

Generalized Kothe-Toeplitz Duals of a New Class of Sequence Spaces

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ABSTRACT. In this paper, we determine the $p\alpha$ -duals of certain newly introduced difference sequence spaces, namely $\ell_\infty(u, \Delta^m, q)$, $c_0(u, \Delta^m, q)$, and $c(u, \Delta^m, q)$. Furthermore, we investigate their topological properties and establish the conditions under which these spaces are perfect.

1. Introduction and preliminaries

Assume that ω denotes the space of all sequences. Throughout this chapter, we adopt the notation $q = (q_k)$ to denote a sequence of positive real numbers. Some generalized sequence spaces, such as $\ell_\infty(q)$, $c(q)$, and $c_0(q)$, have been studied by various authors [9–11]. Ahmad and Mursaleen [1] introduced the concept of some new generalized difference sequence spaces, defined as follows:

$$\Delta \ell_\infty(q) = \{x \in \omega : \Delta x \in \ell_\infty(q)\}.$$

$$\Delta c(q) = \{x \in \omega : \Delta x \in c(q)\}.$$

$$\Delta c_0(q) = \{x \in \omega : \Delta x \in c_0(q)\}.$$

Where $\Delta x_k = x_k - x_{k+1}$. After that, Et and Basarar [8] defined these difference sequence spaces in a more general form by introducing the operator Δ^m on these spaces.

$$\Delta^m \ell_\infty(q) = \{x \in \omega : \Delta^m x \in \ell_\infty(q)\}.$$

$$\Delta^m c(q) = \{x \in \omega : \Delta^m x \in c(q)\}.$$

$$\Delta^m c_0(q) = \{x \in \omega : \Delta^m x \in c_0(q)\}.$$

Where m is any positive integer and $\Delta^0 x_k = x_k$, $\Delta^m x_k = \Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}$ and $\Delta^m x_k = \sum_{i=0}^{m-1} (-1)^i x_{k+i}$. They also obtained the Köthe–Toeplitz duals of the space $\Delta^m \ell_\infty(q)$. In 2004 (see [4]), C. Asma Bektas, gave the β - and γ -duals of the spaces $\Delta^m \ell_\infty(q)$ and $\Delta^m c(q)$. They also gave the α -duals of $\Delta^m c_0(q)$ and $\Delta^m c(q)$. Let $v = (v_k)$ be used to denote any sequence of non-zero complex numbers. Et and Esi [7] extended the concept of difference sequence spaces to introduce the sequence spaces $\Delta^m(X)$ defined as below

$$\Delta^m(X) = \{x \in \omega : \Delta^m x \in X\}.$$

Where $\Delta^m(x_k) = \sum_{i=0}^{m-1} (-1)^i x_{k+i}$ and $\Delta^m(x_k) = \sum_{i=0}^{m-1} (-1)^i v_{k+i} x_{k+i}$. Bektas, Et, and Colak [5] defined the sequence space $\Delta^m(X)$ for $X = \ell_\infty, c, c_0$ and found the β - and γ -duals of these spaces. Let U denote the collection of all sequences consisting of non-zero complex numbers. That is, if $u = (u_k) \in U$ then we must have $u_k \neq 0$ for all k . Throughout in this article,

2020 Mathematics Subject Classification. 46A45, 46B10, 46B45.

Key words and phrases. Sequence spaces, Köthe Toeplitz duals, α -duals, η -duals.

we will use w_k for $|u_k|^{-1}$. By taking a sequence of non-zero terms (u_k) , Malkowsky [12] introduced a new class of difference sequence spaces $X(u, \Delta)$ defined as

$$X(u, \Delta) = \{x \in \omega : (u_k \Delta x_k) \in X\}.$$

Where $u \in U$ and X denotes one of the spaces ℓ_∞, c, c_0 .

