Domination Polynomial of P2 C_4 ?

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ABSTRACT

In this paper, the domination polynomial of the Cartesian product the path p_2 and the cycle c_4, has been discussed. Also a direction to find out the dominating polynomial the Cartesian product of the general paths and cycles has been discussed.

Key words: domination set, domination number, domination polynomials.

Mathematics subject classification: 05C31, 05C69, 05C76

1.INTRODUCTION

Let G = (V,E) be a simple graph with vertex set V and edge set E. A set D $\subseteq$ V is a dominating set of G, if every vertex is V-D is adjacent to atleast one vertex in D. The domination number is the minimum cardinality of a dominating set in G. For detailed treatment of dominations, one can refer [1]. The domination polynomials of graphs, and product of graphs is an interesting area of research in graph theory. SaeidAlikhani, and Yee-hock Peng [2] have studied dominating sets and domination polynomials of cycles.TomerKotek, James Preen, Peter Tittmann [3], studied domination polynomials of graph products. In particular, they studied the domination polynomials of products of complete graphs, with complete graphs, and complete graphs with paths. Abdul Jalilm, M.Khalaf and Salibshayyalkahat, studied [4] dominating sets and domination of complete graphs with missing edges. A.Vijayan and K.Lal Gipson [5], studied domination polynomials of square of cycles. As in any branch of graph theory, there are many open problems which deserve attention, in the usual of domination polynomials.

In this paper, we give an introduction part to study the domination polynomial of the Cartesian product of paths and cycles.

2. PRE-REQUISITIES

Let G=(V,E) be a simple graph. A vertex subset D of Vof G is a dominating set in G, if for each vertex v$\in$V/D is adjacent to atleast one vertex of D. The domination number γ (G) is the cardinality of the smallest domination set.

The domination polynomial D (G,x) is defined as

$$D(G,x)=\sum_{i=1}^{|V|} d_i(G)x^i$$

Where $d_i(G)$ is the number of dominating sets of size 'i' in G.

Example 2.1

Consider the following graph

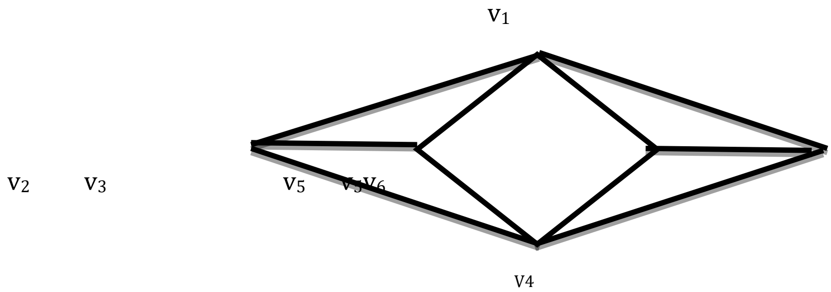


Fig 2.1 A Simple Graph

It can be found that the number of dominating sets of the above graph of sizes 1,2,3,4,5,6 are respectively 0,13,10,6,3,1

Thus, the domination polynomial of the graph given in fig 2.1 is

$D(G,x) = 0x^1+13x^2+10x^3+6x^4+3x^5+1x^6$

SaeidAlikhani and yee-Hock peng(2010) have studied the domination polynomial of paths graphs and obtained

$D(P_{n+1}, \lambda) = x[D(p_n, \lambda) + D(p_{n-1}, \lambda) + D(p_{n-2}, \lambda)]$

Where $D(p_0,x) = 1$; $D(p_1,x) = x$; $D(p_2,x) = x^2+2x$;

They also studied domination polynomial of cycles.[ ] paper 2 Ref.

Abdul Jalil.M.Khalaf and SachinShaygaKahat (2014) have studied the domination polynomial of complete graphs and obtained the recurrence formula

$$D(k_n, x) = D(k_{n-1}, x) + xD(k_{n-1}, x) + x \text{ for } n \geq 3.$$

In the year 2013 "TomerKotek" and others has studied domination polynomial of graph products. In particular, they studied the domination polynomial of products with complete graphs.

In this paper we study the domination polynomial of Cartesian product of path graphs and cycle graphs.

3. DOMINATION POLYNOMIAL OF CARTESIAN PRODUCT OF CYCLES AND PATHS

Let G and H are two simple graphs. The Cartesian product of G and H is a graph with vertex set $\{u_i v_j / u_i \in V(G), v_j \in V(H)\}$ and two vertices (u_i, v_j) and (u_k, v_l) are adjacent if and only if either $u_i$is adjacent to u_k and $v_j = v_l$ or $u_i = u_k$ and v_j is adjacent to v_l.

