Inverse Outer Connected Domination Number of Jump Graph

A. Sugirtha[1]
Research Scholar , Reg.No 22211282092008, PG and Research Department of Mathematics,
St. Xavier's College (Autonomous), Palayamkottai, Tamil Nadu, India.

Y. Therese Sunitha Mary[2]
Assistant Professor and Head,
PG and Research Department of Mathematics, St. Xavier's College (Autonomous), Palayamkottai,
Tamil Nadu, India.

Affiliated to
Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli 627 012, Tamil Nadu.

Abstract

Let G be a graph with no isolated vertices. A collection of vertices D in G is the smallest outer connected dominating set. Consider D' is the smallest inverse dominating set of G with respect to D. If the induced subgraph $< V - D' >$ is connected, then D' is said to be an inverse outer connected dominating set. The inverse outer connected domination number $\tilde{\gamma}^{\prime}(G)$ is the smallest cardinality taken over all the minimal inverse outer connected dominating sets of G. The jump graph $J(G)$ of the graph G is formed on $E(G)$ such that two vertices are adjacent if and only if they are not adjacent in G. In this paper, We investigate the basic properties of inverse outer connected domination of jump graphs and their exact values for common graphs. The relationship between jump graph inverse outer connected domination and other characteristics is also examined.

Keywords : Inverse domination, Jump graph, Outer connected domination, Inverse outer connected domination.

AMS subject classication (2020): 05C69 ,05C70,05C76.

1 INTRODUCTION AND PRELIMINARIES

Consider the graph $G = (p, q)$, where the cardinality of vertices and edges of the graph are indicated by p and q. All of the graphs under study are finite, complex, and connected no loops or many edges. Any unspecfic words or symbols used in this paper can be found in Harary [5].

The diameter of graph is the maximum distance between the pair of vertices. The circumference of a graph is the length of any longest cycle in a graph. For $x, y \in V(G)$, the ditance $d_G(x, y)$ between x and y is the length of the shortest xy-paths in G. The maximum degree of G is denoted by $\Delta(G) = max\{d(x)/x \in V(G)\}$. The minimum degree of G is denoted by $\delta(G) = min\{d(x)/x \in V(G)\}$. A domi- nating collection of vertices D of a graph is one in which every vertex not in D is adjacent to a vertex in D. The least cardinality of a dominating set of a graph G is known as the domination number $\gamma(G)$. If $V - D$ contains another dominating set D^r then D^r is said to be an inverse dominating set with respect to D. The *inverse domination number* $\gamma^r(G)$ of G is the least cardinality of D^r. If $V - D$ is con- nected, then the dominating set D is a outer connected dominating set. The *outer connected domination number* $\tilde{\gamma}_c(G)$ of G is the smallest cardinality taken over all the minimum outer connected dominating set of G. A collection of vertices D in G is the smallest outer connected dominating set. Consider D^r in G is the smallest inverse dominating set with respect to D. If the induced subgraph $< V - D^r >$ is connected, then D^r is said to be an inverse outer connected dominating set. The inverse outer connected domination number $\tilde{\gamma}^r(G)$ is the smallest cardinality taken over all the minimal inverse outer connected dominating sets of G. The n-sunlet graph is a graph on $2n$ vertices is obtained by attaching n-pendant edges to the cycle C_n and it is denoted by S_n. A graph with distinct vertices (u, u^r) and (v, v^r) is adjacent in the strong product graph $G_1 \boxtimes G_2$ if and only if either $u = v$ and u^r is adjacent to v^r, or $u^r = v^r$ and u is adjacent to v, or u is adjacent to v and u^r is adjacent to v^r. This holds true for any two graphs, G_1 and G_2.

$V(G_1 \boxtimes G_2) = \{(u, u^r)/u \in V(G_1) \text{ and } u^r \in V(G_2)\}$

$E(G_1 \boxtimes G_2) = E(G_1 2 G_2) \cup E(G_1 \times G_2)$.

Jump Graph is a graph-valued function that was first introduced by Gary Chartrand [1].

