

Integer Solutions of Bi quadratic Equation $xy + (\sigma^2 + 1)z^2 = 65w^4$

¹P. Sangeetha and ²G. Janaki

¹ Assistant Professor, ² Associate Professor PG and Research Department of Mathematics, Cauvery College for Women (Autonomous), Affiliated to Bharathidasan University, Tiruchirappalli-24, India.

E mail ID: psangeethashree@gmail.com and janakikarun@rediffmail.com

Abstract: This study focuses on the problem of identifying non-zero unique integer solutions to the bi-quadratic equation $xy + (\sigma^2 + 1)z^2 = 65w^4$. A few interesting connections between the solutions are displayed. There are integer solutions for various transformation possibilities given an integer solution to the given problem.

Keywords: Bi-Quadratic Diophantine equation, integral solutions, Polygonal numbers.

Introduction:

A Diophantine equation is a polynomial equation where the solutions are required to be integers. Named after the ancient Greek mathematician Diaphanous, these equations explore the deep relationship between algebra and number theory. A Bi-quadratic Diophantine equation is a specific type of Diophantine equation where the highest degree of the variable is four. These equations are more challenging than linear or quadratic Diophantine equations due to their higher degree and multiple variables. They appear in problems involving Pythagorean quadruples, elliptic curves, and modular arithmetic.

In [1-3], theory of numbers were discussed. In [4-6] problem have been discussed for its integral solutions for basic quadratic Diophantine equations. In [7-13], higher order equations are considered for integral solutions. In this communication, the non-homogeneous Biquadratic equation with five unknowns represented by the equation $xy + (\sigma^2 + 1)z^2 = 65w^4$ is considered and in particular a few interesting relations among the solutions are presented.

Notations

CH_n = Centered Hexagonal number of rank n.

RD_n = Rhombic dodecagonal number of rank n.

TO_n = Truncated octahedral number of rank n.

TT_n = Truncated tetrahedral number of rank n

HO_n = Haüy Octahedral number of rank n

$Star_n$ = Star number of rank n.

CS_n = Centered square number of rank n.

CC_n = Centered cube number of rank n.

Gno_n = Gnomonic number of rank n

Method of Analysis: -

The Diophantine equation representing the bi-quadratic equation under consideration is

$$xy + (\sigma^2 + 1)z^2 = 65w^4 \quad (1)$$

Different solution patterns to (1) be presented below.

Pattern 1:

Introducing the transformations

$$x = u + \sigma v, \quad y = u - \sigma v \quad \text{and} \quad z = v \quad (2)$$

In (1), it simplifies to

$$u^2 + v^2 = 65w^4 \quad (3)$$

Substituting $w = a^2 + b^2$ in (3) and choosing

$$65 = (8+i)(8-i), \text{ it becomes}$$

$$(u + iv)(u - iv) = (8+i)(8-i)(a^2 + b^2)^4 \quad (4)$$

$$\text{Let us define } u + iv = (8+i)(a + ib)^4 \quad (5)$$

by equating real and imaginary parts, we get

$$\left. \begin{aligned} u &= 8a^4 + 8b^4 - 48a^2b^2 - 4a^3b + 4ab^3 \\ v &= a^4 + b^4 - 6a^2b^2 + 32a^3b - 32ab^3 \end{aligned} \right\} \quad (6)$$

Using these values of u and v in (2), the integral solutions to (1) are obtained as,

$$\left. \begin{aligned} x(a, b, \sigma) &= (8 + \sigma)a^4 + (8 + \sigma)b^4 - 6(8 + \sigma)a^2b^2 - 4(1 - 8\sigma)a^3b + 4(1 - 8\sigma)ab^3 \\ y(a, b, \sigma) &= (8 - \sigma)a^4 + (8 - \sigma)b^4 - 6(8 - \sigma)a^2b^2 - 4(1 + 8\sigma)a^3b + 4(1 + 8\sigma)ab^3 \\ z(a, b) &= a^4 + b^4 - 6a^2b^2 + 32a^3b - 32ab^3 \\ w(a, b) &= a^2 + b^2 \end{aligned} \right\} \quad (7)$$