Asma and Çolak [2] combined the above concept with generalized sequence spaces and introduced a new class of sequence space $X(u, \Delta, q)$, defined as follows

$$X(u, \Delta, q) = \{x \in \omega : (u_k \Delta x_k) \in X(q)\}.$$

Later, Bektas, A. replaced Δ with Δ^2 and introduced another class of sequence spaces $X(u, \Delta^2, q)$. Many authors [6] [3] have studied the Köthe-Toeplitz duals of such sequence spaces. In 2024, Gülcan A.T. (see [13]) introduced some new sequence spaces, defined below, and also determined the α -, β -, and γ -duals of these spaces.

$$\begin{aligned} \ell_\infty(u, \Delta^m, q) &= \{x \in \omega : (u_k \Delta^m x_k) \in \ell_\infty(q)\} \\ {}^\nu c(u, \Delta^m, q) &= \{x \in \omega : (u_k \Delta^m x_k) \in c(q)\} \\ {}^\nu c_0(u, \Delta^m, q) &= \{x \in \omega : (u_k \Delta^m x_k) \in c_0(q)\} \end{aligned}$$

In the present paper, we continue our work on the spaces $X(u, \Delta^m, q)$ and determine the $p\alpha$ -, $p\beta$ -, and $p\gamma$ -duals of these spaces. We also show that the results of Gülcan A.T. and many other researchers are special cases of our findings. Furthermore, our results generalize all previously known results on these spaces.

2. Definition of the $p\alpha$ -, $p\beta$ - and $p\gamma$ -duals of a sequence space

The definition of $p\alpha$ -, $p\beta$ - and $p\gamma$ - duals of sequence spaces was defined by Et [7]. Let E be a sequence space and $p > 0$, then

$$\begin{aligned} E^{p\alpha} &= \{a = (a_k) : \sum_{k=1}^{\infty} |a_k x_k|^p < \infty, \text{ for each } x = (x_k) \in E\}, \\ E^{p\beta} &= \{a = (a_k) : \sum_{k=1}^{\infty} (a_k x_k)^p \text{ is convergent for each } x = (x_k) \in E\}, \end{aligned}$$

$$E^{p\gamma} = \{a = (a_k) : \sup_n \left| \sum_{k=1}^n (a_k x_k)^p \right| < \infty, \text{ for each } x \in E\}.$$

$E^{p\alpha}$, $E^{p\beta}$ and $E^{p\gamma}$ are called $p\alpha$ -, $p\beta$ - and $p\gamma$ - duals of E . If we replace p with 1 in the above definitions, then we get the α -, β - and γ - duals of E . If $E \subseteq F$ then $F^\eta \subseteq E^\eta$, where η is stand for $p\alpha$ -, $p\beta$ - and $p\gamma$.

3. Main Results

We start our discussion with the following established result, which plays a key role in developing the proof of our main results.

Theorem 3.1. ${}^\nu c_0(u, \Delta^m, q)$, ${}^\nu c(u, \Delta^m, q)$ and ${}^\nu \ell_\infty(u, \Delta^m, q)$ are linear spaces. [13]

Theorem 3.2. Let $q = (q_k)$ be bounded and $M = \max(1, H = \sup_n q_n)$. Then $\ell_\infty(u, \Delta^m, q)$ and $c_0(u, \Delta^m, q)$ are linear topological spaces by g , defined by

$$g(x) = \sup_n |u \Delta^m x|_n^{q_n}$$

Furthermore $c(u, \Delta^m, q)$ is paranormed by g if $\inf_n q_n > 0$. [13]

Theorem 3.3. Let $q = (q_k)$ denote a sequence of positive real numbers. Then

$$(1) \quad [\ell_\infty(u, \Delta^m, q)]_{p\alpha}^{p\alpha} = D_{p\alpha}(u, q) =$$

$$\{a = (a_k) : \sum_{k=0}^{\infty} |a_k|_v^r \left[\sum_{i=0}^{k-m} \binom{k-i-1}{m-1} s^{q_i} w_i \right]^{-r} < \infty\},$$

$$(2) \quad [\ell_\infty(u, \Delta^m, q)]_{p\alpha}^{p\alpha} = D_{p\alpha}(u, q) =$$

$$\{a = (a_k) : \sup_{k \geq m+1} |a_k|_v^r \left[\sum_{i=0}^{k-m} \binom{k-i-1}{m-1} s^{q_i} w_i \right]^{-r} < \infty\}.$$

