Existence of a Solution for a Nonlinear Integral Equation via Fixed Point Theory

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Abstract

Integral equations serve as a central tool in modelling phenomena across physics, engineering, and applied sciences. Proving existence and uniqueness of solutions for nonlinear integral equations remains a core challenge, especially when kernels depend nonlinearly on the unknown function. In this work we establish novel, easily verifiable conditions under which a nonlinear Fredholm integral equation admits a unique solution by constructing a contraction mapping in a Banach space. The novelty of our approach lies in refining the classical Lipschitz framework to accommodate variable kernels and providing explicit convergence estimates. We also present a completely verified example and relate our method to recent advances in fixed point techniques for integral equations. This combination of rigorous theory and concrete verification makes the paper suitable for applications in mathematical modelling.

Keywords: Integral equation; Fixed point theorem; Banach contraction principle; Existence and uniqueness.

1. Introduction

Integral equations are indispensable in modern mathematical modelling because many phenomena in physics, biology, and engineering naturally admit an integral formulation [7,11,12]. In particular, nonlinear Fredholm integral equations describe nonlocal interactions such as diffusion with fading memory [5].

Classically, the existence of solutions is often shown using compactness methods like Schauder's theorem. In contrast, the Banach contraction principle [6,9] provides a constructive and uniqueness-guaranteeing route, which is especially attractive for numerical implementation.

Recent developments. Recent research has renewed interest in Banach-type techniques for integral equations. Ezquerro and Hern´andez-Ver´on (2024) analyse global convergence of successive approximations for nonlinear Fredholm equations. Matoog et al. (2023) propose new algorithms based on Banach's theorem for mixed Volterra–Fredholm equations. Sahebi et al. (2024) establish existence for nonlinear q-integral equations via fixed point methods. These works show that refined contraction arguments remain a vibrant field of current research.

Contribution of this paper. We present a self-contained treatment that (i) formulates natural hypotheses ensuring that the integral operator is a contraction on $C[0,1]$, (ii) provides a detailed, stepwise proof of existence and uniqueness together with quantitative convergence rates, and (iii) includes a fully verified illustrative example. This combination of classical rigour and practical verifiability is aimed at readers interested both in pure analysis and in applications. The rest of the paper is organised as follows: Section 2 recalls the necessary functional analytic tools. Section 3 formulates the problem and hypotheses. Section 4 presents the main theorem with an extensive stepwise proof. Section 5 provides a verified example demonstrating the hypotheses.

2. Preliminaries

Let $C[0,1]$ be the Banach space of all real-valued continuous functions on $[0,1]$, endowed with the supremum norm

$$\| x \| := \sup_{t\in[0,1]} |x(t)|.$$

Definition 2.1 (Banach space). A normed linear space $(X, \|\cdot\|)$ is a Banach space if it is complete; i.e. every Cauchy sequence converges in X.
Definition 2.2 (Lipschitz and contraction mappings). An operator $T : X \to Y$ is Lipschitz if $\| Tx - Ty \| \leq L \| x - y \|$ for some $L \geq 0$. If X = Y and $L < 1$, T is a contraction.
Definition 2.3 (Fixed point). A point $x^* \in X$ is a fixed point of $T : X \to X$ if $Tx^* = x^*$.

Theorem 2.4 (Banach Contraction Principle [6, 9]). Every contraction on a nonempty complete metric space has a unique fixed point. Moreover, Picard iterates converge to this point with a geometric rate.
If $K : [0,1]2 \times R \to R$ and $g : [0,1] \to R$ are continuous, define

$$(Tx)(t) = g(t) + \int_0^1 K\big(t, s, x(s)\big) ds.$$

By standard arguments, $T : C[0,1] \to C[0,1]$ is well defined and continuous.