Properties: -

(i) The following relations are perfect squares: -

1. $y(a, a, 1) + z(a, a) - x(a, a, 1)$
2. $y(a, a, 1) - 7z(a, a) - x(a, a, 1)$
3. $x(a, a, 1) + z(a, a) - 2y(a, a, 1)$
4. $3[y(a, a, 1) - x(a, a, 1) - z(a, a)]$
5. $2x(a, a, 1) - y(a, a, 1) - 12z(a, a)$
6. $-2x(a, a, 1) - y(a, a, 1)$

$$(ii) \quad x(a, 1, 1) - y(a, 1, 1) - 2z(a, 1, 1) \equiv 0$$

$$(iii) \quad y(a, 1, 1) - 7z(a, 1) + 130 SO_a \equiv 0 \pmod{130}$$

$$(iv) \quad x(a, 1, 1) - 9z(a, 1) + 130 PP_a - 130 T_a \equiv 0$$

$$(v) \quad x(a, 1, 1) - 7z(a, 1) - w(a, 1) + 43(OH_a) + PP_a - Pr o_a - 194 Gno_a \equiv 0 \pmod{193}$$

Pattern 2: -

By taking $65 = \left(\frac{32+74i}{10}\right)\left(\frac{32-74i}{10}\right)$ in (3), we get

$$(u+iv)(u-iv) = \left(\frac{32+74i}{10}\right)\left(\frac{32-74i}{10}\right)(a^2+b^2)^4 \quad (8)$$

Now we define

$$(u+iv) = \left(\frac{32+74i}{10}\right)(a+ib)^4 \quad (9)$$

Since our aim is to find integral solutions,

Let us take $a = 10A$ and $b = 10B$. Then the solutions are given by,

On equating real and imaginary parts we get,

$$\left. \begin{aligned} u &= 32000A^4 + 32000B^4 - 192000A^2B^2 - 296000A^3B + 296000AB^3 \\ v &= 74000A^4 + 74000B^4 - 444000A^2B^2 + 128000A^3B - 128000AB^3 \end{aligned} \right\} \quad (10)$$

Then the integral solutions to (1) are given by

$$\begin{aligned} x(A, B, \sigma) &= (32000 + 74000\sigma)A^4 + (32000 + 74000\sigma)B^4 - (192000 + 444000\sigma)A^2B^2 \\ &\quad - (296000 - 128000\sigma)A^3B + (296000 - 128000\sigma)AB^3 \\ x(A, B, 1) &= 106000A^4 + 106000B^4 - 636000A^2B^2 - 168000A^3B + 168000AB^3 \\ y[A, B, (\sigma = 1)] &= -42000A^4 - 42000B^4 + 252000A^2B^2 - 424000A^3B + 424000AB^3 \\ z(A, B) &= 74000A^4 + 74000B^4 - 444000A^2B^2 + 128000A^3B - 128000AB^3 \\ w(A, B) &= 1000A^2 + 1000B^2 \end{aligned}$$

Properties: -

$$(i) 4[x(A, 1, 1) - z(A, 1)] = 2[x(A, 1, 1) + y(A, 1, 1)]$$

$$(ii) 2z(A, 1) + y(A, 1, 1) - x(A, 1, 1) \equiv 0$$

$$(iii) 106w(A^2, 1) - x(A, 1, 1) - 10500 TO_A - 327500 CH_A - 281250 Gno_A \equiv 0 \pmod{16750}$$

$$(iv) 232000 Gno_A - 106000 RD_A - 64000 Star_A - [y(A, 1, 1) + 42w(A^2, 1)] \equiv 0 \pmod{190000}$$

$$(v) z(A, 1, 1) - 74w(A^2, 1) - 32000 RD_A - 126000 CS_A + 254000 Gno_A \equiv 0 \pmod{96000}$$

Pattern 3:-

$$u^2 + v^2 = 65w^4$$

Consider (1) as $1 = \frac{(4+3i)(4-3i)}{5^2}$ in (3), we get

$$u^2 + v^2 = (65 * 1)w^4$$

We define,

$$(u + iv) = (8 + i) \left(\frac{4 + 3i}{5} \right) (a + ib)^4 \quad (11)$$

equating real and imaginary parts we get

$$\left. \begin{aligned} u &= \frac{1}{5} [29a^4 - 174a^2b^2 - 112a^3b + 112ab^3 + 29b^4] \\ v &= \frac{1}{5} [28a^4 - 168a^2b^2 + 116a^3b - 116ab^3 + 28b^4] \end{aligned} \right\} \quad (12)$$

with these values of u and v , the solutions x, y, z and w of (1) are

$$\left. \begin{aligned} x(a, b, \sigma) &= \frac{1}{5} [(29 + 28\sigma)a^4 - 4(29\sigma - 28)ab^3 + 4(29\sigma - 28)a^3b - 6(29 + 28\sigma)a^2b^2 + (29 + 28\sigma)b^4] \\ y(a, b, \sigma) &= \frac{1}{5} [(29 - 28\sigma)a^4 - 4(29\sigma + 28)a^3b - 6(29 - 28\sigma)a^2b^2 + 4(29\sigma + 28)ab^3 + (29 - 28\sigma)b^4] \\ z(a, b) &= \frac{1}{5} [28a^4 - 168a^2b^2 + 116a^3b - 116ab^3 - 28b^4] \\ w(a, b) &= [a^2 + b^2] \end{aligned} \right\} \quad (13)$$

