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Compact Metric Space and Some Fixed-Point Theorems

Rama Shamrao Tarte[1] , Varsha D. Borgaonkar[2]
[1]*P.G. Department of Mathematics, N.E.S. Science College, Nanded, India - 431 602*
[2]*P.G. Department of Mathematics, N.E.S. Science College, Nanded, India - 431 602*

Abstract: The aim of this paper is to study some fixed point results in compact metric spaces. It discusses the existence and uniqueness of fixed point of a self-map on a metric space. In this paper we defined generalized contraction condition on the continuous self-mappings on compact metric spaces. Our result generalizes the results of Edelstein and Fisher.

Keywords: Compact Space, Metric Space Contraction, Fixed Point, Convergence sequence.

INTRODUCTION

The Banach fixed-point theorem is a very popular and useful theorem in Mathematics as well as in other subjects. Fixed point theory has an enormous number of applications in various fields of Mathematics, such as Differential Equations and Numerical Analysis. It has an important role in Mathematical Economics. First of all, Edelstein [4] has established the fixed point theorem for contractive type mapping on a compact metric space. Later, D. Bailey [5] has obtained the fixed point theorems for the mapping for which the sequence of iterates satisfies the contractive condition on a compact metric space. In 1978, Fisher [6] generalized the contractive condition and gave the fixed point theorem for the mapping satisfying that contractive condition.

In this paper, we generalized the contractive condition and the results of Edelstein and Fisher.

BASIC DEFINITION & EXAMPLES

Definition 2.1: (Metric Space): Let X be a non-empty set. A metric on X is a real-valued function $d: X \times \rightarrow R$ which satisfies the following conditions:

i. $d(x,y) \geq 0, \quad \forall\, x,y \in X,$

ii. $d(x,y) = 0$ if and only if $x = y \quad \forall\, x,y \in X,$

iii. $d(x,y) = d(y,x), \quad \forall\, x,y \in X$ (Symmetry),

iv. $d(x,y) \leq d(x,z) + d(z,y), \quad \forall x,y,z \in X$ (Triangle inequality)

Definition 2.2: (Open Set): A subset G of a metric space (X,d) is said to be open in X, with respect to the metric d, if G is a neighbourhood of each of its points. In other words, if for each $a \in G$, there is an $r > 0$, such that, $s_r(a) \subseteq G$.

Example 2.1: On the real line with the usual metric, the singleton set is open.

Definitions 2.3: (Open Cover): Let (X,d) be a metric space. A family of subsets $\{A_\propto\}$ in X is called a cover of any subset A of X if $A \subseteq \cup_{\propto \in \wedge} A_\propto$, $\wedge$ is any non-empty index set. If each $A_\propto$, $\propto \in \wedge$, is an open set in X, then the cover $\{A_\propto\}$ is called an open cover of A.

Definition 2.4: (Open Subcover): A subfamily of the family $\{A_\propto\}$ which itself is an open cover, is called an open subcover of A.

Definition 2.5: (Finite Subcover): If the number of members in the subfamily is finite, it is called a finite subcover of A.

Definition 2.6: (Compact Metric Space): A subset A of a metric space (X,d) is said to be compact if every open cover of A admits of a finite subcover, i.e., for each family of open subsets $\{G_\propto\}$ of X, for which $A \subseteq \bigcup_{\propto\in\wedge} G_\propto$, there exists a finite subfamily say $\{G_{\propto 1}, G_{\propto 2}, \dots, G_{\propto n}\}$ such that $A \subseteq \bigcup_{i=1}^{n} G_{\propto i}$.

Example 2.2: Any closed interval with the usual metric is compact.

Definition 2.7: Let (X,d) be a metric space. A mapping $T: X \to X$ is called

i.Contraction: If $d(Tx,Ty) \leq \alpha d(x,y)$, $\forall\, x,y \in X$, $\alpha \in [0,1)$.

ii.Contractive: If $d(Tx,Ty) < f(x,y)$, $\forall\, x,y \in X$ with $x \neq y$.

