

Fixed Point Theorems for Self-mappings in Fuzzy Metric Spaces

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ABSTRACT: In this paper, a fixed point theorem for self-mappings is presented by using the concept of occasionally weakly compatible mapping in fuzzy metric spaces.

Keywords: Common fixed points, fuzzy metric space, compatible mappings, occasionally weakly compatible mappings.

2000 AMS Subject Classification: Primary 47H10, Secondary 54H25

1 Introduction:

The concept of fuzzy set was introduced by Zadeh [27] in 1965, and the concept of fuzzy metric space was introduced by Kramosil and Michalek [15] in 1975 as a new way to represent vagueness in everyday life. Subsequently, it was developed by many authors and used in various fields. Pant [19,20,21] introduced the new concept of reciprocally continuous mapping and established common fixed point theorems. Balasubramaniam et al.[5] gives more results on a fuzzy metric space. Aage and Salunke [3] established some results on the fixed point theorem in a fuzzy metric space. The purpose of this paper is to find some common fixed point theorems for occasionally weakly compatible mappings in fuzzy metric spaces. Singh and Jain [28] proved a fixed point theorem for self-mapping in a fuzzy metric space.

2 Preliminaries:

We recall some definitions and known results in fuzzy metric spaces.

Definition 2.1 A fuzzy set A in X is a function with domain X and values in $[0,1]$.

Definition 2.2 A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-norm if $*$ is satisfying conditions:

- 2.2.1 $*$ is a commutative and associative;
- 2.2.2 $*$ is continuous;
- 2.2.3 $a * 1 = a$ for all $a \in [0, 1]$;
- 2.2.4 $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0, 1]$.

Definition 2.3 A 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following condition for all $x, y, z \in X, s, t > 0$,

- 2.3.1 $M(x, y, t) > 0$;
- 2.3.2 $M(x, y, t) = 1$ if and only if $x = y$
- 2.3.3 $M(x, y, t) = M(y, x, t)$;
- 2.3.4 $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
- 2.3.5 $M(x, y, \cdot) : (0, \infty) \rightarrow (0, 1]$ is continuous.

Then M is called a fuzzy metric on X . Then $M(x, y, t)$ denotes the degree of nearness between x and y with respect to t .

Definition 2.4: Let $(X, M*)$ be a fuzzy metric space. Then

- 2.4.1 a sequence $\{x_n\}$ in X is said to converges to x in X if for each $\epsilon > 0$ and each $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - \epsilon$ for all $n \in \mathbb{N}$.
- 2.4.2 a sequence $\{x_n\}$ in X is said to be Cauchy if for each $\epsilon > 0$ and each $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for all $n, m \in \mathbb{N}$.
- 2.4.3 a fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.5: A pair of self-mappings (f, g) of a fuzzy metric space $(X, M, *)$ is said to be

2.5.1 weakly commuting if $M(fgx, gfx, t) \geq M(fx, gx, t)$ for all $x \in X$ and $t > 0$.

2.5.2 R -weakly commuting if there exists some $R > 0$ such that $M(fgx, gfx, t) \geq M(fx, gx, t/R)$ for all $x \in X$ and $t > 0$.

Definition 2.6 Two self-mappings f and g of a fuzzy metric space $(X, M, *)$ are called compatible if

Whenever $\{x_n\}$ is a sequence in X such that for some x in X .

Definition 2.7: Two self mappings f and g of a fuzzy metric space $(X, M, *)$ are called reciprocally continuous on X if

Whenever $\{x_n\}$ is a sequence in X such that for some x in X .

Lemma 2.8: Let $(X, M, *)$ be a fuzzy metric space. If there exists $q \in (0, 1)$ such that $M(x, y, qt) \geq M(x, y, t)$ for all $x, y \in X$ and $t > 0$, then $x = y$.

Definition 2.9 Let X be a set, f and g self-mappings of X . A point x in X is called a coincidence point of f and g if and only if $fx = gx$. We shall call $w = fx = gx$ a point of coincidence of f and g .

Definition 2.10: A pair of maps S and T is called a weakly compatible pair if they commute at coincidence points. The concept, occasionally weakly compatible, is introduced by Thagafi and Shahzad [4].

Definition 2.11: Two self-maps f and g of a set X are occasionally weakly compatible (owc) if and only if there is a point x in X which is a coincidence point of f and g at which f and g commute. Thagafi and Shahzad [4] showed that occasionally weakly is weakly compatible, but the converse is not true.

