in (X,τ),

(v) a generalized α-closed (briefly gα-closed) set [3] if αcl(A)⊆U whenever A⊆U and U is α-open in (X,τ)

(vi) a g*- closed set [10] if cl(A)⊆U whenever A⊆U and U is g-open in (X,τ),

(vii) a g^{**}-closed set [8] if cl(A)⊆U whenever A⊆U and U is g*-open in (X,τ),

(viii) a generalized preclosed(briefly gp-closed) setif pcl(A)⊆U whenever A⊆U and U is open in (X,τ),

(ix) a generalized semi-preclosed (briefly gsp-closed) set [5] if spcl(A)⊆U whenever A⊆U and U is open in(X,τ)

(x) a generalized pre regular closed (briefly gpr-closed) set [6] if pcl(A)⊆U whenever A⊆U and U is regular open in (X,τ),

(xi) a $g^{\#}$-closed set [11] if cl(A)⊆U whenever A⊆U and U is αg-open in (X,τ),

(xii) a generalized α**-closed (briefly gα**-closed) set [3] if αcl(A)⊆int(cl(U)) whenever A⊆U and U is α-open in (X,τ),

(xiii) a μ*-closed set [4] if cl(A)⊆U whenever A⊆U and U is gα**-open in (X,τ),

(xiv) a g*s- closed set [9] if scl(A)⊆U whenever A⊆U and U is gs-open in (X,τ).

The compliment of the above mentioned sets are called their respective open sets.

III .μ^{**}- R_1 Spaces

Definition 3.1. A topological space (X, τ) is said to be μ^{**}-R_1 if for x, y in X with μ^{**}-Cl({x}) ≠ μ^{**}-Cl({y}), there exist disjoint μ^{**}-open sets U and V such that μ^{**}-Cl({x}) is a subset of U and μ^{**}-Cl({y}) is a subset of V.

Theorem 3.1: If (X, τ) is μ^{**}-R_1, then (X, τ) is μ^{**}-R_0.

Proof: Let U be μ^{**}-open and x ∈ U. If y ∉ U, then since x ∉ μ^{**}-Cl({y}), μ^{**}-Cl({x}) ≠ μ^{**}Cl({y}). Hence, there exists a μ^{**}-open v_y such that μ^{**}-Cl({y}) ⊂ v_y and x ∉ v_y, which implies y ∉ μ^{**}-Cl({x}). Thus μ^{**}-Cl({x}) ⊂ U. Therefore (X, τ) is μ^{**}-R0.

Theorem 3.2: A topological space (X, τ) is μ^{**}-R_1 if and only if for x, y ∈ X, μ^{**}- Ker({x}) ≠ μ^{**}-Ker({y}), there exist disjoint μ^{**}-open sets U and V such that μ^{**}-Cl({x}) ⊂ U and μ^{**}-Cl({y}) ⊂ V.

Proof : It follows from Definition.

Theorem 3.3 : If (X, τ) is μ^{**}-T_2, then (X, τ) is μ^{**}-R_1.

Proof. Since X is μ^{**}-T_2, then X is μ^{**}-R_1. If x, y ∈ X such that μ^{**}-Cl({x}) ≠ μ^{**}-Cl({y}), then x ≠ y. There exists disjoint μ^{**}-open sets U and V such that x ∈ U and y ∈ V ; hence μ^{**}-Cl({x}) = {x} ⊂ U and μ^{**}-Cl({y}) = {y} ⊂ V. Hence X is μ^{**}-R_1.

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Definition 3.2: A topological space (X, τ) is called μ^{**}-T_2 if for any distinct pair of points x and y in X, there exist μ^{**}-open sets U and V in X containing x and y, respectively, such that U ∩ V = ∅.

Theorem 3.4. If (X, τ) is gμ^{**}-T_2, then (X, τ) is μ^{**}-R_1.

Proof. Since X is μ^{**}-T_2, then X is μ^{**}-R_1. If x, y ∈ X such that μ^{**}-Cl({x}) ≠ μ^{**}Cl({y}), then x ≠ y. There exists disjoint μ^{**}-open sets U and V such that x ∈ U and y ∈ V ; hence μ^{**}-Cl({x}) = {x} ⊂ U and μ^{**}-Cl({y}) = {y} ⊂ V. Hence X is μ^{**}-R1.

