Fractal Dynamics in Musical Compositions

Abhay Dutt Sharma

INTRODUCTION

The connection between math and music is, to some extent, intuitive and instinctive. In the case of fractals, it becomes even more interesting. A fractal is a complex geometric pattern that is self-similar across different scales. This means that if you zoom in on any part of a fractal, you'll see a smaller version of the entire pattern. Fractals are found in nature, such as in the branching of trees, the shapes of mountains, and coastlines. Fractals are often a result of the behaviour of chaotic systems. Therefore, they are also used in mathematics and art to model complex, irregular shapes and patterns. Chaos theory describes how initial conditions largely influence the outcome of a deterministic system. Music is composed of several smaller parts that come together and channel themselves into a direction or multiple directions. From the arithmetic structure to the harmonic framework, which corresponds to self-similarity, complexity, and sensitivity to initial conditions of a fractal, it becomes possible to connect the musical domain and fractal geometry, providing deeper insights for both.

Bifurcation Diagram

The bifurcation diagram is a visual representation used to show how the behaviour of a system changes as a specific parameter within a mathematical function is varied. It's particularly common in the study of chaotic systems, such as in population dynamics or iterated functions like the logistic map.

Bifurcation points are critical points where the system's behaviour shifts from one to another. For example, in logistic map, it system settles into a stable state but as the value of r grows, its complexity increase, we get to see chaotic regions across it.

Phillip Glass is a music composer and in his music we can see a baseline which is like a stable region and above that we hear something which is complex, unstable and extremely difficult to predict line of music going together with the coherent baseline. I'll talk more about his music further in the paper. The bifurcation diagram also signifies the sensitivity of initial conditions, which if you will look at Steve Reich's musical compositions, decide the course and direction of the piece. I will elaborate more on that further.

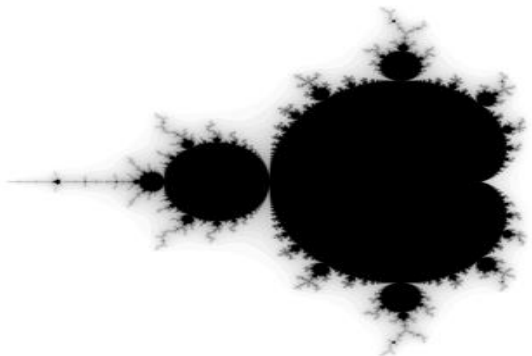
Mandelbrot Set

Another thing that really held my attention is the Mandelbrot Set, an example of fractals that is equally prominent among artists. The Mandelbrot set is defined by a simple iterative equation:

$$z_{n+1} = z_n^{\,2} + c$$

and this equation is repeated, or iterated, infinitely.

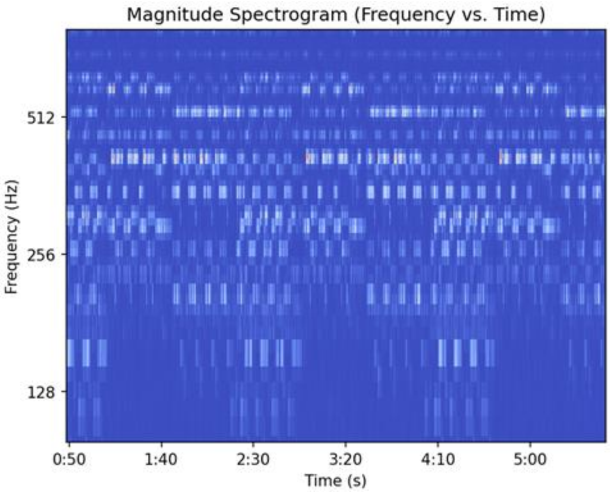
When we visualize this set in the complex plane, what emerges is an infinitely complex, self-similar shape. At the centre is a large cardioid region surrounded by smaller, repeating shapes connected by delicate tendrils. The boundary of the Mandelbrot set is especially intricate and interesting, containing infinite detail no matter how much you zoom in. As you explore its boundary, smaller versions of the entire set appear over and over, showing that the Mandelbrot set exhibits the characteristic self-similarity of fractals.