3. Statement of the Problem

We study the existence and uniqueness of nonlinear Fredholm integral equation

$$x(t) = g(t) + \int_0^1 K\big(t, s, x(s)\big) ds, \quad t \in [0,1],$$

4. Hypotheses:

(H1) K(t, s, u) is continuous on $[0,1] \times [0,1] \times R$.
(H2) There exists $M > 0$ with

$$\sup_{t\in[0,1]} \int_0^1 |K(t, s, u)| ds \leq M$$

for all $u \in R$.
(H3) There exists $m(t, s) \geq 0$ such that

$$|K(t, s, u_1) - K(t, s, u_2)| \leq m(t, s)\, |u_1 - u_2|$$

for all $t, s \in [0,1], u_1, u_2 \in R$.
(H4) The contraction constant

$$\sup_{t\in[0,1]} \int_0^1 m(t,s)ds$$

satisfies $q < 1$.

5. Main Theorem

Theorem 4.1. Under hypotheses (H1)–(H4) the integral equation ion

$$x(t) = g(t) + \int_0^1 K(t,s,x(s))ds,$$

has a unique continuous solution $x^* \in C[0,1]$. Moreover, the Picard iteration $x_{n+1} = Tx_n$ converges to x^* at a geometric rate and satisfies explicit a priori and a posteriori error bounds.

Proof: Let $T : C[0,1] \to C[0,1]$ be defined by

$(Tx)(t) = g(t) + \int_0^1 \mathrm{K(t,s,x(s))}$ ds.

Step 1: T maps $C[0,1]$ into itself. Take any $x \in C[0,1]$. Because K is continuous in all variables (H1) and x is continuous, the function $(t,s) \to K(t,s,x(s))$ is continuous on the compact square $[0, 1]^2$. Continuous functions on compact sets are integrable and bounded. Hence, for each t, the integral $\int_0^1 \mathrm{K(t,s,x(s))}$ ds is well defined. Dominated convergence (with the uniform bound from (H2)) shows that the mapping

$$t \to \int_0^1 K(t,s,x(s))ds.$$

is continuous. Since g is continuous, $(Tx)(t)$ is continuous, proving T indeed maps $C[0,1]$ into itself.

Step 2: Uniform a priori bound on the image of T. For each $t \in [0,1]$,

$|(Tx)(t)| \le |g(t)| + \int_0^1 |\mathrm{K(t,s,x(s))}|$ ds.

By (H2) the integral is bounded by M. Taking supremum in t we obtain

$$\|Tx\| \le \|g\| + M.$$

In particular, any fixed point x^* satisfies $\| x^* \| \le \| g \| + M$, which is a global bound independent of initial guess.

Step 3: Lipschitz estimate for T. Let $x, y \in C[0,1]$ and fix $t \in [0,1]$. Using the linearity of integration and the triangle inequality,

$|(Tx)(t) - (Ty)(t)| \le \int_0^1 |K(t,s,x(s) - K(t,s,y(s)|ds.$

Hypothesis (H3) provides the pointwise Lipschitz bound

$|K(t,s,x(s)) - K(t,s,y(s))| \le m(t,s)\,|x(s) - y(s)|.$

Consequently,

$$|(Tx)(t) - (Ty)(t)| \le \left(\int_0^1 m(t,s)ds\right) \|x - y\|.$$

Now take the supremum over t to obtain

$$\| Tx - Ty \| \le q \ \| x - y \|,$$

where $q := sup_t \int_0^1 m(t,s)ds.$

Step 4: T is a strict contraction. By (H4) we know $q < 1$. Hence T is a strict contraction on the complete metric space $(C[0,1], \|\cdot\|)$.

Step 5: Existence and uniqueness of the fixed point. The Banach contraction principle (Theorem 2.4) asserts that every contraction on a complete metric space possesses a unique fixed point. Therefore there exists a unique $x^* \in C[0,1]$ such that $Tx^* = x^*$. By the definition of T, x^* satisfies the integral equation (1), proving existence and uniqueness of the solution.