Theorem 2.1: (Banach Contraction Theorem): A contraction T on a complete metric space (X,d) has a unique fixed point. If z is the fixed point of the mapping T, then for any $x \in X$, the sequence $\{T^n x\}$ of iterates converges to z.

Definition 2.8: (Fixed point): Let (X,d) be a metric space, then a point is said to be a fixed point of the self-map $f : X \to X$ if $f(x) = x$.

Example2.3: Let $f: R \to R$ defined as, $f(x) = x^3 \;\forall\; x \in R$, then $x = 0$, $x = 1$ and $x = -1$, are the fixed points of the mapping f.

1. MAIN RESULTS

Theorem 3.1: Let T be a continuous self-map of a compact metric space satisfying the conditions,

$$d(Tx,Ty) < \alpha d(x,y) + \beta[d(y,Tx) + d(x,Ty)] + \gamma[d(y,Ty) + d(x,Tx)] \quad \dots (3.1)$$

$\forall x \neq y \in X$, where, $0 < \alpha < 1$, $0 < \beta < 1$ and $0 < \gamma < 1$ are such that $\alpha + \beta + \gamma < \frac{1}{2}$.

Then T has a unique fixed point in X.

Proof: We define $f : X \to [0,\infty]$ as, $f(x) = d(x, T(x))$ $\forall x \in X$.

Case I: If $x = T(x)$ for some $x \in X$ then, x is a fixed point of X.

Case II: Suppose $x \in X$ is such that $T(x) \neq x$.

$$\therefore \quad f(T(x)) = f\,(T(x), T^2(x)).$$

By applying (3.1), we get,

$f(T(x)) = f\,(T(x), T^2(x))$

$< \alpha.d(x,Tx) + \beta\left[d\left(T(x),T(x)\right) + d\left(x,T^2(x)\right)\right] + \gamma\left[d\left(T(x),T^2(x)\right) + d\left(x,T(x)\right)\right]$

$< \alpha.d\left(x,T(x)\right) + \beta\, d\left(x,T(x)\right) + \beta d\left(Tx,T^2(x)\right) + \gamma d\left(T(x),T^2(x)\right) + \gamma d\left(x,T(x)\right)$

$f\left(T(x)\right) < (\alpha+\beta+\gamma)\, d\left(x,T(x)\right) + (\beta+\gamma)\, d\left(T(x),T^2(x)\right)$

$$f\left(T(x)\right) < (\propto + \beta + \gamma)\, f(x) + (\beta+\gamma)\, f\left(T(x)\right) \quad (3.2)$$

Suppose, $f(x) \leq f\left(T(x)\right)$, then, we have, $f\left(T(x)\right) < \propto + 2\beta + 2\gamma\, f\left(T(x)\right)$.

This is not possible, since , $\propto + \beta + \gamma < \frac{1}{2}$. Thus, we have only one possibility, that is

$$f(x) > f\left(T(x)\right) \qquad i.e\; f\left(T(x)\right) < f(x) \qquad \forall x \neq T(x). \quad (3.3)$$

As, T and d are continuous mappings, f is also a continuous mapping on X.
As X is compact, f attains its minimum in X. Suppose, f attains its minimum at $z \in X$.

$$\therefore f(z) = \min\{f(x) : x \in X\} \qquad (3.4)$$

Now, we will claim that, z is fixed point of T i.e $T(z) = z$. If it is not so, we have by (3.3)

$$f(T(z)) < f(z)$$

This contradicts (3.4), therefore, we have, $z = T(z)$. Hence, z is a fixed point of T in X.