The iterative process that generates fractals can also be compared to musical structures that evolve through the repetition and transformation of simple patterns. Talking about Steve Reich's musical composition such as 'Music for 18 Musicians', the quality of repetition and transformation through the iterative process could be seen, very similar to what Mandelbrot Set shows.

Opening, Phillip Glass Analysis

In this section, I have generated the frequency vs time graph of Opening by Philip Glass, which illustrates how the sound frequencies evolve throughout the piece. The x-axis represents time, and the y-axis shows frequency. Color intensity highlights the strength of each frequency. I utilized the Librosa library in Python to compute the short-time Fourier transform (STFT), allowing us to visualize how sound frequencies change over time. This was plotted using Matplotlib, revealing the dynamic fluctuations of frequency throughout the piece.



\

Fig -1: Magnitude Spectrogram of Opening

Just as in fractal structures, we observe self-similarity—repeating patterns at different scales. This mirrors the behavior seen in the logistic map, where initially stable and deterministic patterns become chaotic as parameters shift, reflecting instability in certain sections of the music. Similarly as you zoom into Mandelbrot set, where deeper layers reveal repeating patterns, can be compared to how musical patterns in Opening reappear and evolve over time, reflecting both stability and unpredictability.

Analysing Chaos in Phillip Glass's Composition with Lyapunov Exponent

The Lyapunov exponent was calculated using TISEAN software, where we processed the frequency data to analyze the sensitivity to initial conditions. The different epsilon values reflect perturbations in initial conditions, which are used to estimate the Lyapunov exponent. In this context, the epsilon values serve as a stepping stone to understanding the overall chaotic behavior of the audio signal, but they do not represent the Lyapunov exponent directly. Instead, the exponent is derived from how these epsilon values evolve over time in the analysis.

Epsilon values by TISEAN using the graph between frequency and time in Fig. 1:

- epsilon= 9.887844e-04
- epsilon= 1.758335e-03
- epsilon= 3.126811e-03
- epsilon= 5.560343e-03
- epsilon= 9.887844e-03

These epsilon, such as smaller values (9.887844e-04 and 1.758335e-03) indicate regions of more stability, suggesting that small perturbations in the system lead to similar behavior in future iterations.

Large epsilon values (like 9.887844e-03) may indicate regions where the system becomes more chaotic and sensitive to initial conditions, which is typical in complex systems like music compositions.

The Lyapunov exponent λ can often be estimated using the formula:

$$\boldsymbol{\lambda = \lim_{n\to\infty} \frac{1}{n} \ln\left(\frac{\delta x(n)}{\delta x(0)}\right)}$$

The calculated Lyapunov exponent for the given epsilon values is approximately 0.576.

As we see that the overall lyapunov exponent for the musical composition is positive which indicates that the system exhibits chaotic behaviour. This means that small changes in the composition's frequencies lead to unpredictable changes as time progresses, which is characteristic of chaotic behavior. In the context of music, this chaos could manifest as variations in timing, dynamics, or pitch, making the composition feel unpredictable yet structured.

Music for 18 Musicians by Steve Reich – Analysis

The plot of average frequency versus time for Philip Glass's "Music for 18 Musicians" reveals dynamic fluctuations in sound frequencies. The graph shows how the composition transitions between stable regions and bursts of higher frequencies, indicating shifting musical patterns. (Fig – 2)

Steve Reich's music is repetitive and nuanced which makes it really difficult to predict and to some extent is the reason behind its chaotic behaviour.

These patterns resemble fractals, with self-similarity emerging at different scales. Just as in the logistic map, where stability gives way to chaos, the frequency changes in the music reflect this evolution. The piece's structure mirrors the Mandelbrot set, where zooming in reveals repeating patterns, similarly seen in the recurring and evolving musical themes.

