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Common and Tripled Fixed-Point Theorems for Hybrid Contractions in Perturbed Rectangular b-Metric Spaces

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Abstract

In this paper, we establish novel common fixed point and tripled coincidence point theorems by synthesizing recent advances in perturbed metric fixed point theory, potential-driven contractions, and dual rectangular b-metric spaces. First, we establish a common fixed point theorem for an alternating pair of mappings satisfying a generalized potential-difference contraction condition in complete perturbed metric spaces, thereby unifying the results of Singh and Devi with the discontinuous fixed point framework of Pant et al. Second, we establish common tripled fixed point and tripled coincidence point theorems for nonlinear mappings acting on spaces endowed with two perturbed rectangular b-metrics, generalising the theorems of Gu and Liu under perturbed error structures. Step-by-step detailed proofs are provided, accompanied by three illustrative examples verifying the conditions and highlighting the role of potential and perturbation functions. Finally, a comprehensive conclusion and full declarations are presented.

Keywords: Perturbed metric space; Rectangular b-metric space; Common fixed point; Tripled coincidence point; Potential function; ω-compatibility.

2020 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

Metric fixed point theory plays a pivotal role in functional analysis, operator theory, and applied mathematics, providing powerful analytical tools for studying the existence and uniqueness of solutions to differential, integral, and functional equations. Initiated by the celebrated Banach Contraction Principle [1], the subject has evolved across multiple distinct topological and analytical directions:

(i) Generalization of Topological Structures: Czerwik [4] generalized classical metric spaces to b-metric spaces by relaxing the triangle inequality with a parameter s ≥ 1. Branciari [2] introduced generalized (rectangular) metric spaces by replacing the triangular inequality with a quadrilateral condition. George et al. [5] combined both frameworks into rectangular b-metric spaces. In the realm of generalized distance geometry, Mishra et al. investigated fixed point principles in M-fuzzy metric spaces [9], generalized s-convex metric spaces [12], Polish spaces [13], ordered fuzzy metric spaces [15], and G-metric spaces with applications to fractional differential equations [16]. Recently, Jleli and Samet [7] conceptualized perturbed metric

(ii) Relaxation of Contraction, Commutativity, and Continuity Requirements: Classical results often require the underlying mappings to be continuous and commute. Mishra et al. extensively studied common fixed points under generalized commutativity conditions (such as subcompatible and occasionally non-commuting maps) [10, 11], multi-valued generalized φ-weak contractive mappings [14], and integral type implicit relations [9,11,13,14]. Concurrently, Pant et al. [17] developed a general fixed point theorem incorporating a potential function $\varphi : X \to [0,\infty)$, offering a resolution to Rhoades' open problem concerning the existence of fixed points for mappings discontinuous at the fixed point. Singh and Devi [18] studied common fixed points for pairs of mappings in perturbed metric spaces, while Gu and Liu [6] explored coupled and tripled coincidence and common fixed point theorems in spaces equipped with two rectangular b-metrics.

The objective of this paper is to build a unified mathematical framework combining the perturbation principles of Singh and Devi [18], the potential function formulation of Pant et al. [17], and the multidimensional fixed point theory in dual rectangular b-metric spaces established by Gu and Liu [6].

2. Preliminaries

Definition 2.1 ((7,**18**]). Let X be a non-empty set. Let $\sigma: X \times X \to [0,\infty)$ and $P: X \times X \to [0,\infty)$ be two mappings. The mapping σ is said to be a *perturbed metric* on X with respect to the perturbation function P if the mapping $\rho: X \times X \to [0,\infty)$ defined by

$$\rho(x,y) = \sigma(x,y) - P(x,y), \quad \forall x,y \in X, \qquad (2.1)$$

is a standard metric on X. The triplet (X, σ, P) is called a *perturbed metric space*, and ρ is called the *exact metric*.

Definition 2.2 ([5,6]). Let X be a non-empty set and $s \geq 1$ be a real number. A mapping $d: X \times X \to [0, \infty)$ is called a *rectangular b-metric* on X with coefficient s if for all $x, y \in X$:

1. $d(x, y) = 0 \Leftrightarrow x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, z) \leq s[d(x, u) + d(u, v) + d(v, z)]$ for all distinct $u, v \in X \backslash \{x, z\}$.

The pair (X, d) is called a rectangular b-metric space with coefficient s.

Definition 2.3. Let X be a non-empty set and $s \geq 1$. Let $\sigma, P: X \times X \to [0, \infty)$ be two mappings. The mapping σ is called a *perturbed rectangular b-metric* on X with respect to P with coefficient $s \geq 1$ if the difference mapping

$$\rho(x, y) = \sigma(x, y - P(x, y), \quad \forall x, yX, (2.2)$$

is an exact rectangular b-metric on X with coefficient s. The triplet (X, σ, P) is called a *perturbed rectangular b-metric space*.

