Harshad Number: A Number is Divisible by its Digit Sum, An Divisibility Approach by Considering the Digit-sum to be Seven

Banashree Sen [a,b], Tarit Kr Majumder[b], Payal Sen[b]

[(a)] Department of Commerce and Management, St. Xavier's University, Newtown-700160, WB, India

[(b)] Department of Applied Mathematics, Maulana Abul Kalam Azad University of Technology, Haringhata-741249, WB, India

Email: banashreesen7@gmail.com

Abstract: A Harshad number, also known as a Niven number, is a positive integer that is divisible by the sum of its digits. In other words, if one can add up all the digits of that number and divide the number by that sum, the result will be a whole number. We will exclude the trivial cases where the summative number itself present in that digit; otherwise in most of the cases the number will follow the properties of Harshad number[1,2]. In this article, excluding the trivial cases, we tried to extend our concept which usually maintained in Harshad number, in an advanced way. The advanced concept of Harshad number deals with such kind of expression of the digits where the summative result of combined digit will not be present anywhere. This type of number also known as Harshad number [1,3] with an exclusion of total non-minor cases. This type of number does not occur in units, tens but in hundreds if the choice of digit sums chosen to be seven. For the choice of division factor[2,4] to be 7 we take trial and error method pertaining the outcome to be a non-trivial and non-containing digit sum number. The first digit of this kind satisfies in hundred places and the number in particular is 322. We also found that the next such number is next appearing in thousand digits and that is 1016. And such kind of number used to maintain in a sequential manner. We also tried to exhibit the pattern of Harshad [1,2,3] Sequence in our article.

Keywords: Harshad Number (HN), Digit-Sum, Divisibility.

1. Historical Background with the development of Harshad Number: Harshad numbers are being named after D. R. Kaprekar, a mathematician from India who discovered them in the year 1940. Harshad is a Sanskrit word that means "Harsha-great joy" or "great pleasure," which reflects the fact that Harshad numbers have some interesting mathematical properties [1,2,4]. The word "harshad" comes from the Sanskrit "*harsa"*(joy) + "*da"*(give), meaning joy- giver. For example, the number 18 is a Harshad number because the sum of its digits (1 + 8) is 9, 18 is divisible by 9. Also it will worth noticing that 9 is not present in the digit 18. Other examples of Harshad numbers include 10, 12, 20, and 27. Harshad numbers [2] have some interesting properties and have been studied in mathematics especially in the branch of number theory. The number also has applications in digital root calculations and the impact can be used in some crypto graphical algorithms. The term "Niven Number" arose from a paper delivered by **Ivan M. Niven [3,5]** at a conference on number theory in 1977.

2. Introduction: In this article, we review the theory for Harshad Number divisibility [2, 3, 5] by its digit sum concept as this year 2023 is itself a Harshad Number (HN). A Harshad Number [1] is **a positive integer that is divisible by the sum of its digits**. **In particular, the sum of an integer's digits is called the digit sum**. To explain it a little more mathematically, the digit sum is a divisor (factor) of the number, then that kind of number is technically induced as Harshad Number. The main motivation comes from the year 2023 and involves us to study further. The year 2023 is such a number which is exactly divisible by the sum of its digits. We can notice further that this kind of numbers occur in a sequential manner one may verify that 2022, 2024 & 2025 are all HNs, but in all these cases the sum of the digits is not prime whereas, starting from the year 1014, 1015, 1016, 1017 are also in the same category of that sequence [5,6] after being a part of Harshad number. Four consecutive years turning out to be HN is quite rare. So, let us have HARSH over it. But it becomes quite rare when the sum of the digits is a prime. As in 2023, we have

$$2 + 0 + 2 + 3 = 7, and\ 2023 = 7 \times 289.$$

In this project, we have two different matters to be taken into account. Firstly, we have been aimed to find out the least number, which is special type of **Harshad Number** [1,3,5], whose digit-sum and the number itself is divisible by 7 with a restriction that 7 will not appear in the number. Our second limitation is that we have restricted our study for exposing the divisibility criteria of the digit 'Seven'. There is some sort of significance for choosing this particular number but this study can be extended to any other digit for extracting the generalised divisible criteria of the other numbers.