Example 2.12: Let R be the usual metric space. Define $S, T: R \rightarrow R$ by $Sx = 2x$ and $Tx = x^2$ for all $x \in R$. Then $Sx = Tx$ for $x = 0, 2$, but $ST0 = TS0$, and $ST_2 = TS_2$. S and T are occasionally weakly compatible self-maps, but not weakly compatible.

Lemma 2.13: Let X be a set, f and g occasionally weakly compatible self-mappings of X . If f and g have a unique point of coincidence, $w = fx = gx$, then w is the unique common fixed point of f and g .

MAIN RESULTS

Theorem 3.1: Let $(X, M, *)$ be a fuzzy metric space and let P, Q, R, S be self-mappings of X . Let the Pair $\{P, R\}, \{Q, S\}$ be occasionally weakly compatible. If there exists

$$3.1.1 \quad M(Px, Qy, t) \geq \phi(\min\{M(Rx, Sy, t), M(Px, Sy, \phi(t)), M(Rx, Qy, (2 - \alpha)t)\})$$

for all $x, y \in X$, $\alpha \in (0, 2)$ and $t > 0$, where $\phi: [0, 1] \rightarrow [0, 1]$ is a continuous function such that $\phi(t) > t$ for some $0 < t < 1$. Then P, Q, R and S have a unique common fixed point in X .

Proof: Let the pairs $\{P, R\}$ and $\{Q, S\}$ be two occasionally weakly compatible, so there are points such that. We claim that. If not, now, we put $\alpha = 1 - q$ with $q \in (0, 1)$, we have by 3.1.1

$$\begin{aligned} M(Px, Qy, t) &\geq \phi(\min\{M(Rx, Sy, t), M(Px, Sy, \phi(t)), M(Rx, Qy, (2 - \alpha)t)\}) \\ &\geq \phi(\min\{M(Px, Qy, t), M(Px, Qy, (1-q)t), M(Px, Qy, (1+q)t)\}) \\ &\geq \phi(\min\{M(Px, Qy, t), M(Px, Qy, t), M(Px, Qy, t) * M(Px, Qy, qt)\}) \end{aligned}$$

Since t -norm $*$ is continuous, letting $q \rightarrow 1$, we have

$$M(Px, Qy, t) \geq \phi(\phi(M(Px, Qy, t)))$$

since $\phi(t) > t$ for each $0 < t < 1$.

$$M(Px, Qy, t) \geq \phi(\phi(M(Px, Qy, t)))$$

Therefore $Px = Qy$, i.e $Px = Rx = Qy = Sy$. Suppose that there is another point z such that $Pz = Rz = Qy = Sy$, so $Px = Pz$ and $w = Px = Rx$ is the unique point of coincidence of P and R . By Lemma 2.13, w is the only fixed point of P and R . Similarly, there is a unique point $z \in X$ such that $z = Qz = Sz$. Assume that $w \neq z$. We have

$$\begin{aligned} M(w, z, t) &= M(Pw, Qz, t) \geq \phi(\min\{M(Rw, Sz, t), M(Pw, Sz, \phi(t)), M(Rw, Qz, (2 - \alpha)t)\}) \\ &\geq \phi(\min\{M(w, z, t), M(w, z, \phi(t)), M(w, z, (2 - \alpha)t)\}) \\ &\geq \phi(\phi(M(w, z, t))) \end{aligned}$$

$$M(w, z, t) \geq \phi(\phi(M(w, z, t)))$$

Therefore, we have $z = w$ by Lemma 2.13, and z is a common fixed point of P, Q, R and S . The uniqueness of the fixed point holds.