Definition 2.4 ([6]). Let $F: X \times X \times X \to X$ and $g: X \to X$ be mappings.

1. An element $(x, y, z) \in X^3$ is called a *tripled coincidence point* of F and g if:

$$gx = F(x, y, z), \quad gy = F(y, z, x), \quad gz = F(z, x, y).$$

2. An element $(x, y, z) \in X^3$ is called a *common tripled fixed point* of F and g if:

$$x = gx = F(x, y, z), \quad y = gy = F(y, z, x), \quad z = gz = F(z, x, y).$$

3. The pair $\{F, g\}$ is called ω *-compatible* if $g(F(x, y, z)) = F(gx, gy, gz)$, $g(F(y, z, x)) = F(gy, gz, gx)$, and $g(F(z, x, y)) = F(gz, gx, gy)$ whenever (x, y, z) is a tripled coincidence point of F and g.

3. Main Results

3.1 Alternating Potential Contractions in Perturbed Metric Spaces

Theorem 3.1. *Let (X, σ, P) be a complete perturbed metric space with exact metric $\rho = \sigma - P$. Let $F, G: X \to X$ be two perturbed continuous self-mappings. Suppose there exist a constant $\alpha \in [0,1)$ and a bounded-below potential function $\phi: X \to [0, \infty)$ satisfying $\phi(x) \geq \phi(Gx)$ and $\phi(x) \geq \phi(Fx)$ for all $x \in X$, such that for all $x, y \in X$:*

$$\sigma(Gx, Fy) \leq \alpha\, \sigma(x, y) + [\phi(x) - \phi(Gx)] + [\phi(y) - \phi(Fy)], (3.1)$$
$$\sigma(Fx, Gy) \leq \alpha\, \sigma(x, y) + [\phi(x) - \phi(Fx)] + [\phi(y) - \phi(Gy)]. (3.2)$$

Then F and G possess a unique common fixed point $x^* \in X$ such that $Fx^* = Gx^* = x^*$.

Proof. Let $x_0 \in X$ be an arbitrary initial point. We construct the iterative sequence $\{x_n\}_{n=0}^{\infty} \subset X$ as follows:

$$x_{2n+1} = Gx_{2n}, \quad x_{2n+2} = Fx_{2n+1}, \quad \forall n \geq 0. \qquad (3.3)$$

Step 1: Establishing the convergence of the step-length series.
Setting $x = x_{2n}$ and $y = x_{2n+1}$ in condition (3.1), we obtain:

$$\begin{aligned}\sigma(x_{2n+1}, x_{2n+2}) &= \sigma(Gx_{2n}, Fx_{2n+1}) \\ &\leq \alpha\, \sigma(x_{2n}, x_{2n+1}) + [\phi(x_{2n}) - \phi(Gx_{2n})] + [\phi(x_{2n+1}) - \phi(Fx_{2n+1})] \\ &= \alpha\, \sigma(x_{2n}, x_{2n+1}) + [\phi(x_{2n}) - \phi(x_{2n+1})] + [\phi(x_{2n+1}) - \phi(x_{2n+2})] \\ &= \alpha\, \sigma(x_{2n}, x_{2n+1}) + \phi(x_{2n}) - \phi(x_{2n+2}).\end{aligned}$$

Similarly, by setting $x = x_{2n+1}$ and $y = x_{2n+2}$ in condition (3.2), we have:

$$\begin{aligned}\sigma(x_{2n+2}, x_{2n+3}) &= \sigma(Fx_{2n+1}, Gx_{2n+2}) \\ &\leq \alpha\, \sigma(x_{2n+1}, x_{2n+2}) + [\phi(x_{2n+1}) - \phi(Fx_{2n+1})] + [\phi(x_{2n+2}) - \phi(Gy_{2n+2})] \\ &= \alpha\, \sigma(x_{2n+1}, x_{2n+2}) + \phi(x_{2n+1}) - \phi(x_{2n+3}).\end{aligned}$$

Consequently, for any integer $m \geq 0$, we have the unified recurrence relation:

$$\sigma(x_{m+1}, x_{m+2}) \leq \alpha\, \sigma(x_m, x_{m+1}) + \phi(x_m) - \phi(x_{m+2}). \qquad (3.4)$$

Summing inequality (3.4) over m from 0 to $N-1$, where $N \geq 2$

$$\sum_{m=0}^{N-1} \sigma\,(x_{m+1}, x_{m+2}) \leq \alpha \sum_{m=0}^{N-1} \sigma\,(x_m, x_{m+1}) + \sum_{m=0}^{N-1} (\phi(x_m) - \phi(x_{m+2})). \qquad (3.5)$$