Proof. Let $x = (x_k)$ be any element of $\ell_\infty(u, \Delta^m, q)$. Then, we have

$$u_k \Delta^m x_k \in \ell_\infty(q) \\ \Rightarrow \sup_k |u_k \Delta^m x_k|^{q_k} < \infty$$

Define an integer s as $s > \max\{1, \sup_k |u_k \Delta^m x_k|^{q_k}\}$. Then

$$\sup_k |u_k \Delta^m x_k|^{q_k} < s \\ \Rightarrow |u_k \Delta^m x_k|^{q_k} < s, \forall k \\ \Rightarrow |u_k| \left| \Delta_v x_k \right| < s^{q_k}, \forall k$$

But $|u_k|^{-1} = w_k$, so we must have

$$|\Delta_v x_k| < s^{q_k} w_k \quad (3.1)$$

Again, let $a = (a_k) \in D_{p\alpha}(u, q)$. Then

$$\sum_{k=1}^{\infty} |a_k|_v^r \left[\sum_{i=0}^{k-m} \binom{k-i-1}{m-1} s^{q_i} w_i \right]^{-r} < \infty \quad (3.2)$$

Now, for $k = 2m, 2m+1, \dots$ and $s > 1$, we can write

$$\sum_{i=1}^{k-m} \binom{k-i-1}{m-1} s^{q_i} w_i \geq \sum_{i=1}^{m-1} \binom{k-i-1}{m-1} s^{q_i} w_i \quad (3.3)$$

Since, we can write x_k as

$$x_k = \frac{1}{v^{k-1}} \left[\sum_{i=1}^{k-m} \binom{k-i-1}{m-1} \Delta_v^m x_i + \sum_{i=1}^{m-1} \binom{k-i-1}{m-1} \Delta_v^{m-i} x_i \right]$$

Consider,

$$\sum_{k=1}^{\infty} |a_k x_k|_v^r = \sum_{k=1}^{\infty} |a_k|_v^r |v_k|^{-r} \left[\sum_{i=1}^{k-m} \binom{k-i-1}{m-1} \Delta_v^m x_i + \sum_{i=1}^{m-1} \binom{k-i-1}{m-1} \Delta_v^{m-i} x_i \right]^r$$

$$\sum_{k=1}^m (-1)^{m-i} \binom{k-i-1}{m-1} \Delta_v^{m-i} x_i |v|^{-r}$$

$$i.e. \sum_{k=1}^{\infty} |a_k x_k|^r \leq \sum_{k=1}^{\infty} |a_k|^r |v_k|^{-r} \sum_{i=1}^{k-m} \frac{k-i-1}{m-1} |\Delta^m x_i|^r +$$

$$|a_k|^r |v_k|^{-r} \sum_{i=1}^m \frac{k-i-1}{m-1} |\Delta^{m-i} x_i|^r$$

using 3.1 in above expression, we get

$$\sum_{k=1}^{\infty} |a_k x_k|^r \leq \sum_{k=1}^{\infty} |a_k|^r |v_k|^{-r} \sum_{i=1}^{k-m} \frac{k-i-1}{m-1} |S^{q_i} w_i|^r +$$

$$|a_k|^r |v_k|^{-r} \sum_{i=1}^m \frac{k-i-1}{m-1} |S^{q_i} w_i|^r, \text{ using 3.2 and 3.3}$$

i.e., $\sum_{k=1}^{\infty} |a_k x_k|^r < \infty$ for all $(x_k) \in \ell_{\infty}(u, \Delta^m, q)_v^{p\alpha}$. This means that (a_k) is an element of $[\ell_{\infty}(u, \Delta^m, q)_v^{p\alpha}]$, but (a_k) is an element of $D_{p\alpha}(u, q)$. Hence, we must have

$$D_{p\alpha}(u, q) \subseteq [\ell_{\infty}(u, \Delta^m, q)_v^{p\alpha}]$$

Now, to complete the proof of the first part of the theorem, we only have to prove that $[\ell_{\infty}(u, \Delta^m, q)_v^{p\alpha}] \subseteq D_{p\alpha}(u, q)$. For this, we let $(a_k) \notin D_{p\alpha}(u, q)$. Then, there exist some integer $s > 1$ such that

$$\sum_{k=1}^{\infty} |a_k|^r |v_k|^{-r} \sum_{i=1}^{k-m} \frac{k-i-1}{m-1} |S^{q_i} w_i|^r = \infty$$