Definition 1.1. *The jump graph $J(G)$ of the graph G is formed on $E(G)$ such that two vertices are adjacent if and only if they are not adjacent in G.*

To validate our findings, we require the following theorems.

Theorem 1.2. *If G is a graph with no isolated vertices, then* $\gamma(G) \le \frac{p}{2}$.

Theorem 1.3. *If $G(V, E)$ is a simple graph, then* $2\,|q| \le |p^2| - |p|$

In this paper, we determine exact values of the inverse outer connected domina- tion number of jump graph for some standard graphs. We also obtain bounds and relationship with other graph theoretic parameters for the inverse outer connected domination number of jump graph.

2 INVERSE OUTER CONNECTED DOMINATION NUMBER OF JUMP GRAPH

Definition 2.1. *Let D be a minimum outer connected dominating set of $J(G)$. Let D^r be the minimum inverse dominating set of $J(G)$ with respect to D. Then D^r is called an inverse outer connected dominating set of $J(G)$ if the induced subgraph $< V - D^r >$ is connected. The inverse outer connected domination number is denoted by $\tilde{\gamma}_c^r(J(G))$ and it is the minimum cardinality taken over all the minimal inverse outer connected dominating set of $J(G)$.*

Example 2.2.

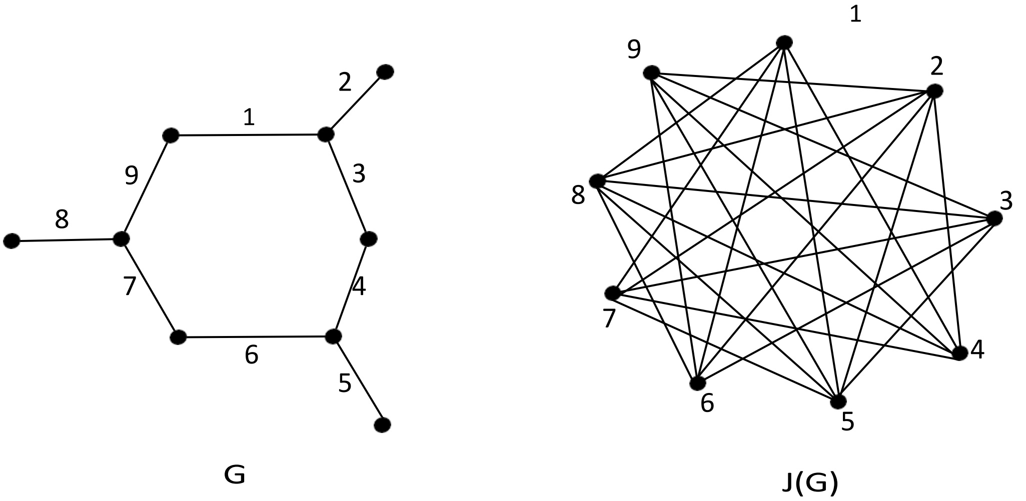

Figure : The Graph G and Jump Graph J(G)

outer connected dominatng set of $J(G)$ is $\tilde{D}_c = \{1, 2\}$ and inverse outer connected dominating set of $J(G)$ is $D^r_c = \{4, 5\}$. Therefore $\tilde{\gamma}_c(J(G)) = 2$ and $\tilde{\gamma}^r_c(J(G)) = 2$.

In the following proposition, we compute the inverse outer connected domination number of the jump graph of standard graphs.