In this project we propose to find a special type of Harshad Number for which sum of the digit and the number as well is divisible by 7 with a restriction that the digit 7 never appears in the number. In fact, otherwise the problem is trivial because all the number of the form 7,77,777... automatically satisfies the condition.

Now we will through light on **"Why 7 is selected?".** As the number 7 has magical appearance in many places. The most enigmatic irrational number π[pi] is taken to approximated as $\frac{22}{7}$. It has its popular appearance in every religious and sociological field as well as in the domain of mathematics. As an example we can say 7 days a week, 7 oceans & continent in world, 7 colours in rainbow, 7 gates of heaven, 7 notes in music. The number 7 is possibly the most interesting digits out of the first 10 digits. If any person is asked to give a number out of the first nine digits [without giving her much time to think] the reply in most of the case 7.Mathematically, 7 has the mystery behaviour. Each of six-digital numbers constructed by one-nine in the adjacent chart is divisible by 7.

111 111
222 222
333 333
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888 888
999 999

Now we will organise the manuscript in two different orientations. Firstly, the details of Harshad Number (HN) have been emphasised. The overview of HN is connected with the theory of divisibility by its digit-sum [3, 4]. In the second part we will elaborate the tricks of division by the digit seven. In the subsection the Harshad Number has been explored in a trial error basis. A generalisation has been termed to create. Why we choose the number and what is the beauty of theory [7,8] behind the division by 'seven' we will discuss in the ending note. The chapter will end with a conclusion highlighting the completeness of Harshad number in the wings of mathematics i.e. Number Theory [4,6]. Also the future study can be distributed using other numbers apart from seven.

3. Mathematical Beauty within Harshad Number:

The target in this article is to find the lowest Harshad Number (HN) [6-8]by the method of trial excluding the trivial cases. The trivial case means all the following $a_i's$ $where$ $a_i \neq 7 \; \forall \; 0 \leq i \leq n$. If 7 occurs then the digit sum probes a tendency to be divisible by 7 and sometimes the property of Harshad number. In the trivial scene the digit often fails the division by 7. The most famous HN is the Ramanujan Number, 1729, which happens to be the least number that can be expressed as the sum of two cubes in two different ways. We can explain the same as follows

$1 + 7 + 2 + 9 = 19, and\; 9^3 + 10^3 = 1729 = 19 \times 91$

As we know any number can be arranged by the expression as follows

$$N = [a_n a_{n-1}, \cdots\cdots\cdots a_1 a_0],\; 0 \leq a_i \leq 9,\; a_n \neq 0$$

then, we also assume that no digit of the number is 7, i.e., $a_n \neq 7$.

Then the number can be rewritten as

$$N = 10^n a_n + 10^{n-1} a_{n-1} + \cdots\cdots + 10a_1 + a_0$$

Our problem is to determine a solution for the digits $a_n,\; a_{n-1},\; \cdots\cdots\cdots a_1, a_0$ for which

Case 1: 7 divides N, i.e.,

$10^n a_n + 10^{n-1} a_{n-1} + \cdots\cdots + 10a_1 + a_0 = 7k$ for some integer k.

Case 2: 7 divides the sum of the digits, i.e.,

$a_n + a_{n-1} + \cdots\cdots + a_1 + a_0 = 7t$ for some integer t.

3.1 Heuristic Approach In Search of Lowest Harshad Number, A Generalised Procedure Starting from Unit Digit Number with a Digit Sum Divisible by 7:

We proceed with considering the situations with different values of n

- ***Situation-1*** ***When n = 1***

Under this situation we have only one digit number. There are two restrictions that the sum of digits and the number as well is divisible by 7 only when the number is itself 7[11, 12].

But we are to ignore this case because the digit 7 cannot appear in the number.

- ***Situation-2*** ***When n = 2***

Under this situation we have two digit numbers.

Let it be $[a_2\,, a_1]$

$\therefore$ The number is $(10a_2 + a_1)$

Sum of the digit $a_2 + a_1$.

$\therefore$ We required to find possible digit $a_2\,, a_1$ such that

(i) $7 | \, 10a_2 + a_1$

(ii) $7|\ a_2 + a_1$

Now, $10a_2 + a_1 = 7a_2 + 2a_2 + a_2 + a_1$

Here $7|\ 7a_2$ and $7|\ a_2 + a_1$

$7|\ 10a_2 + a_1$ if $7|\ 2a_2$.