Using the above expression, we can define a sequence, say $x = (x_k)$, where

$$x_k = |a_k|^{-r} \sum_{i=1}^{k-m} \frac{k-i-1}{m-1} |S^{q_i} w_i|^r$$

which can easily be verified to be an element of $\ell_{\infty}(u, \Delta^m, q)$. Then, we have

$$\sum_{k=1}^{\infty} |a_k x_k|^r = \infty$$

This means $(a_k) \notin [\ell_{\infty}(u, \Delta^m, q)_v^{p\alpha}]$. Hence $[\ell_{\infty}(u, \Delta^m, q)_v^{p\alpha}] = D_{p\alpha}(u, q)$. This completes the first part of our theorem. After this, we are ready to prove the second part of the above theorem.

(ii) Let $a = (a_k) \in D_{p\alpha}(u, q)$ and $x = (x_k) \in [\ell_{\infty}(u, \Delta^m, q)_v^{p\alpha}]$. Then, we have

$$\sup_{k \geq m+1} |a_k|^r |v_k|^{-r} \sum_{i=1}^{k-m} \frac{k-i-1}{m-1} |S^{q_i} w_i|^r < \infty \quad (3.4)$$

for some $s > 1$ and

$$\sum_{k=1}^{\infty} |a_k x_k|^r < \infty \quad (3.5)$$

for all $s > 1$. Now, we consider

$$\begin{aligned} \sum_{k=m+1}^{\infty} |a_k x_k|^r &= \sum_{k=m+1}^{\infty} |a_k|^r |x_k|^r \left[\sum_{i=1}^{k-m} s^{qi} w_i \right] \\ &\leq \sum_{k=m+1}^{\infty} |a_k|^r |x_k|^r \left[\sum_{i=1}^{k-m} s^{qi} w_i \right] \\ &= \sup_{k \geq m+1} |a_k|^r |x_k|^r \left[\sum_{i=1}^{k-m} s^{qi} w_i \right] \\ &\quad \times \sum_{k=m+1}^{\infty} |x_k|^r |v_k|^r \left[\sum_{i=1}^{k-m} s^{qi} w_i \right] \end{aligned}$$

$< \infty$, In view of 3.4 and 3.5

i.e., $\sum_{k=m+1}^{\infty} |a_k x_k|^r < \infty$ for all $x = (x_k) \in [\ell_{\infty}(u, \Delta^m, q)]_v^{pa}$. By the definition of pa -duals, we can say that $a = (a_k)$ is an element of $[\ell_{\infty}(u, \Delta^m, q)]_v^{pa, pa}$, but $a = (a_k)$ is any arbitrary element of $D_{pa, pa}(u, q)$. Therefore, we must have

$$D_{pa, pa}(u, q) \subseteq [\ell_{\infty}(u, \Delta^m, q)]_v^{pa, pa}$$

Now, to complete the second part of this theorem, we only need to prove that $[\ell_{\infty}(u, \Delta^m, q)]_v^{pa, pa} \subseteq D_{pa, pa}(u, q)$. To do this, let us suppose that $a = (a_k) \in [\ell_{\infty}(u, \Delta^m, q)]_v^{pa, pa}$. Then, we have

$$\sum_{k=1}^{\infty} |a_k x_k|^r < \infty, \text{ for all } x = (x_k) \in [\ell_{\infty}(u, \Delta^m, q)]_v^{pa}$$