Theorem 3.5. For a topological space (X, τ), the following statements are equivalent :
(1) (X, τ) is μ^{**}-R_1 ;
(2) If x, y ∈ X such that μ^{**}-Cl({x}) ≠ μ^{**}-Cl({y}), then there exists μ^{**}-closed sets F_1 and F_2such that x ∈ F_1, y ∉ F_1 , y ∈ F2 , x ∉ F_2 and X = F_1 ∪ F_2 .
Proof. (1) ⇒ (2) : Let x, y ∈ X such that μ^{**}-Cl({x}) ∉ μ^{**}-Cl({y}), and hence x ≠ y. Therefore, there exists disjoint μ^{**}-open sets U_1 and U_1such that x ∈ μ^{**}-Cl({x}) ⊂ U_1 and y ∈ μ^{**}-Cl({y}) ⊂ U_1. Then F_1 = X – U_2 and F_2 = X − U_1 are μ^{**}-closed sets such that x ∈ F_1 , y ∉ F_1, y ∈ F_2, x ∉ F_2 and X = F_1 ∪ F_2. **(2) ⇒ (1) :**Suppose that x and y are distinct points of X, such that μ^{**}-Cl({x}) ∉ μ^{**}Cl({y}). Therefore there exist μ^{**}-closed sets F_1 and F_2 such that x ∈ F_1 , y ∉ F_1 , y ∈ F_2 , x ∉ F_2 and X = F_1 ∪ F_2. Now, we set U_1 = X – F_2 and U_2 = X − F_1, then we obtain that x ∈ U_1 , y ∈ U_2, U_1 ∩ U_2 = ∅ and U_1, U_2 are μ^{**}-open. This shows that (X, τ) is μ^{**}- T_2. Therefore (X, τ) is μ^{**}- R_1.
Theorem 3.6. A topological space (X, τ) is μ^{**}- R_1 if and only if for x, y ∈ X, bKer({x}) ≠ μ^{**}-Ker({y}), there exist disjoint μ^{**}-open sets U and V such that μ^{**}- Cl({x}) ⊂ U and μ^{**}-Cl({y}) ⊂ V.
Proof. Proof follows from the definition.

Definition 3.3:A point x of a topological space (X, τ) is a μ^{**}-θ-accumulation point of a subset A ⊂ X, if for each μ^{**}-open U of X containing x, μ^{**}-Cl(U)∩A ≠ ∅. The set μ^{**}-Cl(A) of all μ^{**}-θ-accumulation points of A is called the μ^{**}-θ-closure of A. The set A is said to be μ^{**}-θ-closed if μ^{**}-Clθ(A) = A. Complement of a μ^{**}-θ-closed set is said to be μ^{**}-θ-open.

Lemma 3.1: For any subset A of a topological space (X, τ), μ^{**}-Cl(A) ⊂ μ^{**}-Clθ(A).

Lemma 3.2: Let x and y are points in a topological space (X, τ). Then y ∈ μ^{**}-Clθ({x}) if and only if x ∈ μ^{**}-Clθ({y}).
Theorem 3.7. A topological space (X, τ) is μ^{**}- R_1 if and only if for each x ∈ X, μ^{**}-Cl({x}) =μ^{**}-Clθ({x}).
Proof: Necessity: Assume that X is μ^{**}- R_1 and y ∈ μ^{**}-Clθ({x})−μ^{**}-Cl({x}).
Then there exists a μ^{**}-open set U containing y such that μ^{**}-Cl(U) ∩ {x} ≠ ∅ but U ∩ {x} = ∅. Thus μ^{**}-Cl{y}) ⊂ U, μ^{**}-Cl({x}) ∩ U = ∅. Hence μ^{**}-Cl({x}) ≠ μ^{**}-Cl({y}). Since X is μ^{**}- R_1, there exist disjoint μ^{**}-open sets U_1 and U_2 such that μ^{**}-Cl({x}) ⊂ U_1 and μ^{**}-Cl({y}) ⊂ U_2. Therefore X − U_1 is a μ^{**}-closed μ^{**}-neigbourhood at y which does not contain x.Thus y ∉ μ^{**}-Clθ({x}). This is a contradiction. Sufficiency: Suppose that μ^{**}-Cl({x}) = μ^{**}-Clθ({x}) for each x ∈ X. We first prove that X is μ^{**}- R_0.Let x belong to the μ^{**}-open set U and y ∉U. Since μ^{**}-Clθ({y}) = μ^{**}Cl({y}) ⊂ X − U, we have x ∉ μ^{**}-Clθ({y}) and y ∉ μ^{**}-Clθ({x}) = μ^{**}Cl({x}). It follows that μ^{**}-Cl({x}) ⊂ U. Therefore (X, τ) is μ^{**}- R_0. Now, let a, b ∈