Proposition 2.3. 1. *For any Path P_p with $p \geq 6$, $\tilde{\gamma}_c^r(J(P_p)) = 2$.*

2. *For any Cycle C_p with $p \geq 6$, $\tilde{\gamma}_c^r(J(C_p)) = 2$.*
3. *For any Complete graph K_p with $p \geq 6$, $\tilde{\gamma}_c^r(J(K_p)) = 3$.*
4. *For any Complete Bipartite graph $K_{m,n}$,*

$$\tilde{\gamma}_c^r(J(K_{m,n})) = \begin{cases} 2 & \text{for } k_{2,n},\ n \geq 4. \\ 3 & \text{for } k_{m,n},\ 3 \leq m \leq n. \end{cases} \tag{1}$$

5. *For any Wheel W_p,*

$$\tilde{\gamma}_c^r(J(W_p)) = \begin{cases} 3 & \text{for } p = 5, 6. \\ 2 & \text{for } p \geq 7. \end{cases} \tag{2}$$

Theorem 2.4. *For any connected (p, q) graph G, $2 \leq \tilde{\gamma}_c^r(J(G)) \leq \lfloor \frac{q}{2} \rfloor$. For P_6 and C_6 the limit is acute.*

Proof. Lower bound is trivial. Assume that D is the smallest outer connected dominat- ing set. Since $|V(J(G))| = q$, $|D| \leq \lfloor \frac{q}{2} \rfloor$ then $|V - D| \leq \lfloor \frac{q}{2} \rfloor$. Then $|D^r| \leq |V - D|$ $\Rightarrow |D^r| \leq \lfloor \frac{q}{2} \rfloor$ that is $\tilde{\gamma}_c^r(J(G)) \leq \lfloor \frac{q}{2} \rfloor$. Hence $2 \leq \tilde{\gamma}_c^r(J(G)) \leq \lfloor \frac{q}{2} \rfloor$. Limit is immedi- ate. □

Observation 2.5. $\gamma(J(G)) \leq \tilde{\gamma}_c^r(J(G)) \leq \lfloor \frac{q}{2} \rfloor$
Equality holds if $G \cong P_6, C_6$.

Theorem 2.6. *If G is a connected graph then $\tilde{D}_c^r$ exists for $J(G)$ only if $q \geq 5$.*

Proof. Assume that D is the smallest outer connected dominating set of the jump graph. If $[V(J(G)) - D]$ have one more outer connected dominating set say, D^r then D^r is the inverse outer connected dominating set with respect to D and $|D| \leq |D^r|$. Accord- ing to previous theorem, $|D^r| \geq 2$ also $< V(J(G)) - D^r >$ is connected. Therefore $|V(J(G)) - D^r| - |D| \geq 1$, Hence $q \geq 5$. □

Theorem 2.7. *For any connected $G(p, q)$ graph, $\tilde{\gamma}_c(J(G)) + \tilde{\gamma}_c^r(J(G)) \leq q - 1$, $q \geq 5$
Equality holds if $G \cong P_6$.*

Proof. Since $\tilde{\gamma}_c$ and $\tilde{\gamma}_c^r$ sets of the jump graph has its value ≥ 2, $|V(J(G))| = |E(G)| \geq 4$. Suppose $q = 4$, then $\tilde{\gamma}_c^r$ does not exist for G. Thus $|E(G)| = q \geq 5$. equality occurs immediately. □

Observation 2.8. *Let $J(G)$ be the jump graph of a connected graph G, then $\gamma^r(J(G))+$ $\tilde{\gamma}^r_c(J(G)) \leq q-1$. In addition if $G \cong P_6$ equality continues.*

Theorem 2.9. *If G is k-regular then $J(G)$ is n-k regular and $\tilde{\gamma}^r_c(J(G)) = 2$*

Proof. Since G is k-regular, there are k edges are adjacent to each vertex of G. By definition, $J(G)$ is n-k regular for every v. Choose any two adjacent vertices of $J(G)$, clearly these two vertices form a inverse dominating set of $J(G)$. □

Remark 2.10. *The jump graph $J(G)$of G will have a pendent vertex if there is any one edge e_j in G, such that $deg(e_j) = q - 2$. Any pendent vertex should be a component of the outer connected dominating set and so inverse outer connected is not possible for G.*

Remark 2.11. *The Inverse outer connected domination number does not exist for all jump graph of any graphs. For an example star graph $K_{1,n}$ and frienship graph F_n, $\tilde{\gamma}^r_c$ does not exist.*