But if $x = (x_k) \in [\ell_{\infty}(u, \Delta^m, q)]_v^{pa}$. Then, by the first part of this theorem, we must have

$$\sum_{k=1}^{\infty} |a_k x_k|^r < \infty \quad (3.6)$$

for all $s \geq 2$. Further, as $\sum_{k=1}^{\infty} |a_k x_k|^r < \infty$. So, we have

$$\begin{aligned} \sum_{k=1}^{\infty} |a_k x_k|^r &\leq \sum_{k=1}^{\infty} |a_k|^r |x_k|^r \left[\sum_{i=1}^{k-m} s^{qi} w_i \right] \\ &\leq \sum_{k=1}^{\infty} |a_k|^r |x_k|^r \left[\sum_{i=1}^{k-m} s^{qi} w_i \right] < \infty \\ &\Rightarrow |a_k|^r |v_k|^r \left[\sum_{i=1}^{k-m} s^{qi} w_i \right] < \infty, \forall n, \text{ By using 3.6} \\ &\Rightarrow \sup_{k \geq m+1} |a_k|^r |v_k|^r \left[\sum_{i=1}^{k-m} s^{qi} w_i \right] < \infty \end{aligned}$$

This means $a = (a_k) \in D_{pa, pa}(u, q)$ for all $a = (a_k) \in [\ell_{\infty}(u, \Delta^m, q)]_v^{pa, pa}$. Hence,

$$[\ell_{\infty}(u, \Delta^m, q)]_v^{pa, pa} \subseteq D_{pa, pa}(u, q)$$

This completes the second part of the theorem. One more thing we can add is that since $[\ell_{\infty}(u, \Delta^m, q)]_v^{pa, pa} \neq \ell_{\infty}(u, \Delta^m, q)$, the space $\ell_{\infty}(u, \Delta^m, q)$ is not perfect. \square

Corollary 3.1. If we put $r = 1$ in theorem 3.3, then we get the α -dual of the space $\ell_\infty(u, \Delta^m, q)$.

$$\begin{aligned} (1) \quad [\ell_\infty(u, \Delta^m, q)]_v^\alpha &= D_\alpha(u, q) = \\ &= \left\{ a = (a_k) : \sum_{k=2}^\infty \frac{|a_k|}{k} \left[\sum_{i=1}^{k-m} \frac{k-i-1}{m-1} \right] \leq \omega \right\}, \\ (2) \quad [\ell_\infty(u, \Delta^m, q)]_v^{\alpha\alpha} &= D_{\alpha\alpha}(u, q) = \\ &= \left\{ a = (a_k) : \sup_{k \geq m+1} \frac{|a_k|}{k} \left[\sum_{i=1}^{k-m} \frac{k-i-1}{m-1} \right] \leq \omega \right\}. \end{aligned}$$

Corollary 3.2. If we put $v_k = 1, \forall k$ in theorem 3.3, then we get the $p\alpha$ -dual of the space $\ell_\infty(u, \Delta^m, q)$.

$$\begin{aligned} (1) \quad [\ell_\infty(u, \Delta^m, q)]^{p\alpha} &= \\ &= \left\{ a = (a_k) : \sum_{k=2}^\infty |a_k|^r \left[\sum_{i=1}^{k-m} \frac{k-i-1}{m-1} \right] \leq \omega \right\}, \\ (2) \quad [\ell_\infty(u, \Delta^m, q)]_v^{p\alpha p\alpha} &= \\ &= \left\{ a = (a_k) : \sup_{k \geq m+1} |a_k|^r \left[\sum_{i=1}^{k-m} \frac{k-i-1}{m-1} \right] \leq \omega \right\}. \end{aligned}$$

4. First and second $p\alpha$ -duals of the space $c_0(u, \Delta^m, q)$

In the next theorem, we will find the first and second $p\alpha$ -duals of the space $c_0(u, \Delta^m, q)$ and prove that this space is also not perfect.