The following are some simple observations

1. Let G be a graph of order $'n'$. Let $D \subset V$ where $|D| = 1$ and for $u \in D$,degree of v is n-1. Then D is a dominating set.

2. In cycle C_n, with n vertices if $D \subset V$ where $|D| = r$ is a dominating set, then any set D' with $|D'| > r$ is also a dominating set.

These observations provide the following lemmas

Lemma 3.1

For any graph G $d_r(G) \leq nC_r$

Lemma 3.2

For any cycle C_n,$d_{n-2}(C_n) = nC_{n-2}$

Lemma 3.3

i)$d_{n-1}(C_n) = nC_{n-1}$
ii)$d_n(C_n) = nC_n = n$
iii)$d_{n-3}(C_n) < nC_{n-3}$

Proof

The proof of the first part follows from the fact that, every $D \subset V(C_n)$ is a dominating set, if $|D| = n - 2$.

The proof of (ii),(iii)follows from the fact any subset D'ofC_n,withatleast more than one element is also a dominating set

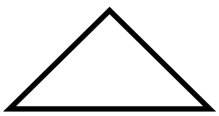
$P_2 \square C_3$ = ({$w_{ij}=(u_i,v_j)$/ i=1,2, j=1,2,3}) ; {$w_{ij}w_{kl}$/either i=k or j=l})

The vertex w_{ij} is said to be contributed by u_i and u_j.

The graph of $P_2 \square C_3$ is given in the fig 3.3

u_1 u_2 v_1

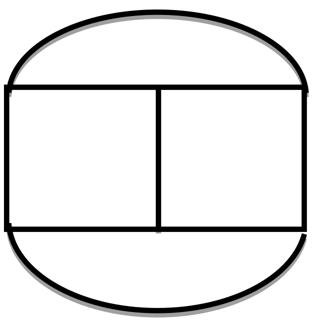
Fig 3.1 P_2



v_2 v_3

Fig 3.2 C_3

$w_{11}w_{12}w_{13}$



w_{21} w_{22} w_{23}

Fig 3.3 $P_2 \square C_3$

we know that $\deg(w_{ij}) = 3$, and $|V(P_2 \square C_3)| = 6$. Hence, there will not be any singleton set as a dominating set for $P_2 \square C_3$, because if $\{w_{ij}\}$ is a dominating set in $P_2 \square C_3$, there $d(w_{ij})$ must be 5,This is not true. Hence $d_1(P_2 \square C_3) = 0$. Now the vertex $w_{ij} \sim w_{ik}$, and also $w_{ij} \sim w_{lj}$, and also the number of vertices which differ in both subscripts of w_{ij} is only two, and hence for every w_{ij}, the number of non- adjacent vertices is 2. Therefore the vertex w_{ij} together with any one of the non-adjacent vertex form a dominating set. The number of dominating sets obtained in this way is 2. This is possible for each w_{ij} for fixed i. Hence total number of dominating sets obtained in this way is 6. Also for every vertex w_{ij}, the set$\{(w_{ij}, w_{kl}) / k=i+1 \text{ or } j=l\}$is also a dominating set. The number of such dominating sets is 3. Hence, there are 9 dominating sets of size 2 in $P_2 \square C_3$. Hence $d_2(P_2 \square C_3) = 9$. The following table gives all the nine dominating sets.

S.No	Dominating set	Distance
1.	w_{11}, w_{22}	2
2.	w_{11}, w_{23}	2
3.	w_{12}, w_{21}	2
4.	w_{12}, w_{23}	2
5.	w_{13}, w_{21}	2
6.	w_{13}, w_{22}	2
7.	w_{11}, w_{21}	1
8.	w_{12}, w_{22}	1
9.	w_{13}, w_{23}	1

It can be observed that $\{w_{ij}, w_{kl}\}$ form a dominating set if $i \neq k$ and $j \neq l$ or if $i \neq k$ and $j \neq l$. ie. The dominating set is obtained when w's are contributed by one vertex from path, and one vertex from cycle, or both the vertices from the path. Therefore,if we consider any three vertices w_{ij},then there should be atleast one vertex contribute from path,and one vertex contribute from cycle.Hence any three vertices form a dominating set. Therefore $d_3(P_2 \square C_3) = 6C_3$ and hence $d_4(P_2 \square C_3) = 6C_4$, $d_5(P_2 \square C_3) = 6C_5$; $d_6(P_2 \square C_3) = 6C_6$.