X with μ^{**}-Cl({a}) ≠ μ^{**}-Cl({b}). (X, τ)is μ^{**}- T_1 and b ∉ μ^{**}-Clθ({a}) and hence there exists a μ^{**}-open set U containing b such that a ∉ μ^{**}-Cl(U). Therefore, we obtain b ∈ U, a ∈ X − μ^{**}-Cl(U) and U ∩ (X − μ^{**}-Cl(U)) = ∅. This shows that (X, τ) is μ^{**}- T_2. It follows that (X, τ) is μ^{**}- R_1.

Theorem 3.8: For a topological space (X, τ) the following are equivalent: (1) (X, τ) is μ^{**}- R_1; (2) (X, τ) is μ^{**}-symmetric.

Proof: (1) ⇒ (2). If x∉ μ^{**}-Cl({y}). Then there exist a μ^{**}-open set U containing x such that y ∉ U. Hence y ∉ μ^{**}-Cl(U). The converse is similarly shown. (2) ⇒ (1) Let U be a μ^{**}-open set and x ∈ U. If y ∉ U, then x ∉ μ^{**}-Cl({y}) and hence y ∉ μ^{**}-Cl({x}) This implies that μ^{**}-Cl({x}) ⊂ U.Hence (X, τ) is μ^{**}- R_1

Theorem 3.9: For a topological space (X, τ), the following statements are equivalent:
(1) (X, τ) is a μ^{**}- R_1 space; (2) If x, y ∈ X, then y ∈ μ^{**}-Cl({x}) if and only if every net in X μ^{**}-converging to y μ^{**}-converges to x.

Proof: (1)→(2): Let x, y ∈ X such that y ∈ μ^{**}-Cl({x}). Suppose that {Xα}α ∈ ∧ be a net in X such that {Xα}α ∈∧ μ^{**}-converges to y. Since y ∈ μ^{**}-Cl({x}), we have μ^{**}-Cl({x}) = μ^{**}-Cl({y}). Therefore x ∈ μ^{**}-Cl({y}). This means that {Xα}α ∈∧ μ^{**}-converges to x. Conversely, let x, y ∈ X such that every net in X μ^{**}-converging to y μ^{**}-converges to x. Then x ∈ μ^{**}-Cl({y}) .we have μ^{**}-Cl({x}) = μ^{**}-Cl({y}). Therefore y ∈ μ^{**}-Cl({x}). (2)→(1): Assume that x and y are any two points of X such that ge-Cl({x}) ∩ geCl({y})≠ ∅. Let z ∈ μ^{**}-Cl({x}) ∩ μ^{**}-Cl({y}). So there exists a net {xα}α∈∧ in μ^{**}Cl({x}) such that {xα}α∈∧ μ^{**}-converges to z. Since z ∈ μ^{**}-Cl({y}), then {xα}α∈∧ μ^{**}-converges to y. It follows that y ∈ μ^{**}-Cl({x}). By the same token we obtain x ∈ μ^{**}-Cl({y}). Therefore μ^{**}-Cl({x}) = μ^{**}-Cl({y}) and (X, τ) is μ^{**}- R_1.