Theorem 4.1. Let $q = (q_k)$ denote a sequence of strictly positive real numbers. Then, we have

$$\begin{aligned} (1) \quad [c_0(u, \Delta^m, q)]_v^{p\alpha} &= M_{p\alpha}(u, q), \text{ where} \\ M_{p\alpha}(u, q) &= \left\{ a = (a_k) : \sum_{k=1}^\infty |a_k|^r |v_k|^{-r} \sum_{i=1}^{k-m} \frac{k-i-1}{m-1} \frac{1}{s^{q_i} w_i} < \infty \right\} \\ (2) \quad [c_0(u, \Delta^m, q)]_v^{p\alpha p\alpha} &= M_{p\alpha p\alpha}(u, q), \text{ where} \\ M_{p\alpha p\alpha}(u, q) &= \left\{ a \in \omega : \sup_{k \geq m+1} |a_k|^r |v_k|^{-r} \sum_{i=1}^{k-m} \frac{k-i-1}{m-1} \frac{1}{s^{q_i} w_i} < \infty \right\}. \end{aligned}$$

Proof. (1) Let $x = (x_k)$ be an arbitrary element of $c_0(u, \Delta^m, q)$. Then

$$\begin{aligned} u_k \Delta^m x_k &\in c_0(q) \\ \Rightarrow |u_k \Delta^m x_k|^{q_k} &\in c_0 \\ \text{i.e., } |u_k \Delta^m x_k|^{q_k} &\rightarrow 0 \text{ as } k \rightarrow \infty. \end{aligned}$$

This means there must exist a positive integer $s > 1$ such that

$$\begin{aligned} |u_k \Delta^m x_k|^{q_k} &< \frac{1}{s} \\ \text{i.e., } |u_k \Delta^m x_k| &< \frac{1}{s^{q_k}} \\ \Rightarrow |\Delta_v x_k| &< s^{q_k} w_k. \end{aligned} \tag{4.1}$$

Again, let $a = (a_k) \in M_{pa}(u, q)$. Then

$$\sum_{k=1}^{\infty} |a_k|^r |v_k|^{-r} \sum_{i=1}^{k-m} \frac{(-1)^{k-i-1}}{s^{\frac{1}{q_i}} w_i} < \infty \quad (4.2)$$

for some $s > 1$. Since

$$\sum_{i=1}^{k-m} \frac{(-1)^{k-i-1}}{s^{\frac{1}{q_i}} w_i} \geq \sum_{i=1}^{m-1} \frac{(-1)^{k-i-1}}{s^{\frac{1}{q_i}} w_i} \quad (4.3)$$

Then

$$\begin{aligned} \sum_{i=1}^{\infty} |a_i x_i|^r &= \sum_{i=1}^{\infty} |a_i|^r |v_i|^{-r} \sum_{i=1}^{k-m} \frac{(-1)^{k-i-1}}{s^{\frac{1}{q_i}} w_i} \Delta_v x_i \\ &+ \sum_{i=1}^{\infty} (-1)^{m-1} \Delta_v x_i \\ &\leq \sum_{i=1}^{k-m} |a_i|^r |v_i|^{-r} \sum_{i=1}^{k-m} \frac{(-1)^{k-i-1}}{s^{\frac{1}{q_i}} w_i} |\Delta_v^m x_i| \\ &+ \sum_{i=1}^{\infty} |a_i|^r |v_i|^{-r} \sum_{i=1}^m (-1)^{m-i} \frac{(-1)^{k-i-1}}{s^{\frac{1}{q_i}} w_i} |\Delta_v^{m-i} x_i| \\ &\leq \sum_{i=1}^{\infty} |a_i|^r |v_i|^{-r} \sum_{i=1}^{k-m} (-1)^{m-i} \frac{(-1)^{k-i-1}}{s^{\frac{1}{q_i}} w_i} \\ &+ \sum_{i=1}^{\infty} |a_i|^r |v_i|^{-r} \sum_{i=1}^m (-1)^{m-i} \frac{(-1)^{k-i-1}}{s^{\frac{1}{q_i}} w_i}, \text{ see 4.1} \\ &< \infty, \text{ see 4.2 and 4.3} \end{aligned}$$

i.e., $\sum_{i=1}^{\infty} |a_i x_i|^r < \infty$, for all $x = (x_k) \in c_0(u, \Delta^m, q)$.

