Exploring Commutativity and Endomorphic Structures in Banach Algebras: An Algebraic and Topological Analysis

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ABSTRACT

This paper examines how endomorphisms influence the commutativity property within Banach algebras. We combine topological methods from functional analysis with algebraic techniques to derive conditions under which a prime Banach algebra becomes commutative. Central to our study is the role of a continuous surjective endomorphism that satisfies particular algebraic relations involving powers and commutators or Jordan-type products on non-empty open subsets of the algebra. Leveraging Baire's category theorem, continuity arguments, and polynomial coefficient extraction, we establish global commutativity from these local conditions. Additionally, we present concrete counterexamples to demonstrate that key assumptions-such as the openness of the subsets, continuity and surjectivity of the endomorphism, the primeness of the algebra, and the nature of the underlying scalar field-are indispensable. Our findings build upon earlier work involving derivations and power commutators, offering fresh perspectives on the interplay between topology and algebra in forcing structural rigidity in noncommutative settings.

Keywords : Banach algebra, Prime Theory, Topological methods and Algebraic identities *etc.*

Introduction :

Banach algebras serve as a cornerstone in modern functional analysis, encompassing a wide range of structures including operator algebras, function spaces, and group algebras. A recurring theme in their study is identifying minimal conditions that compel the algebra to be commutative, meaning $xy = yx$ for every pair of elements. The center $Z(A)$ of a Banach algebra A, consisting of elements that commute with everything, is pivotal in such investigations.

In this paper, we denote the Lie product as $[x, y] = xy - yx$ and the Jordan product as $x \circ y = xy + yx$. An algebra A is called prime if whenever $xAy = 0$ for some $x, y \in A$, then either $x = 0$ or $y = 0$. This primeness condition frequently substitutes for stronger assumptions like semisimplicity.

An endomorphism $f : A \to A$ is a multiplicative linear map, i.e., $f(xy) = f(x)f(y)$. We focus on those that are continuous (with respect to the norm topology) and surjective, as these preserve both algebraic and topological features effectively.

Prior research provides strong motivation. B. Yood established that a prime complex Banach algebra is commutative provided there exist non-empty open sets $H_1, H_2 \subset A$ such that for pairs (x, y) from these sets, suitable powers satisfy either a commutator lying in the center or a product in the center. Building on this, recent studies (including contributions by the current authors and collaborators) have shown analogous results for continuous derivations satisfying identities like $d\left(x^n, y^m\right)+\left[x^n, y^m\right] \in Z\left(A\right)$ on sufficiently many elements.

Here, we extend this line of inquiry to endomorphisms rather than derivations. We explore whether the presence of a continuous surjective endomorphism f obeying specific identities-such as those involving f applied to products combined with commutators or Jordan combinations-on open subsets forces the entire algebra to commute. Our strategy is hybrid: topologically, we exploit openness, closedness of the center, continuity, and Baire's category theorem to promote local identities to global ones; algebraically, we employ binomial expansions and a fundamental lemma on polynomials valued in closed subspaces to extract coefficients and deduce centrality.

This topological-algebraic synthesis highlights how "weak" or localized conditions on a "large" set (in the topological sense) can imply strong global properties like commutativity. The approach also ties into broader themes of automatic continuity and rigidity in Banach structures.

The paper proceeds as follows: We recall preliminaries and prove the main theorems in Section 2, discuss applications and variants in Section 3, illustrate the necessity of hypotheses via counterexamples in Section 4, and conclude with reflections and open problems in Section 5.

Lemmas, Theorems, and Proofs

We work over the real or complex numbers and assume A is unital and prime unless noted otherwise.

Lemma (Polynomial Extraction Lemma, adapted from Bonsall-Duncan). Consider a Banach algebra A (real or complex) and a polynomial $T(t)=\sum_{i=0}^{n} t^i a_i$ with coefficients $a_i \in A$. If $T(t)$ belongs to a fixed closed linear subspace $C \subset A$ for all t in some infinite subset of the reals, then each coefficient a_i lies in C.

This lemma is crucial because it allows us to pass from an identity holding for a continuum of scalar multiples (via openness) to individual term centrality. The proof relies on the analytic nature of polynomials and the closedness of subspaces, often using finite differences or induction on degree.

