

Existence and Local Stability Results for First-Order Nonlinear Random Differential Equations via Random Fixed-Point Theorems

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Abstract: The study of random differential equations plays a crucial role in modeling dynamic systems influenced by uncertainty. Classical fixed point theorems have been extended to random settings to analyze the behavior of such systems. In this paper, we investigate the existence and local stability (attractivity) of solutions for a class of nonlinear first-order random differential equations defined on unbounded intervals of the real line. Using the framework of random fixed point theory in separable Banach spaces, we establish sufficient conditions that guarantee the existence of random solutions under appropriate measurability, continuity, and boundedness assumptions. We further demonstrate that these random solutions are locally attractive when certain contraction-type conditions are satisfied. The analysis unifies and generalizes several known results in the literature related to deterministic and random operator equations. The findings presented here provide a foundation for studying higher-order and perturbed random differential equations and may be applied to a wide range of problems in applied mathematics and engineering.

Keywords: Random differential equation; Random operator; Random fixed point; Compactness; Existence and attractivity; Nonlinear system.

1. Introduction

The class of first-order nonlinear equations plays a significant role in many branches of applied science and engineering, including electromagnetic circuit analysis, control system design, vibration and stability analysis in mechanical systems, and beam bending in structural engineering. In many practical problems, uncertainty and randomness naturally occur in the parameters, which leads to the study of random differential equations. Random fixed point theory provides a useful framework for investigating the existence and stability of solutions under such random influences. In this work, a first-order nonlinear random differential equation is considered to establish existence and local attractivity results on unbounded intervals of the real line. The existence results are obtained using well-known random fixed point theorems under suitable assumptions of measurability, continuity, and compactness.

The problem under consideration is formulated as:

$$x'(t, \omega) = f(t, x(t, \omega), \omega) + g(t, x(t, \omega), \omega), \quad x(0, \omega) = 0, \quad \forall \omega \in \Omega, \quad (1)$$

where $f, g: \mathbb{R}^+ \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ are measurable functions and $(\Omega, \mathcal{A})$ denotes a measurable space with a $\mathcal{A}$. This study focuses on establishing sufficient conditions under which the above random differential equation admits at least one random solution that is locally attractive on $\mathbb{R}^+$. The framework developed here generalizes and extends several existing results on random integral and differential equations based on fixed point theorems.

2. Preliminaries

In this section, we recall some essential definitions and concepts which are used throughout this paper. Let X be a separable Banach space with norm $\|\cdot\|$, and let $(\Omega, \mathcal{A})$ be a measurable space where Ω is a non-empty abstract set and $\mathcal{A}$ is a σ -algebra of subsets of Ω .

Definition 1.1 : [7] A mapping $T: \Omega \times X \rightarrow X$ is called a random operator if, for each $x \in X$, the mapping $\omega \mapsto T(\omega, x)$ is measurable. Such a random operator is denoted by $T(\omega)x$

Definition 1.2: Random Fixed Point [7] A random variable $\mu: \Omega \rightarrow X$ is said to be a random fixed point of a random operator $T(\omega): \Omega \times X \rightarrow X$ if

$$T(\omega)\mu(\omega) = \mu(\omega), \quad \text{for every } \omega \in \Omega.$$

Definition 1.3: Continuous Random Operator [7] A random operator $T : \Omega \times X \rightarrow X$ is said to be continuous at $x_0 \in X$ if $\|x_n - x_0\| \rightarrow 0$ implies $\|T(\omega)x_n - T(\omega)x_0\| \rightarrow 0$, for all sequences $\{x_n\} \subset X$.

Definition 1.4: Totally Bounded Random Operator [7] A random operator $T : \Omega \times X \rightarrow X$ is said to be totally bounded if, for every bounded subset $D \subset X$, the image $T(\omega)D$ is a totally bounded subset of X for each $\omega \in \Omega$.