Step 6: Convergence of Picard iteration and error bounds. Choose any $x_0 \in C[0,1]$ and define the sequence $x_{n+1} = Tx_n$. Applying the contraction estimate recursively,

$$\| x_n - x^* \| \leq q \| x_{n-1} - x^* \| \leq q2 \| x_{n-2} - x^* \| \leq \cdots \leq q^n \| x_0 - x^* \|.$$

Because $q < 1, x_n \to x^*$ geometrically. A practical a posteriori bound, useful for stopping criteria, follows from the standard telescoping series:

$$\| x_n - x^* \| \leq \frac{q^n}{1-q} \|Tx_0 - x_0\|.$$

This shows that after finitely many iterations the error can be made arbitrarily small.

Step 7: Summary of quantitative information. Combining Steps 2–6 we conclude

$$\| x^* \| \leq \| g \| + M, \qquad \| x_n - x^* \| \leq \frac{q^n}{1-q} \|Tx_0 - x_0\|.$$

This complete the proof.

6. Verified Example

To illustrate the applicability of Theorem 4.1, consider

$K(t,s,u) = \lambda e^{-(t-s)2} \sin u, \quad g(t) = t^2, \qquad t,s \in [0,1],$

where λ is a real parameter.

Step 1: Checking hypothesis (H1)

The function $(t,s,u) \to \lambda e^{-(t-s)2}$ sin u is smooth (hence continuous) on the compact set $[0,1]^2 \times R$, so (H1) is satisfied.

Step 2: Uniform integral bound (H2)

For every $t \in [0,1]$ and $u \in R$,

$$\int_0^1 |K(t,s,u)| ds \leq |\lambda| \int_0^1 e^{-(t-s)^2} ds = |\lambda| \int_{-t}^{1-t} e^{-r^2} dr.$$

Since $\int_{-1}^{1} e^{-r^2} dr = \sqrt{\pi}\, \mathrm{erf}\,(1) \approx 1.49365$, we may take

$$M := 1.49365\, |\lambda|.$$

Step 3: Lipschitz continuity in u (H3)

The sine function satisfies $|\sin u_1 - \sin u_2| \leq |u_1 - u_2|$. Hence

$$|K(t,s,u_1) - K(t,s,u_2)| \leq |\lambda| e^{-(t-s)^2} |\mathrm{u}_1 - \mathrm{u}_2|.$$

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Thus (H3) holds with $m(t,s) = |\lambda| e^{-(\mathrm{t-s})^2}$.

Step 4: Contraction constant (H4)

Compute

$$q = \sup_{t\in[0,1]} \int_0^1 m(t,s) ds \leq |\lambda| \int_{-1}^{1} e^{-e^2} dr \approx 1.49365|\lambda|.$$

Hence $q < 1$ whenever

$$|\lambda| < \frac{1}{1.49365} \approx 0.67.$$

Step 5: Existence, uniqueness and iterative approximation

For example, choose $\lambda = 0.5$. Then $M \approx 0.75$ and $q \approx 0.75 < 1$. By Theorem 4.1, the integral equation

$$x(t) = t^2 + 0.5 \int_0^1 e^{-(t-s)^2} \sin\big(x(s)\big)\, ds$$

has a unique continuous solution $x^* \in C[0,1]$.

Starting with $x_0(t) \equiv 0$, compute one Picard step:

$$x_1(t) = t^2,$$

and the next step

$$x_2(t) = t^2 + 0.5 \int_0^1 e^{-(t-s)^2} \sin(s^2)\, ds$$

By the contraction estimate $\| x_2 - x_1 \| \leq 0.75 \| x_1 - x_0 \|$, and in general

$$\| x_{n+1} - x_n \| \leq 0.75^n \| x_1 - x_0 \|,$$

which confirms rapid convergence.

Thus every hypothesis of Theorem 4.1 is met, the solution is unique, and the Picard sequence provides a constructive approximation with guaranteed geometric rate.

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