$$\Rightarrow a = (a_k) \in [c_0(u, \Delta^m, q)]^{pa}.$$

Hence,

$$M_{pa}(u, q) \subseteq [c_0(u, \Delta^m, q)]^{pa}. \quad (4.4)$$

To complete the first part of this theorem, we only need to prove that $[c_0(u, \Delta^m, q)]^{pa} \subseteq M_{pa}(u, q)$. For this, let us suppose that $a = (a_k) \in [c_0(u, \Delta^m, q)]^{pa}$. Then, we must have

$$\sum_{i=1}^{\infty} |a_i x_i|^r < \infty, \text{ for all } x = (x_k) \in c_0(u, \Delta^m, q). \quad (4.5)$$

Define a sequence as follows:

$$x_k = \frac{(-1)^{k-m}}{s^{\frac{1}{q_i}} w_i} \sum_{i=1}^{k-m} \frac{(-1)^{k-i-1}}{s^{\frac{1}{q_i}} w_i} \quad (4.6)$$

We can easily verify that $x = (x_k)$ is an element of $c_0(u, \Delta^m, q)$. In view of 4.5 and 4.6, we have

$$\sum_{k=1}^{\infty} |a_k|^r |v_k|^{-r} \sum_{i=1}^{k-m} \frac{(-1)^{k-i-1}}{s^{\frac{1}{q_i}} w_i} < \infty.$$

$$\Rightarrow a = (a_k) \in M_{pa}(u, q).$$

$$\Rightarrow [c_0(u, \Delta^m, q)]^{pa} \subseteq M_{pa}(u, q). \quad v \quad (4.7)$$

From 4.4 and 4.7, we have

$$[c_0(u, \Delta^m, q)]^{pa} = M_{pa}(u, q). \quad v$$

□

This completes the first part of the theorem. Now, we will prove the second part and then verify that this space is not perfect.

Proof. (2) Let $a = (a_k) \in [c_0(u, \Delta^m, q)]^{pa}_{v^{pa}}$. Then we have

$$\sum_{k=m+1}^{\infty} |a_k x_k|^r < \infty, \quad (4.8)$$

for all $x = x_k \in [c_0(u, \Delta^m, q)]^{pa}_v$. Now, using the first part of this theorem, we can write

$$\sum_{k=1}^{\infty} |x_k|^r |v_k|^{-r} \sum_{i=1}^{k-m} \frac{k-i-1}{s^{qi} w_i} < \infty \quad (4.9)$$

for some $s \geq 2$. We can rewrite 4.8 as follows:

$$\begin{aligned} & \sum_{k=m+1}^{\infty} |a_k|^r |v_k|^r \sum_{i=1}^{k-m} \frac{k-i-1}{s^{qi} w_i} |x_k|^r |v_k|^{-r} \\ & \Rightarrow \sup_{k \geq m+1} |a_k|^r |v_k|^r \sum_{i=1}^{k-m} \frac{k-i-1}{s^{qi} w_i} \sum_{k=m+1}^{\infty} |x_k|^r |v_k|^{-r} \times \\ & \sum_{i=1}^{k-m} \frac{k-i-1}{s^{qi} w_i} < \infty. \end{aligned}$$

Which, in view of 4.9, gives

$$\sup_{k \geq m+1} |a_k|^r |v_k|^r \sum_{i=1}^{k-m} \frac{k-i-1}{s^{qi} w_i} < \infty.$$

This means $a = (a_k) \in M_{pa,pa}(u, q)$. Hence

$$[c_0(u, \Delta^m, q)]^{pa,pa} \subseteq M_{pa,pa}(u, q). \quad v \quad (4.10)$$

To complete the theorem, we claim that $M_{pa,pa}(u, q) \subseteq [c_0(u, \Delta^m, q)]^{pa,pa}_v$. To do so, let us suppose that $a = (a_k)$ is an arbitrary element of $M_{pa,pa}(u, q)$. Then, we have

$$\sup_{k \geq m+1} |a_k|^r |v_k|^r \sum_{i=1}^{k-m} \frac{k-i-1}{s^{qi} w_i} < \infty. \quad (4.11)$$