Definition 1.5: Locally Attractive Random Solution [3] The random equation

$T(\omega)x = x$ is said to have a locally attractive random solution on $\mathbb{R}^+$ if there exists a closed ball $B_r(x_1) \subset C_B(\mathbb{R}^+, \mathbb{R})$ and for all random solutions $x(t, \omega), y(t, \omega) \in B_r(x_1) \cap C$,

$$\lim_{t \rightarrow \infty} (x(t, \omega) - y(t, \omega)) = 0, \quad \forall \omega \in \Omega.$$

If this convergence is uniform with respect to $C \cap B_r(x_1)$, i.e., for every $\varepsilon > 0$, there exists $T > 0$ such that

$$|x(t, \omega) - y(t, \omega)| \leq \varepsilon, \quad \forall t \geq T, \omega \in \Omega,$$

then the solution is said to be uniformly locally attractive on $\mathbb{R}^+$.

Definition 1.6: (Random Carathéodory Function) A function $\beta : \mathbb{R}^+ \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ is said to be a random Carathéodory function if:

1. For each $x \in \mathbb{R}$, the map $t \mapsto \beta(t, x, \omega)$ is measurable for all $\omega \in \Omega$;
2. For almost every $t \in \mathbb{R}^+$, the function $x \mapsto \beta(t, x, \omega)$ is continuous.

Definition 1.7: (Random L^1 -Carathéodory Function) [1] A random Carathéodory function $\beta : \mathbb{R}^+ \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ is called a random L^1 -Carathéodory function if, for each real number $r > 0$, there exists a measurable bounded function $q_r \in L^1(\mathbb{R}^+, \mathbb{R})$ such that

$$|\beta(t, x(t, \omega), \omega)| \leq q_r(t, \omega), \quad \forall t \in \mathbb{R}^+, |x| < r.$$

These definitions will be used to establish the main results in the following sections.

3. Existence Theory

In this section, we establish the main existence and local stability results for the nonlinear first-order random differential equation introduced earlier. The analysis is based on a random version of Schaffer's fixed point theorem.

Theorem 3.1 [1] Let X be a separable Banach space, and let $T(\omega), S(\omega) : X \rightarrow X$ be two random operators satisfying, for each $\omega \in \Omega$:

- i. $S(\omega)$ is a nonlinear contraction;
- ii. $T(\omega)$ is completely continuous;
- iii. The set $\mu = \{x \in X : S(\omega)x + T(\omega)x = \alpha(\omega)x\}$ is bounded for any function $\alpha(\omega) > 1$.

Then the random operator equation $S(\omega)x + T(\omega)x = x$ has at least one random solution.

Hypotheses:

The first-order nonlinear random differential equation

$$x'(t, \omega) = f(t, x(t, \omega), \omega) + g(t, x(t, \omega), \omega), \quad x(0, \omega) = 0, \quad (2)$$

is studied under the following assumptions:

(H0) The functions $f, g : \mathbb{R}^+ \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ are measurable for every $t \in \mathbb{R}^+$.

(H1) The function $\mathcal{G} : \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}$ defined by $\mathcal{G}(t, \omega) := \int_0^t q_r(s, \omega) ds$ is bounded with norm $\|q_r\|_{L^1}$ on $\mathbb{R}^+$ and vanishes at infinity, i.e., $\lim_{t \rightarrow \infty} \mathcal{G}(t, \omega) = 0$.

(H2) The function $g : \mathbb{R}^+ \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ is a random L^1 -Carathéodory function.

(H3) The function $f : \mathbb{R}^+ \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ is continuous, and there exists a bounded function

$\alpha(t, \omega) : \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}^+$ such that

$$|f(s, x(s, \omega), \omega) - f(s, y(s, \omega), \omega)| \leq \alpha(t, \omega) |x(t, \omega) - y(t, \omega)| \quad (3)$$

for every $t \in \mathbb{R}^+$. Furthermore, $|\alpha(t)| < 1$ and $\lim_{t \rightarrow \infty} \alpha(t) = 0$.

