Finite Difference Simulation of the Viscous Burgers' Equation with External Source

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Abstract: This study presents a numerical investigation of the viscous Burgers' equation subjected to an external sinusoidal source term, utilizing an explicit finite difference method. The model incorporates nonlinear advection, viscous diffusion, and spatially varying forcing, representing a physically relevant nonlinear partial differential equation. The numerical scheme uses forward time stepping, upwind discretization for the convective term, and central differencing for the diffusive term. Through simulation, the evolution of the solution profile is visualized using 3D surface plots, space-time contours, and cross-sectional curves. The results demonstrate a consistent amplification near the domain center due to the source term and show the smoothing influence of viscosity over time. The study also addresses the scheme's stability and convergence under suitable discretization parameters.

Keywords: Viscous Burgers' equation, nonlinear PDE, finite difference method, source term, numerical simulation, stability analysis, space-time evolution

1. Introduction:

Nonlinear partial differential equations (PDEs) are fundamental in modeling various complex phenomena in fluid dynamics, wave propagation, traffic flow, and heat transfer. Among them, the Burgers' equation serves as a simplified yet insightful model that captures essential features of convection and diffusion. When modified to include an external forcing term, the equation becomes a versatile tool to simulate scenarios where additional energy or momentum is introduced into the system. In this context, the viscous Burgers' equation with a sinusoidal source provides a compelling model for analyzing how nonlinear effects interact with diffusion and external influences. This work focuses on solving such an equation using an explicit finite difference scheme, enabling visualization of the dynamic behavior of the solution and assessment of numerical stability. The initial and boundary conditions are chosen to reflect physically meaningful constraints, and the simulation results are analyzed through graphical interpretations.

Oruç et al. (2015) proposed a hybrid numerical approach combining the Haar wavelet method with the finite difference method to solve the modified Burgers' equation. Their method effectively captured the nonlinear dynamics and provided improved numerical accuracy. The hybridization strategy helped overcome issues of stability and convergence often encountered in pure finite difference schemes for nonlinear PDEs. **Oruç et al.**

(2016) extended their work by applying the Haar wavelet method to the regularized long wave equation. The study emphasized the efficiency of Haar wavelet-based approaches in handling dispersive wave phenomena and preserving key wave characteristics over long time simulations. **Oruç et al. (2016)** developed a Haar wavelet collocation method for coupled nonlinear Schrödinger–KdV equations. The research highlighted the strength of wavelet-based methods in handling systems of coupled nonlinear PDEs, showcasing high resolution in spatial localization and good computational performance. **Reutskiy (2016)** introduced a meshless radial basis function method for two-dimensional steady-state heat conduction in anisotropic and inhomogeneous media. This study demonstrated the flexibility of meshless techniques in modeling complex geometries and material distributions, thereby offering an alternative to structured grid-based approaches like finite differences. **Mittal and Pandit (2018)** presented a quasilinearized scale-3 Haar wavelet-based algorithm for fractional dynamical systems. By linearizing the nonlinear terms and utilizing multiscale wavelets, they achieved accurate approximations of fractional PDEs. The method demonstrated high accuracy and computational efficiency for a wide class of nonlinear systems. **Mittal and Pandit (2018)** also explored the sensitivity analysis of shock wave solutions in the Burgers' equation using a novel scale-3 Haar wavelet-based algorithm. This contribution underscored the potential of wavelet methods in capturing sharp gradients and shock phenomena, which are typically challenging for classical numerical techniques. **Mittal and Pandit (2019)** proposed a new algorithm based on scale-3 Haar wavelets for the numerical simulation of second-order ordinary differential equations. Although focusing on ODEs, the foundational techniques have implications for PDE simulations, especially in terms of adaptive multiresolution and localized error control. **Haq et al. (2019)** addressed variable-order time fractional advection-dispersion and diffusion models in one and two dimensions. Their work provided robust numerical schemes for capturing anomalous diffusion behavior and emphasized the adaptability of fractional order methods for modeling physical processes with memory effects.