Uniqueness: Suppose T has two fixed points, say z_0 and z in X.
Therefore we have $T(z_0) = z_0$ and $T(z) = z$.
Consider,

$$d(z, z_0) = d\big(T(z), T(z_0)\big)$$
$$d(z, z_0) < \alpha\, d(z, z_0) + \beta\left[d\left(z_{0,} T(z)\right) + d\big(z, T(z_0)\big)\right] + \gamma\big[d(z_0, z_0) + d\big(z, T(z)\big)\big]$$
$$d(z, z_0) < \alpha . d(z, z_0) + \beta\big[d\big(z_{0,} z\big) + d(z, z_0)\big] + \gamma[d(z_0, z_0) + d(z, z)]$$
$$d(z, z_0) < (\alpha + 2\beta)\, d(z, z_0)$$

This is not possible since, $\alpha + 2\beta < 1$. Hence, T has only one fixed point z in X.

Remark 3.1: If we put $\alpha = 1$ $\beta = \gamma = 0$, we obtain Edelstein's Theorem.
Remark 3.2: If we put $\alpha = \gamma = 0$ & $\beta = \frac{1}{2}$, we obtain Fisher's Theorem.

Theorem 3.2: Let T be a self-map on a compact metric space (X, d), satisfying (3.1). Then the Sequence $\{T^n(x)\}$ of iterates converges to the unique fixed point

Proof: By theorem (3.1), T has a unique fixed point, say, z in X. We define,

$$d_n = d(T^n(x), z) \qquad \forall\ x \neq z \in X, \qquad \text{for each } n = 0,1,2\cdots$$

Case I: If $d_n = 0$ for some n, $T^n(x) = 0$ $\forall\, m \geq n$. Hence, the sequence $\{T^n(x)\}$ converges to z.
Case II: If $d_n \neq 0$ $\forall\, n$ then,

$$d_{n+1} = d\big(T^{n+1}(x), T^{n+1}(z)\big)$$
$$d_{n+1} = d\big(T(T^n x), T(T^n z)\big)$$
$$d_{n+1} < \alpha . d\big(T^n(x), T^n(z)\big) + \beta\big[d\big(T^n(z), T^{n+1}(x)\big) + d\big(T^n(x), T^{n+1}(z)\big)\big] +$$
$$\gamma\big[d\big(T^n(z), T^{n+1}(z)\big) + d\big(T^n(x), T^{n+1}(x)\big)\big]$$
$$d_{n+1} < \alpha . d\,(T^n(x), z) + \beta\big[d\big(z, T^{n+1}(x)\big) + d(T^n(x), z)\big] +$$
$$\gamma\big[d(z, z) + d(T^n(x), z) + d\big(z, T^{n+1}(x)\big)\big]$$
$$d_{n+1} < \alpha . d_n + \beta . d_{n+1} + \beta d_n + \gamma d_n + \gamma d_{n+1}$$
$$d_{n+1} < (\alpha + \beta + \gamma) d_n + (\beta + \gamma) d_{n+1}$$
$$[1 - (\beta + \gamma)]\ d_{n+1} < (\alpha + \beta + \gamma) d_n$$
$$d_{n+1} < \left[\frac{\alpha + \beta + \gamma}{1 - (\beta + \gamma)}\right] . d_n$$
$$d_{n+1} < d_n \qquad \because\ \alpha + \beta + \gamma < \frac{1}{2}$$

Hence, $\{d_n\}$ is a strictly decreasing sequence of possible real numbers. Thu, it converges to a real number $r \geq 0$ (say), where, $r = \inf\{d_n \mid n \in N\}$. As, X is compact, the sequence $\{T^n(x)\}$ has a subsequence $\{T^{n_k}(x)\}$ which converges to $z_0 \in X$.
As,T is continuous mapping, $T^{n_{k+1}}(x) = T\big(T^{n_k}(x)\big) \rightarrow T(z_0)$.
By using the continuity of metric d, as $k \rightarrow \infty$, we yield

$$\lim_{k\to\infty} d_{n_k} = \lim_{k\to\infty} d(T^{n_k}(x), z) = d\big(z_{0,} z\big) \qquad (3.4)$$