Since our aim is to find integral solutions,

Let us take $a = 5A$ and $b = 5B$. Then the solutions are given by,

$$\left. \begin{aligned} x(A, B, \sigma) &= 125(29 + 28\sigma)A^4 + 500(29\sigma - 28)A^3B - 750(29 + 28\sigma)A^2B^2 \\ &\quad - 500(29\sigma - 28)AB^3 + 125(29 + 28\sigma)B^4 \\ y(A, B, \sigma) &= 125(29 - 28\sigma)A^4 - 500(29\sigma + 28)A^3B - 750(29 - 28\sigma)A^2B^2 \\ &\quad + 500(29\sigma + 28)AB^3 + 125(29 - 28\sigma)B^4 \\ z(A, B) &= 3500A^4 - 21000A^2B^2 + 14500A^3B - 14500AB^3 + 3500B^4 \\ w(A, B) &= 25A^2 + 25B^2 \end{aligned} \right\} \quad (14)$$

Properties: -

(i) The following equations are all perfect squares: -

- $9y(1, 1, 1) + z(1, 1) - x(1, 1, 1)$
- $24y(1, 1, 1) - x(1, 1, 1) + z(1, 1)$
- $x(1, 1, 1) - 13y(1, 1, 1) + z(1, 1)$ and
- $5[x(1, 1, 1) - 2y(1, 1, 1) - z(1, 1)]$

$$(ii) \quad x(1, 1, 1) - y(1, 1, 1) - 2z(1, 1) \equiv 0$$

$$(iii) \quad x(A, 1, 1) - 57y(A, 1, 1) - 812500 \quad SO_A \equiv 0 \pmod{812500}$$

$$(iv) \quad z(A, 1) - 28y(A, 1, 1) - 406250 \quad P_A^5 \equiv 0 \pmod{1625000}$$

$$(v) \ x(1,1,1) - y(1,1,1) - 2z(1,1) + w(A,1) - 5T_{12,a} \equiv 0 \pmod{20}$$

Pattern 4:-

$$\text{Let us choose (1) as } 1 = \frac{(3+4i)(3-4i)}{25}$$

$$\text{Then (3) can be written as } u^2 + v^2 = (65 * 1)w^4$$

$$(u+iv)(u-iv) = [(8+i)(8-i)] \left[\frac{(3+4i)(3-4i)}{25} \right] (a^2 + b^2)^4 \quad (15)$$

Now we define,

$$u+iv = (8+i) \left(\frac{3+4i}{5} \right) (a+ib)^4$$

By equating real and imaginary parts on both sides, we obtain the values of u and v as

$$\left. \begin{aligned} u &= \frac{1}{5} [20a^4 + 20b^4 - 120a^2b^2 - 140a^3b + 140ab^3] \\ v &= \frac{1}{5} [35a^4 + 35b^4 - 210a^2b^2 + 80a^3b - 80ab^3] \end{aligned} \right\} \quad (16)$$

With these values of u and v , the solutions of x, y, z and w of (1) are,

$$x(a,b,\sigma) = \frac{1}{5} [(20+35\sigma)a^4 + (20+35\sigma)b^4 - 6(20+35\sigma)a^2b^2 - 4(35-20\sigma)a^3b + 4(35-20\sigma)ab^3]$$

$$y(a,b,\sigma) = \frac{1}{5} [(20-35\sigma)a^4 + (20-35\sigma)b^4 - 6(20-35\sigma)a^2b^2 - 4(35+20\sigma)a^3b + 4(35+20\sigma)ab^3]$$

$$z(a,b) = \frac{1}{5} [35a^4 + 35b^4 - 210a^2b^2 + 80a^3b - 80ab^3]$$

$$w(a,b) = a^2 + b^2$$

Since our aim is to find integral solutions.