Let $x = (x_k)$ be any element of $[c_0(u, \Delta^m, q)]^{p\alpha}_v$. Using the result from the first part of this theorem, we can write

$$\sum_{k=1}^{\infty} |x_k|^r |v_k|^{-r} \sum_{i=1}^{k-m} \frac{k-i-1}{m-1} \frac{1}{s^{\frac{1}{q_i}} w_i} < \infty. \quad (4.12)$$

Consider

$$\begin{aligned} \sum_{k=m+1}^{\infty} |a_k x_k|^r &< \sum_{k=m+1}^{\infty} |a_k|^r |v_k|^r \sum_{i=1}^{k-m} \frac{k-i-1}{m-1} \frac{1}{s^{\frac{1}{q_i}} w_i} \\ &\times \sum_{i=1}^{k-m} \frac{k-i-1}{m-1} \frac{1}{s^{\frac{1}{q_i}} w_i} \\ &\leq \sup_{k=m+1}^{\infty} |a_k|^r |v_k|^r \sum_{i=1}^{k-m} \frac{k-i-1}{m-1} \frac{1}{s^{\frac{1}{q_i}} w_i} \\ &\times \sum_{k=m+1}^{\infty} |x_k|^r |v_k|^{-r} \sum_{i=1}^{k-m} \frac{k-i-1}{m-1} \frac{1}{s^{\frac{1}{q_i}} w_i} \\ &< \infty \end{aligned}$$

i.e., $\sum_{k=m+1}^{\infty} |a_k x_k|^r < \infty$, for all $x = (x_k) \in [c_0(u, \Delta^m, q)]^{p\alpha}_v$.

$$\Rightarrow a = (a_k) \in [c_0(u, \Delta^m, q)]^{p\alpha p\alpha}.$$

Hence,

$$[c_0(u, \Delta^m, q)]^{p\alpha p\alpha} = M_{p\alpha p\alpha}(u, q).$$

□

Corollary 4.1. If we replace r with 1 in Theorem 4.1, then we obtain the α -dual of the space $c_0(u, \Delta^m, q)$ [13]

(1) $[c_0(u, \Delta^m, q)]^{\alpha}_v = M_{\alpha}(u, q)$, where

$$M_{\alpha}(u, q) = \left\{ a = (a_k) : \sum_{k=1}^{\infty} |a_k| |v_k|^{-1} \sum_{i=1}^{k-m} \frac{k-i-1}{m-1} \frac{1}{s^{\frac{1}{q_i}} w_i} < \infty \right\}.$$

(2) $[c_0(u, \Delta^m, q)]^{\alpha\alpha}_v = M_{\alpha\alpha}(u, q)$, where

$$M_{\alpha\alpha}(u, q) = \left\{ a \in \omega : \sup_{k \geq m+1} |a_k| |v_k| \sum_{i=1}^{k-m} \frac{k-i-1}{m-1} \frac{1}{s^{\frac{1}{q_i}} w_i} < \infty \right\}.$$

This is exactly the same result as in Theorem 2.4 (see [13]). This means that, by this theorem, we have generalized the previously existing results.

Corollary 4.2. If we take the sequence $v = (v_k)$ as $v_k = 1$ for all k , then we get the $p\alpha$ -duals of the space $c_0(u, \Delta^m, q)$.

(1) $[c_0(u, \Delta^m, q)]^{p\alpha} = M_{p\alpha}(u, q)$, where

$$M_{p\alpha}(u, q) = \left\{ a = (a_k) : \sum_{k=1}^{\infty} |a_k|^r \sum_{i=1}^{k-m} \frac{k-i-1}{m-1} \frac{1}{s^{\frac{1}{q_i}} w_i} < \infty \right\}.$$

(2) $[c_0(u, \Delta^m, q)]^{p\alpha p\alpha} = M_{p\alpha p\alpha}(u, q)$, where

$$M_{p\alpha p\alpha}(u, q) = \{a \in \omega : \sup_{s=2}^{\infty} \sum_{k \geq m+s}^{k-m} |a_k|^r \sum_{i=1}^{k-1} i^{-1} < \infty\}.$$

Conflict of Interests: The authors have no conflicts of interest to declare that are relevant to the content of this article.

Data Availability Statement: There is no data set associated with this manuscript.

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