Remark 2.12. *If the graph G contains atleast an edge e such that $deg(e) = q - 1$, then $J(G)$ of G is not connected graph.*

Theorem 2.13. *Let G be a connected graph, $\tilde{\gamma}^r_c \leq q - \beta_1(G) + 1$. where $\beta_1(G)$ is a edge independent number of G.*

Proof. Assume that D is the smallest outer connected dominating set of the jump graph. Let $V - D = \{u_1, u_2, u_3, ..., u_j\}$ represent the collection of vertices in $J(G)$ that corre- spond to G's independent edges $\beta(G) = \{e_1, e_2, \quad, e_j\}$. According to the jump graph, $< V - D >$ constitutes a graph K_n. Take $u \in V - D$ and let $F \subseteq [V(J(G)) - (V - D)]$ such that $D^r = F \cup \{u\}$ creates an inverse dominating set. Furthermore the comple- ment of D^r is connected. Therefore D^r is the inverse outer connected dominating set of the jump graph. And $\tilde{D}^r = \{V_c(J(G)) - (V - D)\} \cup \{v\}$. Hence $\tilde{\gamma}^r |J(G)|_c \leq$ $q - \beta_1(G) + 1$. □

Theorem 2.14. *For any connected graph G, with diameter ≥ 5 , then $\tilde{\gamma}^r_c(J(G)) = 2$.*

Proof. Let $V_1 = \{u_1, u_2, u_3, ..., u_p\}$ be the set of vertices in G and $E = \{e_1, e_2, e_3, ..., e_q\}$ be the set of edges in G. Let $V = \{v_1, v_2, v_3, ..., v_q\}$ be the set of vertices in $J(G)$. Let $u_1 - u_j$, $5 \leq j \leq p$ be the longest path of G, labelled as $u_1e_1u_2e_2u_3e_3 .... u_{j-2}e_{j-2}u_{j-1}$ $e_{j-1}u_j$. Let e_1, e_{j-2} be the two edges of the $u_1 - u_j$ path. Let v_1, v_{j-2} be the vertices in V corresponding to the edges e_1, e_{j-2} respectively of E. Then $D = \{v_1, v_{j-2}\}$ be the minimal outer connected dominating set of $J(G)$.

Consider $V - D = \{v_2, v_3, ..., v_{j-3}, v_{j-1}, ..., v_q\}$. Let e_2, e_{j-1} be the two edges of the $u_1 - u_j$ path.Let v_2, v_{j-1} be the vertices in $V - D$ correspon ing to the edges e_2, e_{j-1} respectively of E.Then $D^r = \{v_2, v_{j-1}\}$ forms a dominating set of $J(G)$. Also since the diameter of the graph G is greater than equal to 5 it follows that $< V(J(G)) - D^r >$ is connected subgraph. And so D^r forms a minimum inverse outer connected dominating set of $J(G)$. Hence $\tilde{\gamma}^r_c(J(G)) = 2$. □

Theorem 2.15. *Let G be a n-sunlet graph then* $\tilde{\gamma}^r_c(J(G)) = 2$.

Proof. Clearly it has 2n vertices. Take any two non adjacent pendent edges of G, the corresponding vertices of $J(G)$will form an inverse outer connected dominating set G. Hence $\tilde{\gamma}^r_c(J(G)) = 2$. □

Theorem 2.16. *For any Tree T, with diameter ≥ 5, then* $\tilde{\gamma}^r_c(J(T)) = 2$.

Proof. Let $V_1 = \{u_1, u_2, u_3, ..., u_p\}$ be the set of vertices in T and $E = \{e_1, e_2, e_3, ..., e_q\}$ be the set of edges in T. Let $V = \{v_1, v_2, v_3, \quad , v_q\}$ be the set of vertices in $J(T)$. Let $u_1-u_i, 5 \leq i \leq p$ be the longest path of T, labelled as $u_1e_1u_2e_2u_3e_3 .... u_{i-2}e_{i-2}u_{i-1}e_{i-1}u_i$.