The second sum on the right-hand side of (3.5) telescopes:

$$\begin{aligned}\sum_{m=0}^{N-1} (\phi(x_m) - \phi(x_{m+2})) &= (\phi(x_0) - \phi(x_2)) + (\phi(x_1) - \phi(x_3)) + \cdots + (\phi(x_{N-1}) - \phi(x_{N+1})) \\ &= \phi(x_0) + \phi(x_1) - \phi(x_N) - \phi(x_{N+1}).\end{aligned}$$

Since $\phi(x) \geq 0$ for all $x \in X$, it follows that $-\phi(x_N) - \phi(x_{N+1}) \leq 0$. Hence:

$$\sum_{m=0}^{N-1} (\phi(x_m) - \phi(x_{m+2})) \leq \phi(x_0) + \phi(x_1). \qquad (3.6)$$

Now, re-indexing the metric terms in (3.5):

$$\sum_{m=1}^{N} \sigma\,(x_m, x_{m+1}) \leq \alpha \left(\sigma(x_0, x_1) + \sum_{m=1}^{N-1} \sigma\,(x_m, x_{m+1}) \right) + \phi(x_0) + \phi(x_1),$$

which yields:

$$(1 - \alpha) \textstyle\sum_{m=1}^{N-1} \sigma\,(x_m, x_{m+1}) + \sigma(x_N, x_{N+1}) \leq \alpha\, \sigma(x_0, x_1) + \phi(x_0) + \phi(x_1). \qquad (3.7)$$

Since $\alpha \in [0,1)$, we have $1 - \alpha > 0$. Taking the limit as $N \to \infty$, the sequence of partial sums is non-decreasing and bounded from above, which proves that the infinite series converges:

$$\sum_{m=0}^{\infty} \sigma\,(x_m, x_{m+1}) < \infty. \qquad (3.8)$$

Step 2: Proof that $\{x_n\}$ is Cauchy in (X, ρ).
By Definition 1.1, the exact metric ρ satisfies:

$$0 \le \rho(x,y) = \sigma(x,y) - P(x,y) \le \sigma(x,y), \quad \forall x,y \in X,$$

since $P(x,y) \ge 0$. Thus, from [eq:series_sigma], we immediately deduce:

$$\sum_{m=0}^{\infty} \rho\,(x_m, x_{m+1}) \le \sum_{m=0}^{\infty} \sigma\,(x_m, x_{m+1}) < \infty. \tag{3.9}$$

For any integers $n, p \ge 1$, by the triangle inequality of the metric ρ:

$$\rho(x_n, x_{n+p}) \le \sum_{m=n}^{n+p-1} \rho\,(x_m, x_{m+1}). \tag{3.10}$$

Because the series $\sum_{m=0}^{\infty} \rho\,(x_m, x_{m+1})$ converges, the Cauchy remainder tends to zero:

$$\lim_{n\to\infty} \sum_{m=n}^{\infty} \rho\,(x_m, x_{m+1}) = 0.$$

Therefore, $\lim_{n\to\infty} \sup_{p\ge1} \rho(x_n, x_{n+p}) = 0$, establishing that $\{x_n\}$ is a Cauchy sequence in (X,ρ).

Step 3: Existence of the common fixed point.

Since (X,σ,P) is perturbed complete, the exact metric space (X,ρ) is complete. Thus, there exists a limit point $x^* \in X$ such that:

$$\lim_{n\to\infty} \rho(x_n, x^*) = 0. \tag{3.11}$$

Consequently, every subsequence of $\{x_n\}$ also converges to x^* in (X,ρ):

$$\lim_{n\to\infty} \rho(x_{2n}, x^*) 0 \quad \text{and} \quad \lim_{n\to\infty} \rho(x_{2n+1}, x^*) = 0. \tag{3.12}$$

Since F and G are perturbed continuous, they are continuous with respect to the exact metric ρ. Utilizing the metric properties and the construction (3.3):

$$\rho(x^*, Gx^*) \le \rho(x^*, x_{2n+1}) + \rho(x_{2n+1}, Gx^*) = \rho(x^*, x_{2n+1}) + \rho(Gx_{2n}, Gx^*).$$

Taking $n \to \infty$, since $\lim_{n\to\infty} \rho(x^*, x_{2n+1}) = 0$ and $\lim_{n\to\infty} \rho(Gx_{2n}, Gx^*) = 0$ by the ρ-continuity of G, we obtain:

$$\rho(x^*, Gx^*) = 0 \Longrightarrow Gx^* = x^*.$$

Similarly, for the mapping F:

$$\rho(x^*, Fx^*) \le \rho(x^*, x_{2n+2}) + \rho(x_{2n+2}, Fx^*) = \rho(x^*, x_{2n+2}) + \rho(Fx_{2n+1}, Fx^*).$$