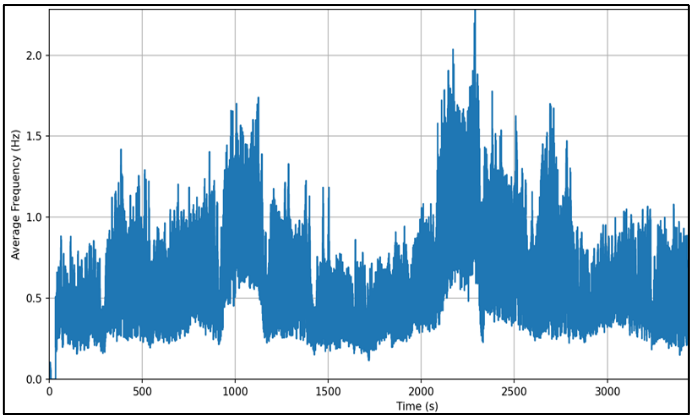


Fig -2: Magnitude Spectrogram of Opening

Analysing Chaos in Steve Reich's Composition with Lyapunov Exponent

Epsilon values by TISEAN using the graph between frequency and time in Fig. 1:

- epsilon = 2.282245e-03
- epsilon = 4.058469e-03
- epsilon = 7.217091e-03
- epsilon = 1.283400e-02
- epsilon = 2.282245e-02

Smaller epsilon values (e.g., 2.282245e-03 and 4.058469e-03) correspond to regions of stability, suggesting that small perturbations in the system lead to similar future behavior.

Larger epsilon values (e.g., 2.282245e-02) indicate regions where the system becomes more chaotic, which is typical of complex systems like music compositions.

For these epsilon values, the calculated Lyapunov exponent is approximately 0.543, confirming that the overall system exhibits chaotic behavior. A positive Lyapunov exponent like this suggests that small differences in initial conditions will lead to exponentially divergent outcomes, a hallmark of chaos. This analysis further emphasizes the intricate, dynamic nature of Music for 18 Musicians, where patterns may evolve unpredictably, much like other chaotic systems.

CONCLUSION

By analyzing the works of composers like Philip Glass and Steve Reich, we observe that their compositions exhibit characteristics similar to mathematical fractals and chaotic systems. The self-similarity in their musical patterns shows the recursive nature of fractals. Through the use of tools such as the Lyapunov exponent and frequency analysis, we were able to quantify the chaotic behavior in both "Opening" and "Music for 18 Musicians." The positive Lyapunov exponents confirmed the presence of chaotic elements. This entire process was not just about technical analysis, but appreciating different layers of mathematics and music around.

Bibliography

- Application of Chaos Theory and Fractal Geometry in the Music of Gyorgy Ligeti by Brian J. Lefresne – May 2005
- Cao, L. (2003). Fractals and Chaos: Introduction for Scientists and Engineers. Cambridge University Press.
- Devaney, R. L. (2012). An Introduction to Chaotic Dynamical Systems. Westview Press.
- Farnell, L. (2016). The Fractal Geometry of Nature. W. H. Freeman and Company.
- Fractal Foundation. "Fractal Foundation." https://fractalfoundation.org – Mandelbrot Set, Logistic Map, Fractals.
- Glass, P. (1993). Opening. In Music in Twelve Parts (pp. 1-10). Universal Edition.
- Reich, S. (1978). Music for 18 Musicians. New York: Daniel Charles.
- Mandelbrot, B. B. (1983). The Fractal Geometry of Nature. W. H. Freeman and Company.
- Strogatz, S. H. (1994). Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering. Westview Press.
- TISEAN. (n.d.). Retrieved from TISEAN Documentation
- Voss, R. F. (1986). Fractals in Nature: From Characterization to Simulation. Proceedings of the National Academy of Sciences, 83(14), 4133-4137.
- Wiggins, S. (2003). Introduction to Applied Nonlinear Dynamical Systems and Chaos. Springer.