4. OTHER PROPERTIES OF μ^{**}-OPEN SETS

Definition 4.1: A subset A of a topological space X is called a μ^{**}D -set if there are two U, V ∈ μ^{**}o (X, τ) such that U ≠ X and A=U − V . One can observe that every μ^{**}-open set U different from X is a μ^{**}D -set if A = U and V = ∅.

Definition 4.2: A topological space (X, τ) is called: (i) $\mu^{**}D_0$ if for any distinct pair of points x and y of X there exists a μ^{**}D -set of X containing x but not y or a μ^{**}D -set of X containing y but not x. (ii) $\mu^{**}D_1$ if for any distinct pair of points x and y of X there exists a μ^{**}D -set of X containing x but not y and a μ^{**}D -set of X containing y but not x. (iii) $\mu^{**}D_2$if for any distinct pair of points x and y of X there exists disjoint μ^{**}D -sets G and E of X containing x and y, respectively. (iv) μ^{**}-T_0 if for any distinct pair of points in X, there is a μ^{**}-open set containing one of the points but not the other.

Remark 4.1: (i) If (X, τ) is μ^{**}- T_i , then it is μ^{**}- T_i −1, i = 1, 2. (ii) If (X, τ) is μ^{**}- T_i , then (X, τ) is μ^{**}- D_i , i = 0, 1, 2. (iii) If (X, τ) is μ^{**}- D_i , then it is μ^{**}- D_i −1 , i = 1, 2.

Theorem 4.1: For a topological space (X, τ) the following statements are true: (1) (X, τ) is μ^{**}- D_0 if and only if it is μ^{**}- T_0. (2) (X, τ) is μ^{**}- D_1if and only if it is μ^{**}- D_2.

Proof:(1) We prove only the necessity condition since the sufficiency condition is stated in Remark 4.1(ii). Necessity. Let (X, τ) be μ^{**}- D_0. Then for each distinct pair x, y ∈ X, at least one of x, y, say x, belongs to a μ^{**}D -set G but y ∉ G. Let G = U_1 \ U_2 where U_1 ≠ X and U_1, U_2 ∈ μ^{**}O (X, τ). Then x ∈ U_1, and for y ∉ G we have two cases: (a) y ∉ U_1 ; (b) y ∈ U_1 and y ∈ U_2. In case (a), U1 contains x but not y ; In case (b), U2 contains y but not x. Hence X is μ^{**}- T_0. (2) Sufficiency. Remark 5.1(iii). Necessity. Let X be a

μ^{**}- D_1 topological space. Then for each distinct pair x, y ∈ X , we have μ^{**}D -sets G_1, G_1 such that x ∈ G_1, y ∉ G_1 ; y ∈ G_2 , x ∉ G_2 . Let $G_1= U_1\setminus U_2$, $G_2 = U_3\setminus U_4$. From x ∉ G_2, we have either x ∉ U_3or x ∈ U_3 and x ∈ U_4. Now we consider the following two cases separately (1) x ∉ U_3. From y ∉ G_1we have two subcases: (a) y ∉ U_1. From x ∈ $U_1\setminus U_2$ we have x ∈ $U_1\setminus (U_2 \cup U_3)$ and from y ∈ $U_3\setminus U_4$we have y ∈ $U_3\setminus(U_1 \cup U_4)$. Therefore, $(U_1\setminus(U_2 \cup U_3)) \cap (U_3\setminus(U_1 \cup U_4) = \emptyset$. (b) y ∈ $U_1$and y ∈ U_2. We have x ∈ $U_1\setminus U_2$, y ∈ U_2. $(U_1\setminus U_2) \cap U_2 = \emptyset$. (2) x ∈ U_3 and x ∈ U_4. We have y ∈ $U_3\setminus U_4$, x ∈ U_4. $(U_3\setminus U_4) \cap U_4 = \emptyset$. From the discussion above we know that the space X is μ^{**}- D_2 .

Theorem 4.2: For a μ^{**}-T_0 topological space (X, τ) each pair of distinct points x, y of X, μ^{**}-Cl({x}) ≠ μ^{**}-Cl({y}).