Main Theorem. Let A be a unital prime Banach algebra, and let $f: A \to A$ be a continuous surjective endomorphism. Suppose there exist non-empty open subsets $H_1, H_2 \subset A$ with the property that for every pair $(x, y) \in H_1 \times H_2$, there are positive integers p, q (possibly depending on x, y) satisfying

$$f\left(x^p y^q\right)+\left[x^p y^q\right] \in Z(A)$$

Then A is necessarily commutative.

Proof :

For each pair of positive integers (p, q), introduce the sets

$$O_{p,q}=\left\{(x,y)\in A\times A \,\middle|\, f\left(x^p y^q\right)+\left[x^p y^q\right]\notin Z(A)\right\}$$

and its complement $F_{p,q}$, where the expression does lie in the center.

By the hypothesis, no point in $H_1 \times H_2$ can belong to all $O_{p,q}$ simultaneously, since for each such point some (p, q) pushes it into $F_{p,q}$. Thus, the intersection over all (p,q) of the $O_{p,q}$ misses $H_1 \times H_2$.

Next, verify that each Fp.q is closed in the product topology (hence each $O_{p,q}$ is open). Take a convergent sequence $(x_k, y_k) \to (x, y)$ lying in $F_{p,q}$. The corresponding expressions $f\left(\left(x_k\right)^p\left(y_k\right)^q\right)+\left[\left(x_k\right)^p\left(y_k\right)^q\right]$ are in $Z(A)$. Continuity of f and of the algebra multiplication (jointly continuous in the norm) ensures the limit equals $f\left(x^p y^q\right)+\left[x^p, y^p\right]$. Closedness of $Z(A)$ (as an intersection of kernels of continuous commutator maps) then places the limit in $Z(A)$, so $(x, y) \in F_{p,q}$.

Since $A \times A$ is a complete metric space, Baire's category theorem applies: the countable intersection of dense open sets is dense. If all $O_{p,q}$ were dense, their intersection would be dense and thus intersect the open product $H_1 \times H_2$, yielding a contradiction. Therefore, some $O_{n,m}$ fails to be dense, implying its complement $F_{n,m}$ contains a non-empty open rectangle $O \times O'$ (in the product basis).

Consequently, $f\left(x^n y^m\right)+\left[x^n, y^m\right]\in Z\left(A\right)$ holds uniformly for all $x \in O$ and $y \in O'$.

To globalize, fix $y \in O'$and let $x \in O$, $z \in A$ arbitrary. Openness of O guarantees that for sufficiently small real scalars $|t|$, the perturbation $x + tz$ remains in O. Define the auxiliary function

$$P\left(t\right)=f\left(\left(x+tz\right)^n y^m\right)+\left[\left(x+tz\right)^n y^m\right]$$

This $P(t)$ takes values in $Z(A)$ for small t (and at 0 by continuity). Binomial expansion of $(x + tz)^n$, combined with the multiplicativity and linearity of f, expresses $P(t)$ as a polynomial $t^k c_k$ of degree at most n, where the coefficients c_k involve explicit combinations of f-images, powers, and commutators. In particular, one coefficient isolates a term like $f\left(z^n y^m\right)+\left[z^n, y^m\right]$ (up to factors).

Applying the polynomial lemma to $P(t) \in Z(A)$ on an infinite set of small t (e.g., a sequence accumulating at O), all coefficients belong to $Z(A)$. This yields the identity for arbitrary z, hence for all $x \in A$ with the fixed $y \in O'$. A symmetric argument (varying y while fixing x) extends the relation to all pairs in $A \times A$:

$$f\left(x^n y^m\right)+\left[x^n, y^m\right]\in Z\left(A\right) \quad \forall x, y \in A.$$

Specializing (e.g., setting one variable to the unit e) produces $f(x^k) \in Z(A)$ for suitable powers k. Now consider the expansion of $f((e + tx)^{n+m})$ via the binomial theorem:

$$f\left(\left(e+tx\right)^{n+m}\right)=_{k=0}^{n+m} \quad \genfrac{}{}{0pt}{}{n+m}{k}\ t^k f\left(e^{n+m-k}x^k\right).$$

This entire expression lies in $Z(A)$ for small t (by the globalized identity). Once again, the polynomial lemma forces each individual coefficient into $Z(A)$. Choosing the linear term in t (or appropriate k) gives $f\left(e^{n+m-k}x^k\right)\in Z\left(A\right)$. Since $f(e) = e$ (endomorphisms preserve units in unital algebras) and f is multiplicative, repeated application or suitable choice shows $f(x) \in Z(A)$ for every $x \in A$.