REFERENCES

- [1]. C. T. Aage, J. N. Salunke, "Common Fixed Point Theorems in Fuzzy Metric Spaces", International Journal of Pure and Applied Mathematics, 56(2), 2009, pp 155-164.
- [2]. C. T. Aage, J. N. Salunke, "Some Fixed Point Theorems in Fuzzy Metric Spaces", International Journal of Pure and Applied Mathematics, 56(3)009, pp 311-320.
- [3]. C. T. Aage, J. N. Salunke, "On Fixed Point Theorems in Fuzzy Metric Spaces, Int. J. Open Problems Compt. Math., Vol. 3, No. 2, June 2010.
- [4]. A. Al-Thagafi and Naseer Shahzad, "Generalized I-Nonexpansive Self maps and Invariant Approximations", Acta Mathematica Sinica, English Series, 2008, Vol. 24, No. 5, pp. 867876.
- [5]. P. Balasubramaniam, S. Muralisankar, R.P. Pant, "Common fixed points of four mappings in a fuzzy metric space", J. Fuzzy Math. 10(2) (2002), 379-384.
- [6]. Y.J. Cho, H.K. Pathak, S.M. Kang, J.S. Jung, "Common fixed points of compatible maps of type (A) on fuzzy metric spaces", Fuzzy Sets and Systems 93 (1998), 99-111.
- [7]. A. George, P. Veeramani, "On some results in fuzzy metric spaces", Fuzzy Sets and Systems, 64 (1994), 395-399.
- [8]. M. Grabiec, "Fixed points in fuzzy metric spaces", Fuzzy Sets and Systems 27 (1988), 385-389.
- [9]. O. Hadzic, "Common fixed point theorems for families of mapping incomplete metric spaces", Math. Japon. 29 (1984), 127-134.
- [10]. Mohd. Imdad and Javid Ali, "Some common fixed point theorems in fuzzy metric spaces", Mathematical Communications 006), 153-163 153.
- [11]. G. Jungck, "Compatible mappings and common fixed points (2)", Internat. J. Math. Math. Sci. (1988), 285-288.
- [12]. [12] G. Jungck and B. E. Rhoades, "Fixed Point for Set Valued functions without Continuity", Indian J. Pure Appl. Math., 29(3), (1998), pp.771-779.
- [13]. G. Jungck and B. E. Rhoades, "Fixed Point Theorems for Occasionally Weakly compatible Mappings", Fixed Point Theory, Volume 7, No. 2, 2006, 287-296.
- [14]. G. Jungck and B. E. Rhoades, "Fixed Point Theorems for Occasionally Weakly compatible Mappings", Erratum, Fixed Point Theory, Volume 9, No. 1, 2008, 383-384.
- [15]. O. Kramosil and J. Michalek, "Fuzzy metric and statistical metric spaces", Kybernetika, 11 (1975), 326-334.
- [16]. S. Kutukcu, "A fixed point theorem for contraction type mappings in Menger spaces", Am. J. Appl. Sci. 4(6) (2007), 371-373.
- [17]. Servet Kutukcu, Sushil Sharma¹ and Hanifi Tokgoz, "A Fixed Point Theorem in Fuzzy Metric Spaces", Int. Journal of Math. Analysis, Vol. 1, 2007, 2007, No. 18, 861 - 872.
- [18]. S.N. Mishra, "Common fixed points of compatible mappings in PM spaces", Math. Japon. 36 (1991), 283-289.
- [19]. R.P. Pant, "Common fixed points of four mappings", Bull. Cal. Math. Soc. 90 (1998), 281-286.
- [20]. R.P. Pant, "Common fixed point theorems for contractive maps", J. Math. Anal. Appl. 226 (1998), 251-258.
- [21]. R.P. Pant, K. Jha, "A remark on common fixed points of four mappings in a fuzzy metric space", J. Fuzzy Math. 12(2) (2004), 433-437.
- [22]. H. K. Pathak and Prachi Singh, "Common Fixed Point Theorem for Weakly Compatible Mapping", International Mathematical Forum, 2, 2007, no. 57, 2831 - 2839.
- [23]. B.E. Rhoades, "Contractive definitions and continuity", Contemporary Math. 72 (1988), 233-245.
- [24]. B. Schweizer and A. Sklar, "Statistical metric spaces", Pacific J. Math. 10(1960), 313-334.
- [25]. Seong Hoon Cho, "On common fixed point in fuzzy metric space", Int. Math. Forum, 1, 2006, 10, 471-479.
- [26]. R. Vasuki, "Common fixed points for R-weakly commuting maps in fuzzy metric spaces", Indian J. Pure Appl. Math. 30 (1999), 419-423.
- [27]. L.A. Zadeh, "Fuzzy sets, Inform and Control" 8 (1965), 338-353.
- [28] Singh B & Jain S, "Semi-compatible and fixed point theorems in fuzzy metric space", Chungcheong Math. Soc., 18(2005), 1.