Proof: Let x, y be any two distinct points of X. Since, X is μ^{**}-T_0 , there exists a μ^{**}- open set G containing x or y, say x but not y. Then G^C is a μ^{**}-closed set which does not contain x but contains y. Since μ^{**}-Cl({y}) is the smallest μ^{**}-closed set containing y, Clb({y}) ⊂ G^C , and so x ∉ μ^{**}-Cl({y}). Consequently μ^{**}-Cl({x}) ≠ μ^{**}-Cl({y}).

Theorem 4.3: A topological space X is μ^{**}-T_2 if and only if the intersection of all μ^{**}-closed μ^{**}-neighbourhood of each point of X is reduced to that point.

Proof: Necessity: Let X be μ^{**}-$T_2$and x ∈ X. Then for each y ∈ X which is distinct from x, there exist μ^{**}-open sets G and H such that x ∈ G, y ∈ H and G ∩ H = ∅. Since x ∈ G ⊂ H^C , hence H^C is a μ^{**}-closed μ^{**}-neighbourhood of x to which y does not belong . Consequently, the intersection of all μ^{**}-closed μ^{**}-neighbourhood of x is reduced to {x}. Sufficiency: Let x, y ∈ X and x ≠ y. Then by hypothesis there exists a μ^{**}-closed μ^{**}-neighbourhood U of x such that y ∉ U. Now there is a μ^{**}-open set G such that x ∈ G ⊂ U. Thus G and G^C are disjoint μ^{**}-open sets containing x and y respectively. Hence X is μ^{**}-T_2.

Definition 4.3: A function f : (X, τ) → (Y, σ) is μ^{**}-irresolute if the inverse image of each μ^{**}-open set is μ^{**}-open.

Theorem 4.4: If f : (X, τ) → (Y, σ) is a μ^{**}-irresolute surjective function and E is a μ^{**}D -set in Y, then the inverse image of E is a μ^{**}D -set in X.

Proof: Let E be a μ^{**}D -set in Y . Then there are μ^{**} -open sets U_1 and U_2in Y such that $E = U_1\setminus U_2$ and $U_1 \neq Y$. By theμ^{**}- irresoluteness of f, $f^{-1} (U_1)$ and $f^{-1} (U_2)$ are μ^{**}-open in X. Since $U_1 \neq Y$, we have $f^{-1} (U_1) \neq X$. Hence $f^{-1} (E) = f^{-1} (U_1)\setminus f^{-1} (U_2)$ is a μ^{**}D -set.

Theorem 4.5: If (Y, σ) is μ^{**}- D_1 and f : (X, τ) → (Y, σ) is g˜-irresolute and bijective, then (X, τ) is μ^{**}- D_1.

Proof: Suppose that Y is a μ^{**}- D_1 space. Let x and y be any pair of distinct points in X. Since f is injective and Y is μ^{**}- D_1, there exist μ^{**}D -sets G_X and G_Y of Y containing f(x) and f(y) respectively, such that f(y) ∈/ G_X and f(x) ∈/ G_Y, $f^{-1} (G_X)$ and $f^{-1} (G_Y)$ are μ^{**}D -sets in X containing x and y respectively. This implies that X is a μ^{**}- D_1space.

Theorem 4.6:A topological space (X, τ) is μ^{**}- D_1 if and only if for each pair of distinct points x, y ∈ X, there exists a g˜-irresolute surjective function f : (X, τ) → (Y, σ), where Y is a μ^{**}- D_1space such that f(x) and f(y) are distinct.

Proof: Necessity. For every pair of distinct points of X, it suffices to take the identity function on X. Sufficiency. Let x and y be any pair of distinct points in X. By hypothesis, there exists a μ^{**}-irresolute, surjective function f of a space X onto a μ^{**}- D_1 space Y such that f(x) ≠ f(y). Therefore, there exist disjoint μ^{**}D -sets G_X and G_Y in Y such that f(x) ∈ G_X and f(y) ∈ G_Y . Since f is μ^{**}-irresolute and surjective, f^{-1} (G_X) and f^{-1} (G_Y) are disjoint μ^{**}D -sets in X containing x and y, respectively. Hence X is μ^{**}- D_1space **.**

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