Taking $n \to \infty$, the ρ-continuity of F gives:

$$\rho(x^*, Fx^*) = 0 \Longrightarrow Fx^* = x^*.$$

Hence, $x^* \in X$ is a common fixed point of F and G:

$$Fx^* = Gx^* = x^*.$$

Step 4: Uniqueness of the common fixed point.
Suppose there exists another point $y^* \in X$ such that $Fy^* = Gy^* = y^*$. Applying the contractive condition(3.1)with $x = x^*$ and $y = y^*$:

$$\begin{aligned}\sigma(x^*, y^*) &= \sigma(Gx^*, Fy^*) \\ &\le \alpha\,\sigma(x^*, y^*) + [\phi(x^*) - \phi(Gx^*)] + [\phi(y^*) - \phi(Fy^*)] \\ &= \alpha\,\sigma(x^*, y^*) + [\phi(x^*) - \phi(x^*)] + [\phi(y^*) - \phi(y^*)] \\ &= \alpha\,\sigma(x^*, y^*).\end{aligned}$$

This implies:

$$(1-\alpha)\,\sigma(x^*, y^*) \le 0. \tag{3.13}$$

Since $\alpha \in [0,1)$, we have $1 - \alpha > 0$. Moreover, by non-negativity of σ, $\sigma(x^*, y^*) \ge 0$. Thus, (3.13)forces:

$$\sigma(x^*, y^*) = 0.$$

Recall that $\rho(x^*, y^*) = \sigma(x^*, y^*) - P(x^*, y^*)$. Since $\rho(x^*, y^*) \ge 0$ and $P(x^*, y^*) \ge 0$, we have:

$$0 \le \rho(x^*, y^*) \le \sigma(x^*, y^*) = 0 \Longrightarrow \rho(x^*, y^*) = 0.$$

By axiom (2) of the metric ρ, it follows that $x^* = y^*$. This completes the proof of uniqueness.

3.2 Common Tripled Fixed Points in Dual Perturbed Rectangular *b*-Metric Spaces

Theorem 3.2. *Let X be a non-empty set. Let (σ_1, P_1) and (σ_2, P_2) be two perturbed rectangular b -metrics on X defining exact rectangular b -metrics $\rho_1 = \sigma_1 - P_1$ (with parameter $s \ge 1$) and $\rho_2 = \sigma_2 - P_2$, such that:*

$$\sigma_2(x,y) \le \sigma_1(x,y) \quad \text{and} \quad \rho_2(x,y) \le \rho_1(x,y), \quad \forall x,y \in X. \tag{3.14}$$

Let $F: X \times X \times X \to X$ and $g: X \to X$ be mappings such that $F(X^3) \subseteq g(X)$, and suppose that $(g(X), \rho_1)$ is a complete rectangular b-metric space. Assume there exist non-negative real constants k_1, k_2, k_3 satisfying:

$$k_1 + k_2 + k_3 < 1 \quad \text{and} \quad sk_3 < 1 \tag{3.15}$$

such that for all

$(x,y,z), (u,v,w) \in X^3$:
$$\begin{aligned}&\sigma_1(F(x,y,z), F(u,v,w)) + \sigma_1(F(y,z,x), F(v,w,u)) + \sigma_1(F(z,x,y), F(w,u,v)) \\ &\quad \le k_1[\sigma_2(gx,gu) + \sigma_2(gy,gv) + \sigma_2(gz,gw)] \\ &\quad\quad k_2[\sigma_2(gx, F(x,y,z)) + \sigma_2(gy, F(y,z,x)) + \sigma_2(gz, F(z,x,y))] \\ &+k_3[\sigma_2(gu, F(u,v,w)) + \sigma_2(gv, F(v,w,u)) + \sigma_2(gw, F(w,u,v))].\end{aligned} \tag{3.16}$$

Then:

1. *F and g possess a tripled coincidence point $(x^*, y^*, z^*) \in X^3$.*

2. *If, in addition, the pair $\{F, g\}$ is ω-compatible, then F and g possess a unique common tripled fixed point of the form $(u^*, u^*, u^*) \in X^3$ satisfying $u^* = gu^* = F(u^*, u^*, u^*)$.*

Proof. **Step 1: Construction of the iterative sequences.**
Let $(x_0, y_0, z_0) \in X^3$ be arbitrary. Since $F(X^3) \subseteq g(X)$, we can choose $(x_1, y_1, z_1) \in X^3$ such that:

$$gx_1 = F(x_0, y_0, z_0), \quad gy_1 = F(y_0, z_0, x_0), \quad gz_1 = F(z_0, x_0, y_0).$$

Continuing this process inductively, for each $n \geq 0$, we construct the sequence $\{gx_n, gy_n, gz_n\}$ satisfying:

$$gx_{n+1} = F(x_n, y_n, z_n), \quad gy_{n+1} = F(y_n, z_n, x_n), \quad gz_{n+1} = F(z_n, x_n, y_n). \tag{3.17}$$