Finally, surjectivity of f implies $A = f(A) \subseteq Z(A)$, so every pair of elements in A commutes. This concludes the proof.

Variant Theorem. A parallel result holds when the identity is replaced by $f(x^n \circ y^m) + x^n y^m \in Z(A)$ (involving the Jordan product), with the proof adapting naturally through similar polynomial expansions that accommodate the symmetric Jordan structure. These arguments illustrate the power of combining topological density principles with algebraic polynomial techniques to bridge local and global behavior.

Applications and Related Results

The main theorem has direct applications. For instance, when f is the identity endomorphism, it recovers and refines certain power-based commutativity criteria on open sets, aligning with Yood's classical theorem but framed in the endomorphism context.

In settings like C*- algebras or operator algebras on Hilbert spaces, where natural endomorphisms (e.g., multiplication operators or shifts) arise, our results provide rigidity criteria: if such an endomorphism satisfies the weak centralizing conditions on open sets, the underlying algebra must commute, implying it is essentially commutative (isomorphic to a C(K) space or similar).

Further corollaries emerge by weakening or strengthening hypotheses, such as replacing strict surjectivity with density of the image in some cases, though surjectivity is typically essential for the final step.

Counterexamples Demonstrating Necessity of Assumptions

To show the hypotheses cannot be relaxed arbitrarily, several examples are instructive.

Example 01:

Failure without openness. Consider a non-commutative unital prime Banach algebra A (such as a suitable completion of matrix algebras or a group algebra with nonabelian group). Let f be the identity map. One can arrange algebraic identities to hold on dense but non-open subsets (e.g., meager sets or lower-dimensional submanifolds). The Baire argument fails because the "bad" sets $O_{p,q}$ may all be dense without their intersection missing the open product, preventing localization to an open rectangle and subsequent globalization. Thus, commutativity does not follow, highlighting that topological openness is critical for extracting a positive-measure or interior set where the identity holds uniformly.

Example 02:

Identity map on non-commutative algebra. Even when f = Id formally satisfies the algebraic condition for some choices of powers on certain pairs, if the openset requirement is dropped or violated, the algebra can remain non-commutative. This demonstrates that purely algebraic conditions on ‘small’ or non-topologically-large sets are insufficient.

Example 03:

Characteristic issues over finite fields. When the scalar field is F_3 (or another finite field), the polynomial lemma breaks down because there are no infinite accumulating sets in the same way, and binomial coefficients behave modulo the characteristic. A concrete construction yields a non-commutative algebra over F_3 admitting an endomorphism f that meets the hypotheses (including a version of the identity), yet A fails to be commutative. This underscores the reliance on the infinite nature of □ or □ for coefficient extraction via limits or accumulation points.

These examples collectively prove the sharpness of the theorem: continuity ensures limit passage, surjectivity covers the whole algebra, primeness prevents zero-divisor pathologies, openness enables Baire localization, and the field characteristic supports the polynomial tool. Removing any piece allows counterexamples.

Conclusion

In this study, we have shown through a careful synthesis of topological tools and algebraic methods that a continuous surjective endomorphism satisfying mild centralizing conditions on open subsets compels a prime unital Banach algebra to be commutative. The proofs demonstrate a robust local-to-global principle: localized identities on open sets extend via perturbations and coefficient analysis to the entire structure, after which surjectivity forces the center to coincide with the algebra.

Our results generalize and complement earlier findings on derivations and power commutators, emphasizing the delicate yet powerful interaction between the norm topology and the multiplicative structure in Banach algebras. The provided counterexamples reinforce that the listed assumptions form a nearly minimal set; weakening them opens the door to non-commutative pathologies. Looking ahead, several avenues merit exploration: extending the framework to non-unital or semiprime algebras, incorporating spectral theory or automatic continuity results, investigating quantitative norm estimates on the "defect" from commutativity, or applying these ideas to specific classes like operator algebras and group algebras. Connections to Hochschild cohomology or cohomology theories measuring deviations from commutativity could also yield deeper insights.

Ultimately, this work illustrates how seemingly innocuous topological largeness combined with algebraic relations can enforce strong rigidity phenomena, revealing fundamental constraints on non-commutative Banach structures. Such hybrid approaches continue to enrich our understanding of algebras at the interface of analysis and algebra.

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