Let e_1, e_{i-2} be the two edges of the $u_1 - u_i$ path. Let v_1, v_{i-2} be the vertices in V corre- sponding to the edges e_1, e_{i-2} respectively of E. Then $D = \{v_1, v_{i-2}\}$ be the minimal outer connected dominating set of $J(T)$.

Consider $V - D = \{v_2, v_3, ..., v_{i-3}, v_{i-1}, ..., v_q\}$. Let e_2, e_{i-1} be the two edges of the $u_1 - u_i$ path. Let v_2, v_{i-1} be the vertices in $V - D$ corresponding to the edges e_2, e_{i-1} respectively of E. Then $D^r = \{v_2, v_{i-1}\}$ forms a inverse dominating set of $J(T)$. Also since the diameter of the graph T is greater than equal to 5 it follows that $< V(J(T)) - D^r >$ is connected subgraph. And so D^r forms a minimum inverse outer connected dominating set of $J(T)$. Hence $\tilde{\gamma}^r_c(J(T)) = 2$. □

Remark 2.17. *If the diameter is ≤ 3, then the jump graph $J(T)$ will be disconnected. If diameter = 4,*

$$\tilde{\gamma}^r_c(J(T)) = \begin{cases} 3 & for, p = 6. \\ 2 & for, p \geq 7. \end{cases} \quad (3)$$

Theorem 2.18. *For any connected graph G, with circumference ≥ 5, then* $\tilde{\gamma}^r_c(J(G)) = 2$.

Proof. Let $V_1 = \{u_1, u_2, u_3, ..., u_p\}$ be the set of vertices in G and $E = \{e_1, e_2, e_3, ..., e_q\}$ be the set of edges in G. Let $V = \{v_1, v_2, v_3, ..., v_q\}$ be the set of vertices in $J(G)$. Let

$u_1e_1u_2e_2u_3e_3 \ldots\ldots u_{j-2}e_{j-2}u_{j-1}e_{j-1}u_je_ju_1$ be a longest cycle in G. Let e_1, e_{j-1} be the two edges of the longest cycle. Let v_1, v_{j-1} be the vertices in V corresponding to the edges e_1, e_{j-1} respectively of E. Then $D = \{v_1, v_{j-1}\}$ be the minimal outer connected dominating set of $J(G)$. Consider $V - D = \{v_2, v_3, \ldots, v_{j-2}, v_j, \ldots, v_q\}$. Let e_2, e_j be the two edges of the longest cycle. Let v_2, v_j be the vertices in $V - D$ corresponding to the edges e_2, e_j respectively of E.Then $D^r = \{v_2, v_j\}$ forms a dominating set of $J(G)$. Also $< V(J(G)) - D^r >$ is connected subgraph. And so D^r forms a minimum inverse outer connected dominating set of $J(G)$. Hence $\tilde{\gamma}^r_c(J(G)) = 2$. □

Theorem 2.19. *For any connected (p, q) graph G, $\tilde{\gamma}^r_c(J(G)) \leq q - \Delta(G)$, where $\Delta(G)$ is the maximum degree of G.*

Proof. Let $V = \{v_1, v_2, v_3, \ldots, v_p\}$ be the set of vertices in $J(G)$. Let $D = \{v_1, v_2, v_3, \ldots\ldots, v_k\}$ be a minimum outer connected dominating set of G. Then $V - D = \{v_{k+1}, v_{k+2}, \ldots\ldots, v_p\}$ in G. Now consider $V_1 = (V - D) - v_i$, where $deg(v_i) = \Delta(G), v_i \in V - D$ .Since $E(G) = V(J(G))$ let $I = \{e_1, e_2, \ldots.., e_j\}$ be the set of edges adjacent to v_i in G.
Let $H \subseteq V(J(G))$ be the set of vertices of $J(G)$ such that $H \subseteq E - I$. Then H itself forms a minimal inverse outer connected dominating set.Therefore $\tilde{\gamma}^r_c(J(G)) \leq |E| - |I|$
Hence $\tilde{\gamma}^r_c(J(G)) \leq q - \Delta(G)$. □