Step 2: Consecutive step-length estimation.
For each $n \geq 0$, define:

$$\Delta_n = \sigma_1(gx_n, gx_{n+1}) + \sigma_1(gy_n, gy_{n+1}) + \sigma_1(gz_n, gz_{n+1}).$$

Applying the contractive condition (3.16) with $x = x_n, y = y_n, z = z_n$ and $u = x_{n+1}, v = y_{n+1}, w = z_{n+1}$:

$$\begin{aligned}\Delta_{n+1} &= \sigma_1(gx_{n+1}, gx_{n+2}) + \sigma_1(gy_{n+1}, gy_{n+2}) + \sigma_1(gz_{n+1}, gz_{n+2}) \\ &= \sigma_1(F(x_n, y_n, z_n), F(x_{n+1}, y_{n+1}, z_{n+1})) + \sigma_1(F(y_n, z_n, x_n), F(y_{n+1}, z_{n+1}, x_{n+1})) \\ &\quad + \sigma_1(F(z_n, x_n, y_n), F(z_{n+1}, x_{n+1}, y_{n+1})) \\ &\leq k_1[\sigma_2(gx_n, gx_{n+1}) + \sigma_2(gy_n, gy_{n+1}) + \sigma_2(gz_n, gz_{n+1})] \\ &\quad + k_2[\sigma_2(gx_n, gx_{n+1}) + \sigma_2(gy_n, gy_{n+1}) + \sigma_2(gz_n, gz_{n+1})] \\ &\quad + k_3[\sigma_2(gx_{n+1}, gx_{n+2}) + \sigma_2(gy_{n+1}, gy_{n+2}) + \sigma_2(gz_{n+1}, gz_{n+2})].\end{aligned}$$

Utilizing the dominance relation $\sigma_2(u, v) \leq \sigma_1(u, v)$ from (3.14), this inequality becomes:

$$\Delta_{n+1} \leq (k_1 + k_2)\Delta_n + k_3\Delta_{n+1}. \tag{3.19}$$

Since $k_3 < 1$, subtracting $k_3\Delta_{n+1}$ from both sides yields:

$$(1 - k_3)\Delta_{n+1} \leq (k_1 + k_2)\Delta_n \Longrightarrow \Delta_{n+1} \leq \lambda\Delta_n, \tag{3.20}$$

where

$$\lambda = \frac{k_1 + k_2}{1 - k_3}. \tag{3.21}$$

Condition (3.15) guarantees that $k_1 + k_2 < 1 - k_3$, which implies $\lambda \in [0,1)$. By mathematical induction:

$$\Delta_n \leq \lambda^n\Delta_0, \quad \forall n \geq 0. \tag{3.22}$$

Since $\rho_1(u, v) = \sigma_1(u, v) - P_1(u, v) \leq \sigma_1(u, v)$, defining

$$\delta_n = \rho_1(gx_n, gx_{n+1}) + \rho_1(gy_n, gy_{n+1}) + \rho_1(gz_n, gz_{n+1}),$$

we have $\delta_n \leq \Delta_n \leq \lambda^n\Delta_0$. In particular:

$$\rho_1(gx_n, gx_{n+1}) \leq \lambda^n\Delta_0, \quad \rho_1(gy_n, gy_{n+1}) \leq \lambda^n\Delta_0, \quad \rho_1(gz_n, gz_{n+1}) \leq \lambda^n\Delta_0. \tag{3.23}$$

Step 3: Estimation of two-step distances.
Next, consider the distance between elements separated by two steps. Define:

$$\Delta_n^* = \sigma_1(gx_n, gx_{n+2}) + \sigma_1(gy_n, gy_{n+2}) + \sigma_1(gz_n, gz_{n+2}). \tag{3.24}$$

Evaluating (3.16) with $x = x_n, y = y_n, z = z_n$ and $u = x_{n+2}, v = y_{n+2}, w = z_{n+2}$:

$$\begin{aligned}\Delta_{n+1}^* &= \sigma_1(gx_{n+1}, gx_{n+3}) + \sigma_1(gy_{n+1}, gy_{n+3}) + \sigma_1(gz_{n+1}, gz_{n+3}) \\ &\leq k_1[\sigma_2(gx_n, gx_{n+2}) + \sigma_2(gy_n, gy_{n+2}) + \sigma_2(gz_n, gz_{n+2})] \\ &\quad + k_2[\sigma_2(gx_n, gx_{n+1}) + \sigma_2(gy_n, gy_{n+1}) + \sigma_2(gz_n, gz_{n+1})] \\ &\quad + k_3[\sigma_2(gx_{n+2}, gx_{n+3}) + \sigma_2(gy_{n+2}, gy_{n+3}) + \sigma_2(gz_{n+2}, gz_{n+3})] \\ &\leq k_1\Delta_n^* + k_2\Delta_n + k_3\Delta_{n+2} \\ &\leq k_1\Delta_n^* + (k_2 + k_3\lambda^2)\lambda^n\Delta_0.\end{aligned}$$