Let us take $a = 5A$ and $b = 5B$. Then the solutions are given by,

$$\begin{aligned} x(A,B,\sigma) &= 125(20+35\sigma)A^4 + 125(20+35\sigma)B^4 - 750(20+35\sigma)A^2B^2 \\ &\quad - 500(35-20\sigma)A^3B + 500(35-20\sigma)AB^3 \end{aligned}$$

$$\begin{aligned} y(A,B,\sigma) &= 125(20+35\sigma)A^4 + 125(20+35\sigma)B^4 - 750(20-35\sigma)A^2B^2 \\ &\quad - 500(35+20\sigma)A^3B + 500(35+20\sigma)AB^3 \end{aligned}$$

$$z(A,B) = 125[A^4 + B^4 - 210A^2B^2 + 80A^3B - 80AB^3]$$

$$w(A,B) = 25A^2 + 25B^2$$

Properties: -

$$(i) \ y(1,1,1) - x(1,1,1) + 2z(1,1) = 0$$

(ii) $z(1,1,1) - x(1,1,1) - y(1,1,1)$ and $-[z(1,1,1) + y(1,1,1)]$ is a perfect square.

(iii) $x(1,1,1) - z(1,1) = -[y(1,1,1) + z(1,1)]$

(iv) $x(1,1,1) + 4y(1,1,1) = -[z(1,1) + 2y(1,1,1)] = z(1,1) - x(1,1,1) - y(1,1,1)$

(v) $2[y(1,1,1) + z(1,1)] = x(1,1,1) + y(1,1,1)$

(vi) $\frac{1}{100}[y(1,1,1) + z(1,1) - x(1,1,1)]$ is a disarium number of 175 in order (3).

Pattern 5: -

Let us choose 65 as $65 = (4 + 7i)(4 - 7i)$

Then (3) can be written as

$$(u + iv)(u - iv) = (4 + 7i)(4 - 7i)(a^2 + b^2)^4 \quad (17)$$

Now we define

$$(u + iv) = (4 + 7i)(a + ib)^4 \quad (18)$$

By equating real and imaginary parts on both sides, we obtain the values of u and v as

$$u = 4a^4 + 4b^4 - 24a^2b^2 - 28a^3b + 28ab^3$$

$$v = 7a^4 + 7b^4 - 42a^2b^2 + 16a^3b - 16ab^3$$

Using these values of u and v in (2) the integral solutions to (1) are obtained as,

$$x(a, b, \sigma) = (4 + 7\sigma)a^4 + (4 + 7\sigma)b^4 - 6(4 + 7\sigma)a^2b^2 - 4(7 - 4\sigma)a^3b + 4(7 - 4\sigma)ab^3$$

$$y(a, b, \sigma) = (4 - 7\sigma)a^4 + (4 - 7\sigma)b^4 - 6(4 - 7\sigma)a^2b^2 - 4(7 + 4\sigma)a^3b + 4(7 + 4\sigma)ab^3$$

$$z(a, b) = 7a^4 + 7b^4 - 42a^2b^2 + 16a^3b - 16ab^3$$

$$w(a, b) = a^2 + b^2$$

Properties: -

(i) $x(1,1,1) + y(1,1,1) - z(1,1) + 2w(1,1) \equiv 0$

(ii) $z(a,1) - 7w(a^2,1) - To_a + T_{20,a} + 24Gno_a \equiv 0 \pmod{18}$

(iii) $y(a,1,1) + 3w(a^2,1) - 22CC_a - 14Star_a + 13Gno_a \equiv 0 \pmod{5}$

(iv) $11w(a^2,1) - x(a,1,1) - 3RD_a - 14Star_a - 30Gno_a \equiv 0 \pmod{19}$

(v) $[y(a,1,1) + z(a,1,1)] - 4w(a^2,1) - 15RD_a - 22CH_a + 54a \equiv 0 \pmod{7}$

Pattern 6:-

$$\text{Let us consider, } (1) \text{ as } l = \frac{(8+6i)(8-6i)}{10^2}$$

For this choice, the corresponding integer solutions are found to be,

$$\begin{aligned} x = x(A, B, \sigma) &= 1000(70\sigma - 40)A^4 - 1000(70\sigma - 40)B^4 + 6000(40 - 70\sigma)A^2B^2 \\ &\quad + 4000(70 + 40\sigma)AB^3 - 4000(70 + 40\sigma)A^3B \\ y = y(A, B, \sigma) &= 1000(70\sigma + 40)A^4 - 1000(70\sigma + 40)B^4 + 6000(40 + 70\sigma)A^2B^2 \\ &\quad + 4000(70 - 40\sigma)AB^3 - 4000(70 - 40\sigma)A^3B \end{aligned}$$

$$\begin{aligned} z = z(A, B) &= 70000A^4 + 70000B^4 - 420000A^2B^2 + 160000AB^3 - 160000A^3B \\ w = w(A, B) &= 100A^2 + 100B^2 \end{aligned}$$