Theorem 2.20. *For any connected graph G without pendent vertex $\tilde{\gamma}^r_c(J(G)) \leq \delta(G)$.*

Proof. Let $V = \{v_1, v_2, v_3, \ldots, v_p\}$ be the set of vertices in $J(G)$. Let $D = \{v_1, v_2, v_3, \ldots\ldots, v_k\}$ be a minimum outer connected dominating set of G. Then $V - D = \{v_{k+1}, v_{k+2}, \ldots\ldots, v_p\}$ in G. Now consider $V_1 = (V - D) - v_i$, where $deg(v_i) = \delta(G), v_i \in V - D$. Since $E(G) = V(J(G))$, let $E_1 = \{e_1, e_2, \ldots.., e_j\}$ be the set of edges adjacent to v_i in G. Then $E_1 \subseteq V(J(G)) - D$ will forms a minimal inverse outer connected dominating set of $J(G)$. Therefore $|E_1| = \delta(G)$. Hence $\tilde{\gamma}^r_c(J(G)) \leq \delta(G)$. □

3 STRONG PRODUCT OF JUMP GRAPH

Theorem 3.1. *Let K_m and K_n are two complete graphs,* $\tilde{\gamma}^r_c(J(K_m \boxtimes K_n)) = 2$.

Proof. Let $\{a_1, a_2, \ldots., a_m\}$ be the vertices of K_m and $\{b_1, b_2, \ldots\ldots, b_n\}$ be the vertices of K_n. First we have to show that $J(K_m \boxtimes K_n) = J(K_{mn})$. By the definition of strong product, $J(K_m \boxtimes K_n)$ is a complete graph with mn vertices.
Take any two vertices (a_1, b_1) and (a_2, b_2) from $J(K_{mn})$. we consider the following.

According to the definition of K_m, a_1 and a_2 are adjacent if $b_1 = b_2$ and $a_1 /= a_2$. similarly by the definition of K_n, b_1 and b_2 are adjacent if $a_1 = a_2$ and $b_1 /= b_2$. If both the vertices are distinct in $J(K_{mn})$, by the definition of K_m, a_1 and a_2 are adjacent and by K_n, b_1 and b_2 are adjacent. Hence by every pair of vertices in $J(K_{mn})$ are adjacent Also it has mn vertices. Hence $J(K_{mn}) = J(K_m \boxtimes K_n)$. Since $J(K_{mn}) = 2$, we get $\tilde{\gamma}^{r}_{c}(J(K_m \boxtimes K_n)) = 2$.

□

4 CONCLUSION

In this paper, we defined the notions of inverse outer connected domination in jump graphs. we got many bounds on inverse outer connected numbers.

REFERENCES

[1] G. Chartrand, H. Hevia, E.B. Jarrett, M. Schultz, *Subgraph distances in graphs defined by edge transfers*, Discrete Math.,170 , 63-79 (1997).

[2] G. Chartrand and L. Lesniak, *Graphs and Digraphs,*CRC (2004).

[3] G.S Domke, J.E. Dunbar and L.R. Markus, *The inverse domination number of graph*, Ars.Combin., **72** 149-160 (2004).

[4] A. Frendrup, M.A.Henning, B. Randerath and P.D Vestergoard, *On con- jecture about inverse domination in graphs*, Ars Combinatoria, **95A**, 103- 111(2010).

[5] F. Harary, *Graph Theory*, Addision-Wesley, Reading Mass.(1969).

[6] T.H. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in graphs,*Marcel Dekkaer Inc, New York (1998).

[7] V. R. Kulli, *Theory of domination in Graphs*, Vishwa International Publica- tions, India (2012).

[8] V. R. Kulli and S. C. Sigarkanti, *Inverse domination in Graphs,* Nat.Acad.Sci.Lett.,**14** ,473-475 (1991).

[9] O. Ore, *Theory of graphs*, Amer.Math.Soc.Colloq.Public., 38,provi- dence(1962).