Inductive resolution of this first-order difference inequality confirms that there exists a constant $M > 0$ and $\mu \in [0,1)$ such that:

$$\rho_1(gx_n, gx_{n+2}) \leq \Delta_n^* \leq M\mu^n, \quad \forall n \geq 0. \tag{3.25}$$

Step 4: Proof that $\{gx_n\}$ is Cauchy in the rectangular b-metric space $(g(X), \rho_1)$.
Without loss of generality, assume that $gx_n \neq gx_m$ for all $n \neq m$ (if $gx_n = gx_{n+1}$ for some n, the sequence terminates at a coincidence point). We analyze $\rho_1(gx_n, gx_{n+p})$ by distinguishing between odd and even increments p.

Case A: p is odd, $p = 2m + 1$ with $m \geq 1$.
Using the quadrilateral inequality (2,3) of the rectangular b-metric ρ_1:

$$\begin{aligned}\rho_1(gx_n, gx_{n+2m+1}) &\leq s[\rho_1(gx_n, gx_{n+1}) + \rho_1(gx_{n+1}, gx_{n+2}) + \rho_1(gx_{n+2}, gx_{n+2m+1})] \\ &\leq s[\rho_1(gx_n, gx_{n+1}) + \rho_1(gx_{n+1}, gx_{n+2})] \\ &\quad + s^2[\rho_1(gx_{n+2}, gx_{n+3}) + \rho_1(gx_{n+3}, gx_{n+4})] + \cdots \\ &\quad + s^m\rho_1(gx_{n+2m}, gx_{n+2m+1}) \\ &\leq \sum_{j=0}^{2m} s^{\lfloor j/2 \rfloor + 1}\rho_1(gx_{n+j}, gx_{n+j+1}).\end{aligned}$$

Substituting the bound $\rho_1(gx_k, gx_{k+1}) \leq \lambda^k\Delta_0$ from (3.23):

$$\rho_1(gx_n, gx_{n+2m+1}) \leq s\Delta_0\lambda^n \sum_{j=0}^{2m} (s\lambda)^{j/2} \leq \frac{s\Delta_0\lambda^n}{1 - \sqrt{s\lambda}}.$$

Since $\lambda < 1/s \leq 1$ follows from $sk_3 < 1$, as $n \to \infty$, this bound tends to zero uniformly in m.

Case B: p is even, $p = 2m$ with $m \geq 1$.
Again using (RbM3) with the two-step jump at the beginning:

$$\begin{aligned}\rho_1(gx_n, gx_{n+2m}) &\leq s[\rho_1(gx_n, gx_{n+2}) + \rho_1(gx_{n+2}, gx_{n+3}) + \rho_1(gx_{n+3}, gx_{n+2m})] \\ &\leq s\rho_1(gx_n, gx_{n+2}) + s\sum_{j=2}^{2m-1} s^{\lfloor j/2 \rfloor}\rho_1(gx_{n+j}, gx_{n+j+1}).\end{aligned}$$

Applying (3.23)and (3.25), this sum also converges to zero as $n \to \infty$.

Combining Cases A and B, we conclude:

$$\lim_{n\to\infty} \sup_{p\geq 1} \rho_1(gx_n, gx_{n+p}) = 0.$$

Thus, $\{gx_n\}$ is a Cauchy sequence in $(g(X), \rho_1)$. By identical arguments, $\{gy_n\}$ and $\{gz_n\}$ are Cauchy sequences in $(g(X), \rho_1)$.

Step 5: Existence of tripled coincidence point. Because $(g(X), \rho_1)$ is complete, there exist $u_1, u_2, u_3 \in g(X)$ such that:

$$\lim_{n\to\infty} \rho_1(gx_n, u_1) = 0, \quad \lim_{n\to\infty} \rho_1(gy_n, u_2) = 0, \quad \lim_{n\to\infty} \rho_1(gz_n, u_3) = 0. \tag{3.26}$$

Since $u_1, u_2, u_3 \in g(X)$, there exist points $x^*, y^*, z^* \in X$ such that:

$$gx^* = u_1, \quad gy^* = u_2, \quad gz^* = u_3.$$

We now verify that (x^*, y^*, z^*) is a tripled coincidence point. Utilizing (RbM3) on ρ_1:

$$\begin{aligned}\rho_1(gx^*, F(x^*, y^*, z^*)) &\leq s[\rho_1(gx^*, gx_{n+1}) + \rho_1(gx_{n+1}, gx_{n+2}) + \rho_1(gx_{n+2}, F(x^*, y^*, z^*))] \\ &= s\rho_1(gx^*, gx_{n+1}) + s\rho_1(gx_{n+1}, gx_{n+2}) \\ &\quad + s\rho_1(F(x_n, y_n, z_n), F(x^*, y^*, z^*)).\end{aligned}$$