Shukla and Kumar (2022) developed a numerical solution method for the Burgers–Huxley equation using 3-scale Haar wavelets, along with an in-depth error analysis. The study demonstrated the accuracy and efficiency of Haar wavelets in resolving nonlinearities and reaction-diffusion dynamics within fractional frameworks. **Shukla and Kumar (2022)** also extended their investigation to time and space fractional PDEs, employing the same 3-scale Haar wavelet methodology. Their findings confirmed the capability of wavelets to model complex fractional dynamics with high spatial and temporal resolution. **Vallejo-Sánchez and Villegas (2022)** proposed a meshless method for solving coupled Burgers equations, utilizing radial basis functions. The work demonstrated the flexibility of meshless methods in solving coupled nonlinear systems, especially in domains where meshing is computationally expensive or geometrically complex. **Kumar et al. (2023)** developed a two-dimensional uniform and non-uniform Haar wavelet collocation method for nonlinear PDEs. The study emphasized how non-uniform grids could enhance solution accuracy near regions of rapid change and showed how Haar wavelets could be effectively adapted to complex spatial features. **Ghafoor et al. (2024)** analyzed the nonlinear Burgers' equation incorporating a time fractional Atangana–Baleanu–Caputo derivative. Their approach introduced a nonlocal and nonsingular kernel to better reflect real-world diffusion phenomena, improving solution accuracy and modeling capabilities for systems with memory. **Khan et al. (2024)** investigated scalar reaction-diffusion equations with cubic nonlinearity and time-dependent coefficients using the wavelet method of lines. The study underscored the effectiveness of combining wavelet transforms with time discretization in simulating strongly nonlinear and time-varying processes. **Bilal et al. (2025)** developed a comprehensive numerical scheme integrating finite difference methods with scale-3 Haar wavelets to solve two-dimensional diffusion and Burgers systems. Their work stands out for providing stability and error estimates and demonstrates how hybrid methods can yield accurate, stable solutions to nonlinear PDEs. The research further validates the robustness of wavelet-FDM combinations in multi-dimensional, nonlinear simulations involving advection and diffusion.

2. Nonlinear PDE: The Viscous Burgers' Equation with a Source Term:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} + \lambda sin(\pi x) \tag{1}$$

where:

$u(x,t)$: velocity or field variable

$\nu > 0$: viscosity coefficient (diffusion-like term)

λ: strength of the external source term

$x \in [0,1], \mathrm{t} \in [0,0.1]$

Initial Condition: $u(x,0) = sin\pi x; 0 \leq x \leq 1$

This gives a smooth sine wave as the initial velocity profile.

Boundary Conditions (Dirichlet type): $u(0,t) = u(l,t) = 0$

These homogeneous boundary conditions represent a scenario such as a velocity that is pinned to zero at both ends (like fixed endpoints in a pipe or channel).

3. Finite Difference Method for the Solution of Proposed PDE: Let's define:

$x_i = i\Delta x$, Where $i = 0,1,2,\ldots,N, \Delta x = \frac{1}{N}$

$t_n = n\Delta t$, for time steps $n = 0,1,2,\ldots$

We denote: $u_i^n \approx u(x_i,t_n)$

$$\frac{\partial u}{\partial t} \approx \frac{u_i^{n+1} - u_i^n}{\Delta t} \tag{2}$$

$$\frac{\partial u}{\partial x} \approx \frac{u_i^n - u_{i-1}^n}{\Delta x} \tag{3}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} \tag{4}$$

Putting it all together, the explicit scheme becomes:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + u_i^n \left[\frac{u_i^n - u_{i-1}^n}{\Delta x}\right] = \nu \left[\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}\right] + \lambda sin(\pi x)$$

$$u_i^{n+1} = u_i^n - \Delta t\, u_i^n \left[\frac{u_i^n - u_{i-1}^n}{\Delta x}\right] + \nu\, \Delta t \left[\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}\right] + \Delta t \lambda sin(\pi x_i) \tag{5}$$

Initialize: Set $u_i^0 = sin(\pi x_i)$ (6)

Apply boundary conditions:

For all n, set $u_0^n = 0, u_0^N = 0$ (7) **Loop over time:**

For $n = 0$ to max time steps:

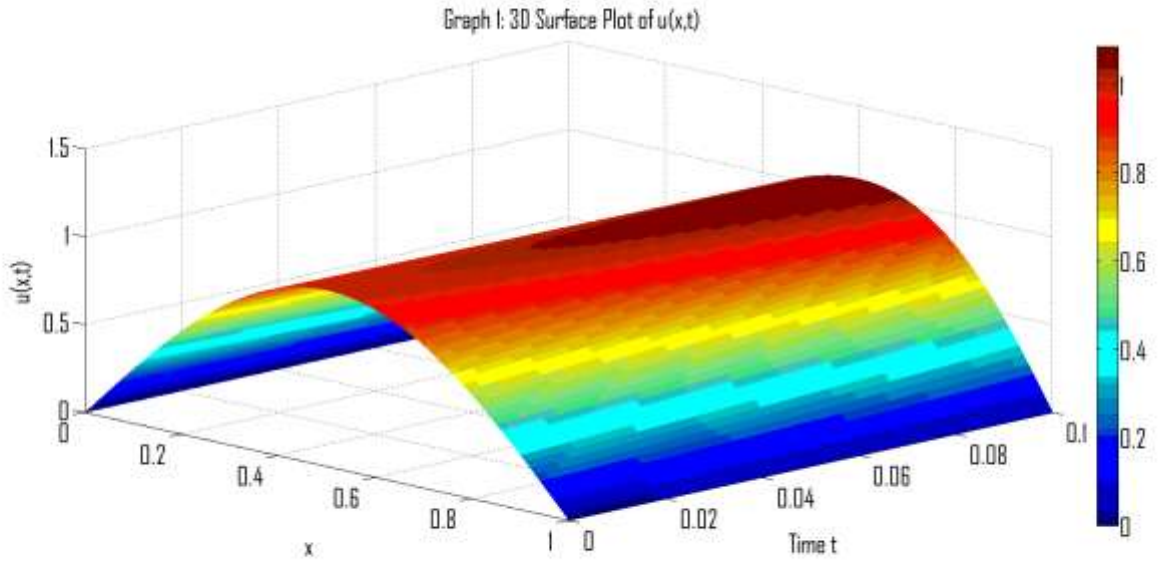
Loop over $i = 1$ to $N - 1$

Update u_i^{n+1} using the update formula

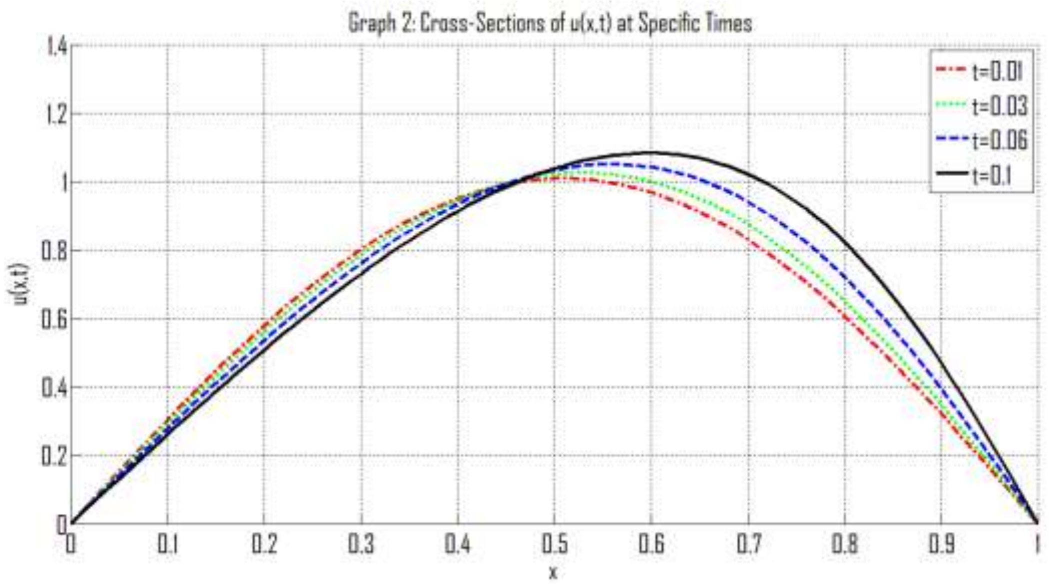
4. Stability Condition: The scheme is explicit, so stability requires:

$$\Delta t \leq min\left(\frac{\Delta x}{max|u|}, \frac{(\Delta x)^2}{2\nu}\right) \tag{8}$$

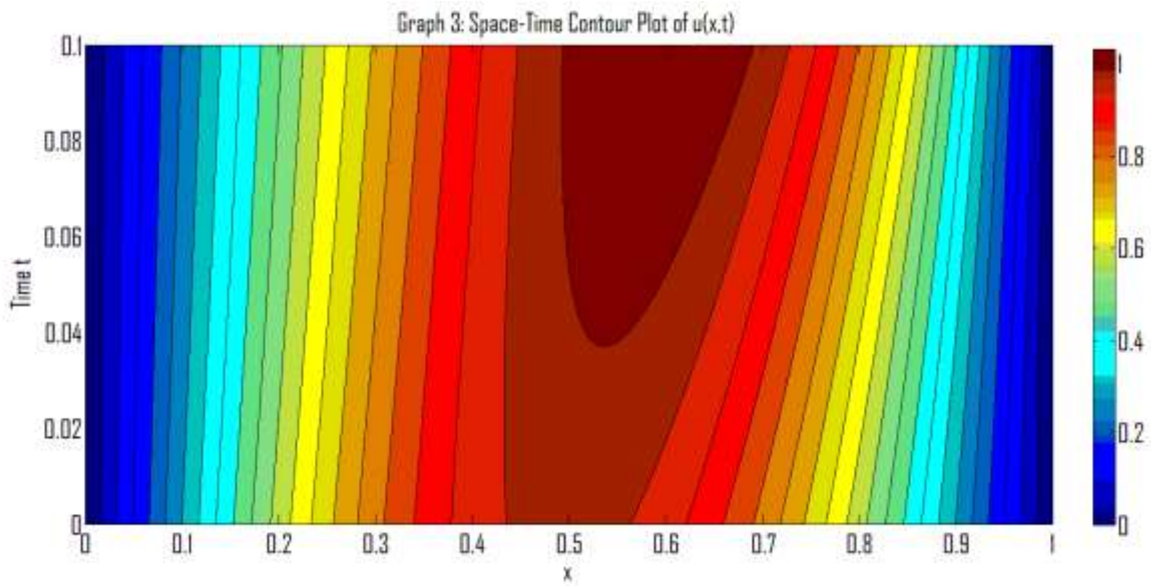
5. Results and Discussion: Let's choose: $\nu = 0.01, \lambda = 1, L = 1, N = 100, \Delta x = 0.01, \Delta t = 0.0001$