But we can't find any value of a_2 for which $7|\ 2a_2$.

So, we are to ignore this case.

- ***Situation-3:*** ***When n = 3***

Under this situation we have three digit numbers.

Let it be [a_3 , a_2, a_1]

∴ *The number is* $(100a_3 + 10a_2 + a_1)$

Sum of the digit $a_3 + a_2 + a_1$.

∴ *We required to find possible digit* a_3 , a_2 , a_1 *such that*

(i) $7|\ 100a_3 + 10a_2 + a_1$

(ii) $7|\ a_3 + a_2 + a_1$

Now, $100a_3 + 10a_2 + a_1 = 98a_3 + a_3 + 7a_2 + 2a_2 + a_1 + a_3 + a_2$

Here $7|98a_2$, $7|\ 7a_2$ *and* $7|\ a_3 + a_2 + a_1$

$7|\ 100a_3 + 10a_2 + a_1$ *if* $7|a_3 + 2a_2$.

Now we can find the value of a_3 , a_2, a_1 *by trial and error by imposing all kind of combination for the numbers we can conclude 322 is the right combination and lowest number divisible by its digit sum(7) and claimed itself to the category of Harshad number*.322 is the number in 3 digit which is divisible by 7 and the sum of the digit [13, 14] also 7, Also 7 did not appear in the digit 322 as well. So, 322 is the first number whose sum of the digits and the number is also divisible by 7 with a restriction that no digit is 7.

3.2 Specified Harshad Number with Sum of Digit Theorised as a Multiplier of 7:

In this article, we have been made to find the least number, which is special type of **Harshad Number** [2,3,6], whose digit sum and the number itself initially is divisible by 7 with a restriction that 7 will not appear in the number. In this distribution we shall show that 322 is the least such number.

Another interesting but worthy matter of this number theoretical experiment lies in the choosing of 'digit-7' for pursuing the divisibility. We tried to show that, If a number actually any number can be expressed in the form $10a + b$ and additionally [2,3] it is divisible by 7,where a is the root number and b is the last digit ,then $a - 2b$ is also divisible by 7. And we see that any integer is divisible by 7 in easily in the next proof.

- ***If $10a + b$ is divisible by7, where a is the root number and b is the last digit, then prove that $a - 2b$ is also divisible by 7.***

Proof : Let a is the root number and b is the last digit. We can express any length of number after its unit place as a multiple of 10 so, $10a + b$ is the whole number as desired.

If the whole number $10a + b$ is divisible by 7 , $then$ $(3a + b)$ is also multiple of 7.

$$10a + b = 7a + (3a + b)$$

where 1st part '7a' trivially and 2nd part '(3a+b)' by assumption follows the divisibility by 7. Now,

$(3a + b) - 7b = 3a - 6b$ is also a multiple 7.

That proves $3a - 6b$ is divisible by 7.since 3 and 6 are both multiplies of 3. So we can claim that, $a - 2b$ is also divisible by 7 {since gcd $(3,7) = 1$}.

Example Algorithm: To test if an integer is divisible by 7 or not,

1. Take off the last digit and multiply it by 2.
2. Subtract that product from the root number (the integer without the last digit).
3. Repeat the second steps again and again until it is clearly that the number is divisible by 7 or not.

4 Concluding Remarks:

Some interesting facts about Harshad Numbers were discussed in this article with examples. We have seen how it was possible to develop and propose a mathematical formula capable of providing Harshad numbers. To achieve this result we use some elementary properties [8-10] of natural numbers and their implications by the method of direct demonstration. It is worth emphasizing that this formula allows to obtain Harshad numbers by moving from any multiple of a given positive integer 7 provided that the number of digits[8,11,12] of this multiple does not exceed 7. The digit seven has been used for the generalisation of divisible property.

5 Acknowledgement

PS is thankful to Maulana Abul Kalam Azad University of Technology (MAKAUT) for providing her an opportunity to pursue a term project in the curriculum under the MSc program. BS is thankful to MAKAUT to undergo this sort of project work. BS is also grateful to St. Xavier's University. TKM is thankful to MAKAUT for research and project support. The author PS is grateful for the financial support from 'Swami Vivekananda Merit-Cum- Means (SVMCM)' and BS, Prof TKM declare that they have no competing interest.

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