Hence $D(P_2 \square C_3, x) = 0x^1 + 9x^2 + 6C_3x^3 + 6C_4x^4 + 6C_5x^5 + 6C_6x^6$.

$= 0x^1 + 9x^2 + 20x^3 + 15x^4 + 6x^5 + 6x^6$.

Hence the lemma.

In the above proof , we consider the dominating set of consisting of vertices at distance 2, and also another set of vertices at distance 1. It is not always easy to consider the vertices at given distance. Hence, we transform the graphical structure into an another structure by placing the vertices which are at same distance at same level from any one of the structure at entry level. We assume that the vertex at entry level on a vertex at level 0. Similarly the vertices at distance ‘i’ are taken as vertices at level ‘i’ in the structure.

As an illustration, the graph of $P_2\square C_4$ and its distance based structure are given in the following diagram

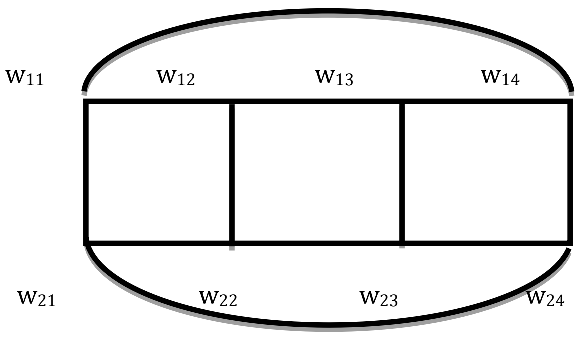


Fig 3.4 $P_2\square C_4$

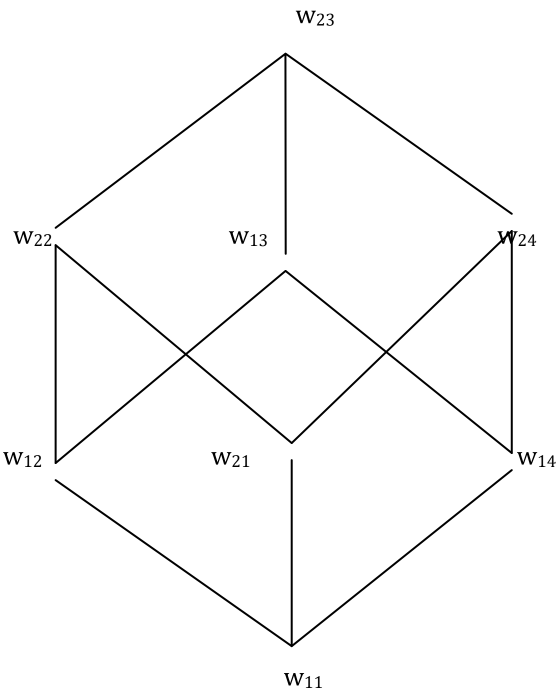


Fig 3.5 Distance based structure of $P_2\square C_4$

In the distance based structure, the vertex at level 0 is a root, and vertex at last level is top of the structure. It can be easily observed that no single vertex form a dominating set. Hence $d_1(P_2\square C_4) = 0$

Lemma 3.4

$d_2(P_2\square C_4) = 4$; $d_3(P_2\square C_4) = 24$

Proof:

It can also be found that only the vertices at root and top will form a dominating set in $P_2 \square C_4$. There are four such combinations in the set of 8 vertices. Therefore there are 4 dominating sets of size 2. Hence $d_2(P_2 \square C_4) = 4$.

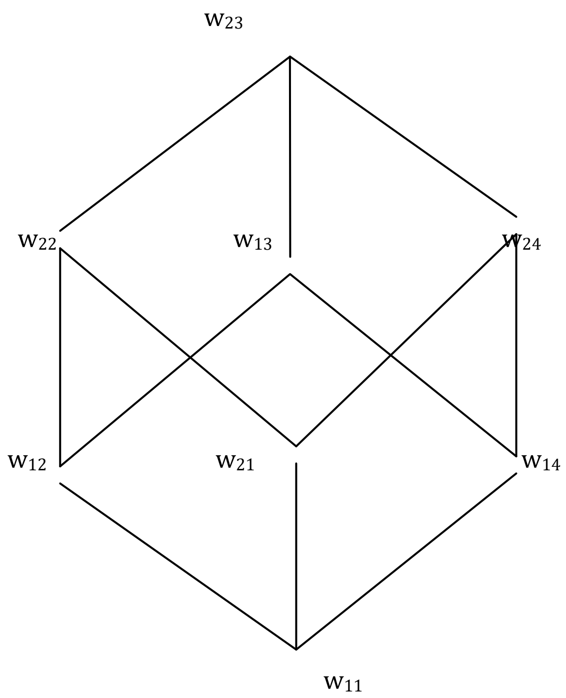
Any dominating set of two elements can also be made as a dominating set of three elements, four elements, five elements and soon …… The following is a list of 2- vertices that form a dominating set.