(H4) The function $f : \mathbb{R}^+ \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ is bounded and there exists a function

$$\alpha_k(t, \omega) = \int_0^t \alpha(s, \omega) ds, \quad \omega \in \Omega.$$

If hypotheses (H3) and (H4) hold, then there exists a constant $M_1 > 0$ such that

$$M_1 = \sup \{\mathcal{G}(t, \omega) : t \in \mathbb{R}^+, \omega \in \Omega\}.$$

Lemma 3.1 : Let $f, g : \mathbb{R}^+ \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ satisfy the first-order nonlinear random differential equation

$$x'(t, \omega) = f(t, x(t, \omega), \omega) + g(t, x(t, \omega), \omega), \quad x(0, \omega) = 0 \quad (4)$$

Then x is a solution of the equivalent random integral equation

$$x(t, \omega) = \int_0^t f(s, x(s, \omega), \omega) ds + \int_0^t g(s, x(s, \omega), \omega) ds, \quad t \in \mathbb{R}^+. \quad (5)$$

Proof: Integrating both sides of the differential equation with respect to t , we obtain

$$x(t, \omega) = \int_0^t f(s, x(s, \omega), \omega) ds + \int_0^t g(s, x(s, \omega), \omega) ds,$$

which establishes the result.

4. Main Result

Theorem 4.1 : Suppose that the hypotheses (H3) and (H4) hold. Further, if there exists a real number $r > 0$ such that $\|\alpha_k\| + \|q_r\|_{L^1} = r$, then the first-order nonlinear random differential equation

$$x'(t, \omega) = f(t, x(t, \omega), \omega) + g(t, x(t, \omega), \omega),$$

has at least one random solution. Moreover, the random solution is locally attractive on $\mathbb{R}^+$

Proof: Let $X = B_M(\mathbb{R}^+, \mathbb{R})$ be a separable Banach space. Define two random operators $A(\omega)$ and $B(\omega)$ on X as follows:

$$A(\omega)x(t) = \int_0^t f(s, x(s, \omega), \omega) ds,$$

$$B(\omega)x(t) = \int_0^t g(s, x(s, \omega), \omega) ds.$$

Then the given differential equation can be written equivalently as the operator equation:

$$A(\omega)x(t) + B(\omega)x(t) = x(t, \omega), \quad t \in \mathbb{R}^+.$$

$$\text{Let } \mu = \{x \in X : \|x\| \leq r\}, \quad \text{where } r = \|\alpha_k\| + \|q_r\|_{L^1}.$$

Step I: (The operators $A(\omega)$ and $B(\omega)$ are random operators on X): Since the functions f and g are measurable (by hypotheses (H0)–(H3)), the mappings

$$\omega \mapsto f(s, x(s, \omega), \omega), \quad \omega \mapsto g(s, x(s, \omega), \omega)$$

are measurable for all $s \in \mathbb{R}^+$. As the integral of measurable functions is measurable, both operators $A(\omega)$ and $B(\omega)$ are random operators on X .

Step II: (The operator $A(\omega)$ is a contraction on X): For any $x, y \in X$, using hypothesis (H3), we have

$$|f(s, x(s, \omega), \omega) - f(s, y(s, \omega), \omega)| \leq \alpha(t, \omega) |x(t, \omega) - y(t, \omega)|.$$

Then

$$\begin{aligned} |A(\omega)x(t) - A(\omega)y(t)| &\leq \int_0^t \alpha(s, \omega) |x(s, \omega) - y(s, \omega)| ds \\ &\leq \|\alpha_k(t, \omega)\| \|x - y\|. \end{aligned}$$

Since $\|\alpha_k(t, \omega)\| < 1$ it follows that $A(\omega)$ is a contraction on X .

Step III: (The operator $B(\omega)$ is continuous on (X)): Let $\{y_n\} \subset X$ be a sequence such that $y_n \rightarrow y$ in X . By the Lebesgue Dominated Convergence Theorem,

$$\begin{aligned} \lim_{n \rightarrow \infty} B(\omega)y_n(t) &= \int_0^t \lim_{n \rightarrow \infty} g(s, y_n(s, \omega), \omega) ds \\ &= \int_0^t g(s, y(s, \omega), \omega) ds \\ &= B(\omega)y(t). \end{aligned}$$

Hence, $B(\omega)$ is continuous on X .