The 3D surface plot shown in Graph 1 illustrates the numerical solution $u(x,t)$ of the viscous Burgers' equation with a sinusoidal source term, computed over the spatial domain $x \in [0,1]$ and time interval $\in [0,0.1]$. The plot reveals how the solution evolves smoothly over time due to the combined effects of nonlinear advection, diffusion (viscosity), and external forcing. The initial condition $u(x,0) = \lambda sin\pi x$ develops into a surface where the solution increases in amplitude before reaching a nearly steady state. The color gradient, ranging from blue (low values) to red (high values), indicates that the central region of the domain maintains the highest values of u, while the boundary conditions enforce zero values at both ends $x = 0$ and $x = 1$. The plot clearly demonstrates the balance between the nonlinear steepening effect (from the advection term) and the smoothing influence of viscosity, alongside the contribution of the **sinusoidal source term**, which sustains the solution's amplitude near the center.



Graph 2 shows the cross-sectional profiles of the solution $\boldsymbol{u}(\boldsymbol{x},\boldsymbol{t})$ at four different time instances: $t = 0.01, 0.03, 0.06$ and 0.10 , corresponding to the red dotted, green dashed, blue dash-dotted, and black solid curves respectively. The initial sine-shaped profile gradually evolves due to the interplay of nonlinear convection, viscosity, and the external source term. Over time, the solution becomes more pronounced and shifts slightly upward in the central region $(x = 0.5)$, reflecting the growth in amplitude contributed by the sinusoidal source term $\lambda sin\pi x$. The profiles remain symmetric about the midpoint, consistent with the source and boundary conditions, and the peak value of $u(x,t)$ increases with time while the solution maintains smoothness, indicating that the diffusion term successfully prevents the formation of sharp gradients. This graph effectively illustrates how the solution evolves over time and converges toward a steady-state influenced by persistent external forcing.



Graph 3 presents a space-time contour plot of the solution $\boldsymbol{u}(\boldsymbol{x},\boldsymbol{t})$, offering a 2D visual representation of how the solution evolves over both spatial and temporal dimensions. The horizontal axis represents the spatial coordinate $\in [0,1]$, the vertical axis shows time $t \in [0,0.1]$, and the contour colors indicate the magnitude of $\boldsymbol{u}(\boldsymbol{x},\boldsymbol{t})$. Initially, the solution starts with a sinusoidal profile, and as time progresses, we observe a clear amplification of the central region (near $x = 0.5$), where the solution reaches its maximum due to the persistent influence of the sinusoidal source term $\lambda sin\pi x$. The color bands evolve smoothly and maintain symmetry about the center, reflecting the boundary conditions and the source profile. The contour lines become denser near the edges and more spread near the center as time increases, highlighting the balance between diffusive smoothing and source-driven growth. This plot effectively captures the continuous evolution of the solution and provides insight into the space-time interaction of nonlinear effects, diffusion, and forcing.

6. Concluding Remarks: Nonlinear partial differential equations (PDEs) are fundamental in modeling various complex phenomena in fluid dynamics, wave propagation, traffic flow, and heat transfer. Among them, the Burgers' equation serves as a simplified yet insightful model that captures essential features of convection and diffusion. When modified to include an external forcing term, the equation becomes a versatile tool to simulate scenarios where additional energy or momentum is introduced into the system. In this context, the viscous Burgers' equation with a sinusoidal source provides a compelling model for analyzing how nonlinear effects interact with diffusion and external influences. This work focuses on solving such an equation using an explicit finite difference scheme, enabling visualization of the dynamic behavior of the solution and assessment of numerical stability. The initial and boundary conditions are chosen to reflect physically meaningful constraints, and the simulation results are analyzed through graphical interpretations.

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