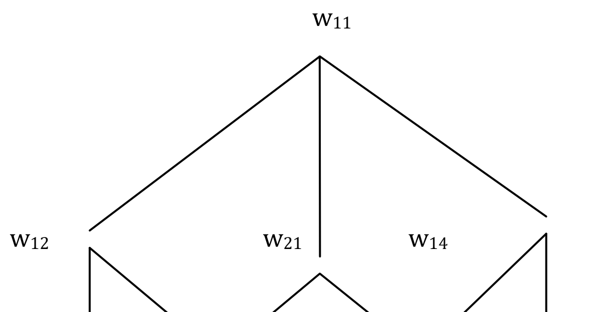
i) w_{11}, w_{23}
ii) w_{12}, w_{24}
iii) w_{13}, w_{21}
iv) w_{14}, w_{22}

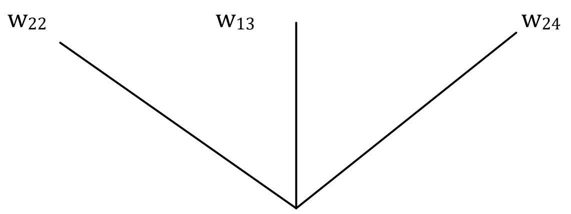
Inclusion of any one vertex in any one of the lists give us a dominating set of size 3. For each list there are 6 possible vertices. Hence $d_3(P_2 \square C_4) = 24$.

Observation 3.5

The graph obtained replacing top to root or root to top will give same structure providing the same dominating sets.







The set of four elements which does not form the dominating set in $P_2 \square C_4$ given in the above diagram are

1. $w_{11}, w_{12}, w_{21}, w_{14}$
2. $w_{23}, w_{22}, w_{13}, w_{24}$
3. $w_{11}, w_{14}, w_{13}, w_{24}$
4. $w_{11}, w_{21}, w_{22}, w_{24}$
5. $w_{11}, w_{12}, w_{22}, w_{13}$
6. $w_{23}, w_{22}, w_{12}, w_{21}$
7. $w_{23}, w_{13}, w_{12}, w_{14}$
8. $w_{23}, w_{24}, w_{21}, w_{14}$

From the above observations, we get

Lemma 3.5

There are 4 distance based structure for $P_2 \square C_4$,with different pair of root and top.

Lemma 3.6

If D is a dominating set in a distance based structure of $P_2 \square C_4$.

with root u and top v, if and only if D is also a dominating set in distance based structure of $P_2 \square C_4$, with root v and top u.

From the above lemma we get

Lemma 3.7

$d_4(P_2 \square C_4)=38$

Lemma 3.8

There are eight 4-sets of vertices in the distance based structure of $P_2 \square C_4$ which are not dominating sets.

Theorem 3.7

The dominating polynomial of $P_2 \square C_4$ is

$D(P_2 \square C_4, x) = 0x^1+4x^2+24x^3+38x^4+56x^5+28x^6+8x^7+x^8$

4. CONCLUSION

In this paper, the domination polynomial of $p_2 \square c_4$ has been discussed. It gives a way to find the domination polynomial of $p_n \square c_m$.

5. REFERENCES

[1] Abdul Jalilm, M. Khalaf and Salibshayyalkahat, Dominating sets and Domination polynomial of complete graphs with missing edges, Journal of Kufa for mathematics and computer, vol 2.No.1, May 2014, pp 64-68

[2] T.W. Haynes, S.T. Hedetniemi, P.J. Slater fundamentals of Domination in graphs, Mareeldekker. New yark, 1998.

[3] SaeidAlikhani, and Yee-hock Peng, Electronic notes, arxiv:0905.3268v1 [math.co] 20, may.2009

[4] TomerKotek, James Preen, Peter Tittmann, Domination polynomials of graph products,arxiv:1305.1475v2 [math.co] 23 Dec 2013