Step IV: The operator $B(\omega)$ is compact on (X) : Let $Y \subset X$ be bounded, i.e., $\|x\| \leq r$ for all $x \in Y$.

Then

$$\begin{aligned} |B(\omega)x(t)| &\leq \int_0^t |g(s, x(s, \omega), \omega)| ds \\ &\leq \int_0^t |q_r(s, \omega)| ds \\ &= \mathcal{G}(t, \omega). \end{aligned}$$

Taking supremum over t , we get $\|B(\omega)x\| \leq M_1$, so $\{B(\omega)Y\}$ is uniformly bounded.

For $t_1, t_2 \in \mathbb{R}^+$ with $t_1 > t_2$,

$$|B(\omega)x(t_1) - B(\omega)x(t_2)| = \left| \int_{t_2}^{t_1} g(s, x(s, \omega), \omega) ds \right| \leq |\mathcal{G}(t_1, \omega) - \mathcal{G}(t_2, \omega)|.$$

Since $\mathcal{G}$ is uniformly continuous, $B(\omega)Y$ is equicontinuous in X .

Hence, by the Arzelà–Ascoli theorem, $B(\omega)$ is compact on X .

Step V: Boundedness of solutions: Let $x \in X$ be a solution of $A(\omega)x + B(\omega)x = x$. Then

$$\begin{aligned} |x(t, \omega)| &= |A(\omega)x(t) + B(\omega)x(t)| \\ &\leq \int_0^t \alpha(s, \omega) ds + \int_0^t q_r(s, \omega) ds \\ &\leq \|\alpha_k\| + \|q_r\|_{L^1} \\ &= r. \end{aligned}$$

Thus, $\|x(t, \omega)\| \leq r$ for all $t \in \mathbb{R}^+$, and the hypotheses of Theorem 4.1 are satisfied. Therefore, the nonlinear Volterra-type equation has a random solution in X .

Step VI: Local attractivity of the solution: Let x, y be two random solutions of the equation. Then

$$|x(t, \omega) - y(t, \omega)| \leq \int_0^t |f(s, x(s, \omega), \omega) - f(s, y(s, \omega), \omega)| ds + 2 \int_0^t q_r(s, \omega) ds.$$

Using (H3),

$$|x(t, \omega) - y(t, \omega)| \leq \alpha(t, \omega) |x(t, \omega) - y(t, \omega)| + 2\mathcal{G}(t, \omega).$$

Taking the limit superior as $t \rightarrow \infty$, we obtain

$$\limsup_{t \rightarrow \infty} |x(t, \omega) - y(t, \omega)| \leq \limsup_{t \rightarrow \infty} \alpha(t, \omega) |x(t, \omega) - y(t, \omega)| + 2 \limsup_{t \rightarrow \infty} \mathcal{G}(t, \omega).$$

Since $\lim_{t \rightarrow \infty} \alpha(t, \omega) = 0$ and $\lim_{t \rightarrow \infty} \mathcal{G}(t, \omega) = 0$, it follows that

$$\lim_{t \rightarrow \infty} |x(t, \omega) - y(t, \omega)| = 0.$$

Hence, for every $\varepsilon > 0$, there exists $T > 0$ such that

$$|x(t, \omega) - y(t, \omega)| \leq \varepsilon, \quad \forall t \geq T, \omega \in \Omega.$$

Thus, the random solution is locally attractive on $\mathbb{R}^+$.

5. Conclusion

In this paper, we have established sufficient conditions for the existence and local stability (attractivity) of solutions to a class of nonlinear first-order random differential equations by employing random fixed point theorems. Under appropriate hypotheses involving measurability, boundedness, and contraction conditions, it is shown that the corresponding random integral equation admits at least one random solution. Furthermore, this solution is locally attractive on $\mathbb{R}^+$. The proposed framework extends and generalizes several existing deterministic and random operator results available in the literature. These findings can serve as a foundation for further studies on higher-order random differential equations, systems with stochastic perturbations, and random integral equations in Banach spaces.

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