Properties: -

- (i) $y(1, 1, 1) - x(1, 1, 1) + 2z(1, 1) \equiv 0$
- (ii) $11x(A, 1, 1) + 3y(1, 1, 1) + 2600000 SO_A \equiv 0 \pmod{2600000}$
- (iii) $x(A, 1, 1) - w(A^2, 1) + 220000 CC_A + 280000 CH_A - 130000 Gno_A \equiv 0 \pmod{190000}$
- (iv) $w(A^2, 1) - z(A, 1) - 160000 P_A^5 - 65000 T_{10,A} \equiv 0 \pmod{195000}$
- (v) $10y(1, 1, 1) = -11[x(1, 1, 1) + z(1, 1)]$
- (vi) $y(1, 1, 1) = 200w(1, 1) - x(1, 1, 1) - z(1, 1)$

Conclusion

In this work, we have shown that the non-homogeneous biquadratic equation $xy + (\sigma^2 + 1)z^2 = 65w^4$ has an unlimited number of non-zero integral solutions. Given the variety of biquadratic equations, the corresponding integer solutions for the other types of biquadratic equations with variables larger than or equal to 4 can be obtained.

References

- [1] L.E.Dickson, History of Theory of Numbers, Chelsea Publishing company, New York, Vol.11, (1952).
- [2] L.J.Mordell, Diophantine equations, Academic Press, London(1969).
- [3] Andreescu, T.Andrica, D., An Intrduction to Diophantine Equations, GIL Publishing house, 2002.
- [4] Janaki G and Vidhya S. On the Integer Solutions of the Homogeneous Biquadratic Diophantine Equation $x^4 - y^4 = 82(z^2 - w^2)p^2$. ISSN 2321 3361 © 2016 IJESC Research Article Volume 6 Issue No. 6. 7275-7278.
- [5] Saranya, C., & Janaki, G. (2021). Integral Solutions of the Binary Quadratic Diophantine Equation $x^2 - 2xy - y^2 + 2x + 14y = 72$. Recent Advances in Mathematical Research and Computer Science Vol. 2, 83-90.

- [6] Saranya P , Poorani K. Intrinsic solutions of Diophantine Equation Involving Centered Square Number. International Journal for Research in Applied Science & Engineering Technology (IJRASET) ISSN: 2321-9653; Volume 11 Issue III Mar 2023
- [7] Janaki, G., & Saranya, C. (2020). Integral Solutions of the Non-Homogeneous Sextic Equation with three Unknowns $3(x^2 + y^2) - 2xy = 972z^6$. Int. J. Sci. Res. in Multidisciplinary Studies Vol, 6(3).
- [8] Shankari, A. G., Janaki, G., & Saranya, A. (2025). Integral Solutions of the Homogeneous Quinary Quintic Diophantine Equation Involving Palindrome Number $p^5 + q^5 - p^2q^2(p + q) = 939(p + q)(r - s)^2$. Arya Bhatta Journal of Mathematics and Informatics, 17(1), 43-48.
- [9] Janaki G, Sangeetha P, On the Ternary Quadratic Diophantine Equation $19(x^2 + y^2) - 37xy = 100z^2$, Arya Bhatta Journal of Mathematics and Informatics, Vol (16), Issue 01, 49-54, 2024.
- [10] Saranya, P., & Janaki, G. (2017). On the Quintic Non-Homogeneous Diophantine Equation. International Journal of Engineering Science, Volume 11 (III), 4685
- [11] Janaki G & Sangeetha P, Integer Solutions on Ternary Quadratic Diophantine Equation, International journal for Research in Applied Science & Engineering Technology, Vol 12, Issue 2321-9653, March 2024
- [12] Janaki G & Sangeetha P, Integral Solutions on Ternary Qubic Equation, Asian Journal of Science and Technology, Vol 14, Issue 05, pp 12509-12512 May 2023.
- [13] Janaki G and Sangeetha P. Non-Homogeneous Solution of Quintic Equation with Five Unknowns $p^4 - q^4 = 82(l^2 - m^2)\sigma^3$. Bodhi International Journal of Research in Humanities, Arts and Science Vol. 9 No. 3 July 2025 E-ISSN: 2456-5571 .293-295.