By (3.3) and (3.4), we have,

$$\lim_{k\to\infty} d_{n_k} = d(z_{0,}z) = r \tag{3.5}$$

Moreover, $\lim_{k\to\infty} d_{n_k+1} = \lim_{k\to\infty} d(T^{n_k+1}x, z) = d(T(z_0), z)$ (3.6)

Hence, (3.5) and (3.6) gives $\lim_{k\to\infty} d_{nk} = \lim_{k\to\infty} d_{n_{k+1}} = r$

$$\therefore d(z_{0,}z) = d(T(z_0), z) = r \tag{3.7}$$

Now, we claim that, $r = 0$. Suppose, $r \neq 0$. We have,

$d(z_0, z) = d(T(z_0), z)$

$d(z_0, z) = d(T(z_0), T(z))$

$d(z_0, z) < \alpha.d(z_0, z) + \beta[d(z, T(z_0)) + d(z_0, T(z))] + \gamma[d(z, T(z)) + d(z_0, T(z_0))]$

$d(z_0, z) < \alpha.d\,(z_{0,}z) + \beta d(z, T(z_0)) + \beta d(z_{0,}z) + \gamma d\left(z_{0,}T(z_0)\right)$

$d(z, Tz_0) < \alpha.d\,(z_{0,}z) + \beta d(z, T(z_0)) + \beta d(z_0, z) + \gamma d(z_0, z) + \gamma d(z, Tz_0)$

$[1 - (\beta + \gamma))d(z, T(z_0)) < (\alpha + \beta + \gamma)\, d(z_1, z_0)$

$d(z, T(z_0)) < \left[\dfrac{\alpha + \beta + \gamma}{1 - (\beta + \gamma)}\right] d(z, z_0)$

This is a contradiction to (3.7), since, $\alpha + \beta + \gamma < \frac{1}{2}$. Hence, by (3.3), we have, $r = 0$.

$$\therefore \lim_{n\to\infty} d_n = r = 0 \Rightarrow \lim_{n\to\infty} d(T^n(x), z) = 0.$$

The sequence $\{T^n(x)\}$ converges to the unique fixed point z of mapping T.

2. APPLICATIONS

Example 4.1: Let $X = [0,1]$ be a compact metric space with respect to the usual metric d, defined as, $d(x, y) = |x - y|\ \forall\, x, y \in X$. Define, $T: X \to X$ as, $T(x) = \frac{x+1}{4}\ \ \forall\, x \in X$.

Clearly, T is continuous mapping on X.

Choose, $\alpha = \frac{1}{4}$, $\beta = \frac{1}{8}$ & $\gamma = \frac{1}{16}$, such that, $\alpha + \beta + \gamma = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} < \frac{1}{2}$. Moreover,

$$\begin{aligned} d(Tx, Ty) &= \left|\tfrac{x+1}{4} - \tfrac{y+1}{4}\right| \\ &= \tfrac{1}{4}\,|(x+1) - (y+1)| \\ &= \tfrac{1}{4}|(x - y)| \\ &= \alpha.d(x, y) \end{aligned}$$

$$\therefore\ d(Tx, Ty) < \alpha.d\,(x, y) + \beta[d(y, Tx) + d(x, Ty)] + \gamma[d(y, Ty) + d(x, Ty)]$$

Hence, all the conditions of Theorem 3.1 are satisfied. Therefore, by Theorem (3.1), T has a unique fixed point in $X = [0, 1]$. The unique fixed points $x = \frac{1}{3}$.

3. CONCLUSIONS

In this paper, we introduce a contraction-type condition for continuous mappings on compact metric spaces. We establish the existence of a unique fixed point for mappings that satisfy this contraction condition, thereby generalizing certain well-known fixed point theorems applicable to compact metric spaces. Our findings demonstrate that the conclusions of the contraction mapping principle remain valid when considering compact spaces instead of complete spaces. However, replacing contraction mappings with merely contractive mappings does not uphold the principle's conclusions. Throughout this study, we focus exclusively on continuous mappings.

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