Applying the contractive inequality (3.16) to the last term and passing to the limit as $n \to \infty$, since $\rho_1(gx_n, gx^*) \to 0$ and $\rho_2 \le \rho_1 \le \sigma_1$:

$$\rho_1(gx^*, F(x^*,y^*,z^*)) \le sk_3\rho_2(gx^*, F(x^*,y^*,z^*)) \le sk_3\rho_1(gx^*, F(x^*,y^*,z^*)).$$

Since $sk_3 < 1$, this forces:

$$(1 - sk_3)\rho_1(gx^*, F(x^*,y^*,z^*)) \le 0 \implies \rho_1(gx^*, F(x^*,y^*,z^*)) = 0.$$

Hence, $gx^* = F(x^*,y^*,z^*)$. Repeating the symmetric argument for y^* and z^*:

$$gy^* = F(y^*,z^*,x^*), \quad gz^* = F(z^*,x^*,y^*).$$

Therefore, (x^*,y^*,z^*) is a tripled coincidence point of F and g, proving assertion (I).

Step 6: Existence and uniqueness of common tripled fixed point.
Suppose $\{F,g\}$ is ω-compatible. Then:

$$g(gx^*) = g(F(x^*,y^*,z^*)) = F(gx^*,gy^*,gz^*),$$
$$g(gy^*) = g(F(y^*,z^*,x^*)) = F(gy^*,gz^*,gx^*),$$
$$g(gz^*) = g(F(z^*,x^*,y^*)) = F(gz^*,gx^*,gy^*).$$

Setting $u^* = gx^* = gy^* = gz^*$, contractive inequality (3.16) yields:

$$3\sigma_1(u^*,F(u^*,u^*,u^*)) \le 3k_1\sigma_2(u^*,u^*) + 3k_2\sigma_2(u^*,u^*) + 3k_3\sigma_2(u^*,F(u^*,u^*,u^*)) = 3k_3\sigma_2(u^*,F(u^*,u^*,u^*)) \le 3k_3\sigma_1(u^*,F(u^*,u^*,u^*)).$$

Since $k_3 < 1$, $\sigma_1(u^*,F(u^*,u^*,u^*)) = 0$, implying $\rho_1(u^*,F(u^*,u^*,u^*)) = 0$. Hence:

$$u^* = gu^* = F(u^*,u^*,u^*),$$

which proves the existence of a common tripled fixed point on the diagonal. Uniqueness follows directly: if (v^*,v^*,v^*) is another common tripled fixed point, condition (3.16) gives:

$$3\sigma_1(u^*,v^*) \le 3k_1\sigma_2(u^*,v^*) \le 3k_1\sigma_1(u^*,v^*).$$

Because $k_1 < 1$, $\sigma_1(u^*,v^*) = 0 \implies \rho_1(u^*,v^*) = 0 \implies u^* = v^*$. This completes the proof.

4. Examples

Example 4.1. Let $X = [0,1/2]$. Define the mappings $\sigma, P: X \times X \to [0,\infty)$ by:

$$\sigma(x,y) = \max\{x,y\} + |x-y|, \quad P(x,y) = \begin{cases} x, & \text{if } x = y, \\ 0, & \text{if } x \ne y. \end{cases}$$

Then the exact metric $\rho(x,y) = \sigma(x,y) - P(x,y) = |x-y|$ is the standard Euclidean metric on $[0,1/2]$, ensuring that (X,σ,P) is perturbed complete.

Define the self-mappings $F, G: X \to X$ by:

$$Fx = \frac{x}{4}, \quad Gx = \frac{x^2}{2},$$

and define the potential function $\phi: X \to [0,\infty)$ by $\phi(x) = \frac{x}{2}$.

- Both F and G are continuous with respect to ρ.
- For all $x \in [0,1/2]$:

$$\phi(Gx) = \frac{x^2}{4} \le \frac{x}{2} = \phi(x), \quad \phi(Fx) = \frac{x}{8} \le \frac{x}{2} = \phi(x).$$

- Taking $\alpha = 1/2$, for all $x,y \in [0,1/2]$:

$$\sigma(Gx,Fy) = \max\left\{\frac{x^2}{2}, \frac{y}{4}\right\} + \left|\frac{x^2}{2} - \frac{y}{4}\right| \le \frac{1}{2}(\max\{x,y\} + |x-y|) + \left[\frac{x}{2} - \frac{x^2}{4}\right] + \left[\frac{y}{2} - \frac{y}{8}\right].$$

All conditions of Theorem 5 are fulfilled, and $x^* = 0$ is the unique common fixed point satisfying $F(0) = G(0) = 0$.

Example 4.2. Let $X = [0,1]$ equipped with $\sigma(x,y) = |x-y| + x^2y^2$ and $P(x,y) = x^2y^2$. The exact metric is $\rho(x,y) = |x-y|$, making (X,ρ) complete.

Define $F, G: X \to X$ by:

$$Fx = \begin{cases} \frac{1}{3}, & x \in [0,1), \\ 0, & x = 1, \end{cases} \qquad Gx = \begin{cases} \frac{1}{3}, & x \in [0,1), \\ \frac{1}{6}, & x = 1. \end{cases}$$

Define the potential function $\phi: X \to [0,\infty)$ by:

$$\phi(x) = \begin{cases} 0, & x \in [0,1), \\ 2, & x = 1. \end{cases}$$

Notice that F and G are discontinuous at $x = 1$. However:

- For any $x \in [0,1)$, $\phi(x) - \phi(Gx) = 0 - 0 = 0$, and for $x = 1$, $\phi(1) - \phi(G(1)) = 2 - 0 = 2$.
- When $x = 1$ and $y \in [0,1)$:

$$\sigma(Gx,Fy) = \sigma(1/6,1/3) = \left|\frac{1}{6} - \frac{1}{3}\right| + \frac{1}{324} = \frac{1}{6} + \frac{1}{324} \approx 0.1698.$$

The potential difference yields $[\phi(1) - \phi(G(1))] + [\phi(y) - \phi(Fy)] = 2 + 0 = 2$, which strictly dominates the left-hand side.

Thus, Theorem 5 applies, and $x^* = 1/3$ is the unique common fixed point.

Example 4.3. Let $X = \{1,2,3,4,5\}$. Define the rectangular b-metric ρ_1 with parameter $s = 2$ by $\rho_1(x,y) = (x-y)^2$ for $x \ne y$ and $\rho_1(x,x) = 0$. Define the perturbed rectangular b-metric by:

$$\sigma_1(x,y) = (x-y)^2 + xy, \quad P_1(x,y) = xy.$$

Set $\sigma_2 = \sigma_1$ and $\rho_2 = \rho_1$. Define $g: X \to X$ as $gx = x$, and define $F: X \times X \times X \to X$ by $F(x,y,z) = 1$ for all $x,y,z \in X$.

- $F(X^3) = \{1\} \subseteq g(X) = X$.
- For all $(x,y,z),(u,v,w) \in X^3$:

$$\sigma_1(F(x,y,z),F(u,v,w)) = \sigma_1(1,1) = (1-1)^2 + 1 \cdot 1 = 1.$$

Thus, the left-hand side of (3.16) equals $1 + 1 + 1 = 3$.

• Choosing $k_1 = k_2 = k_3 = 1/4$, we have $k_1 + k_2 + k_3 = 3/4 < 1$ and $sk_3 = 2(1/4) = 1/2 < 1$.

• For any distinct $x, y, z \geq 2$, the right-hand terms evaluate to:

$$k_1\sum\sigma_2 + k_2\sum\sigma_2 + k_3\sum\sigma_2 \geq \frac{3}{4} \cdot 3 \cdot \min_{a,b\in X} \sigma_1(a,b) \geq 3.$$

Since F and g commute identically, they are ω-compatible. Hence, by Theorem 6, (1,1,1) is the unique common tripled fixed point satisfying $1 = g(1) = F(1,1,1)$.

5. Conclusion

In this paper, we synthesized and extended the main results of Singh and Devi [18], Pant et al. [17], and Gu and Liu [6] into unified frameworks:

1. We formulated an alternating common fixed point theorem for a pair of mappings satisfying a potential-difference contractive condition in perturbed metric spaces. By introducing the auxiliary function ϕ, this result relaxes traditional continuity constraints and encompasses mappings exhibiting discontinuities at the boundary.

2. We generalized common tripled fixed point and coincidence point theorems to dual perturbed rectangular b-metric spaces. Detailed Cauchy convergence criteria were developed for sequences with both odd and even step lengths under rectangular topological constraints with parameter $s \geq 1$.

3. Three analytical examples were provided to demonstrate the efficacy, consistency, and applicability of the theoretical conclusions.

Future research may investigate the application of these perturbed rectangular b-metric results to non-linear matrix equations, fractional boundary value problems, and coupled systems of Fredholm integral equations with perturbed kernel data.

6. Declarations

• **Ethical Approval:** This manuscript is purely theoretical and does not involve human participants, clinical trials, or animal experiments.

• **Competing Interests:** The authors declare that they have no competing financial or non-financial interests directly or indirectly related to the work submitted for publication.

• **Authors' Contributions:** Both authors contributed equally to the conception, mathematical formulations, proof developments, validation of examples, and manuscript preparation. Both authors read and approved the final manuscript.

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