

Mathematical Modelling and Optimal Control of Measles Epidemics: A Review of Compartmental, Fractional, Stochastic and Spatial Approaches

Vandana Sharma¹, Dr. Jaspreet Kaur², Dr. Vikas Gupta³

¹Research scholar, Department of Mathematics, Guru Kashi University, Bathinda, Punjab, 151302

²Associate Professor, Department of Sciences, Guru Kashi University, Bathinda, Punjab, 151302

³Associate Professor, Faculty of Engg & Technology, Guru Kashi University, Bathinda Punjab, 151202

Abstract - Measles is one of the most contagious vaccine-preventable infectious diseases, and outbreaks continue to occur because of inadequate vaccination coverage, geographical inequalities, incomplete immunization, waning immunity and heterogeneous transmission patterns. Mathematical modelling provides an important framework for analysing the transmission dynamics of measles and evaluating effective control strategies. This review examines recent mathematical approaches for measles modelling, including deterministic SEIR and SVEIR compartmental models, double-dose vaccination models, stochastic differential equation models, fractional-order models, spatially heterogeneous vaccination models, network-based models, and optimal-control frameworks. Particular attention is given to the mathematical formulation of the models, positivity and boundedness of solutions, disease-free and endemic equilibria, basic reproduction numbers, stability analysis, sensitivity analysis, numerical methods and optimal control theory. The reviewed studies demonstrate the increasing integration of epidemiological data with advanced mathematical and computational methods. However, few studies simultaneously incorporate age structure, double-dose vaccination, waning immunity, spatial heterogeneity, fractional or stochastic dynamics, real epidemiological data and optimal control. This gap provides a significant opportunity for further mathematical research in the computational modelling and optimal control of measles epidemics.

Keywords: Measles; mathematical modelling; SEIR; SVEIR; fractional differential equations; stochastic modelling; vaccination; waning immunity; Haar wavelets; optimal control; stability analysis; reproduction number.

1. INTRODUCTION

Measles is a highly contagious infectious disease caused by the measles virus. Despite the availability of highly effective vaccines, measles outbreaks continue to occur in different regions of the world. The persistence of measles is associated with insufficient vaccination coverage, geographical inequalities, incomplete vaccination schedules, vaccine hesitancy, population mobility and the accumulation of susceptible individuals.

Mathematical modelling has become an important tool for understanding infectious disease transmission. A mathematical model converts biological and epidemiological

assumptions into mathematical equations that can be analysed theoretically and solved numerically.

The general modelling process can be represented as:

Epidemiological Data → Mathematical Model → Parameter Estimation → Theoretical Analysis → Numerical Simulation → Control Strategy

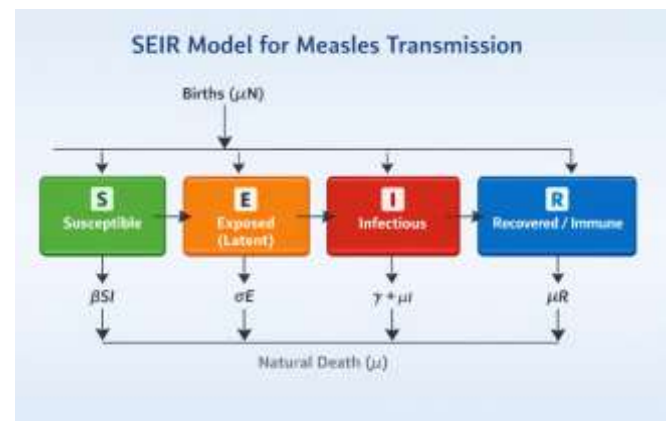


Fig.1. schematic diagram for the flow of the measles disease.

The classical compartmental approach divides the population into epidemiological classes. The basic SEIR structure is:

$$S \rightarrow E \rightarrow I \rightarrow R$$

where S represents susceptible individuals, E exposed individuals, I infectious individuals and R recovered individual

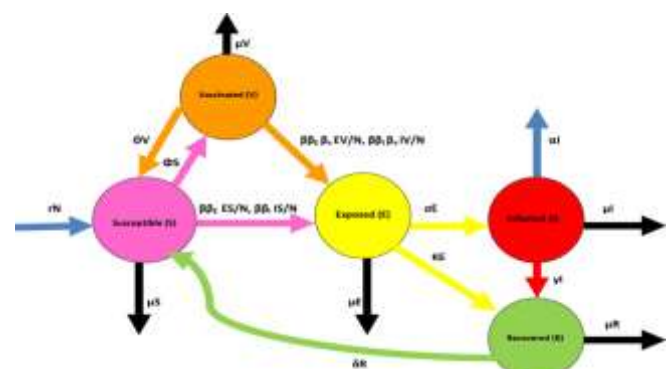


Fig. 2. Schematic illustration for stream of the measles disease

With vaccination, the model can be extended to:

$$S \rightarrow V \rightarrow E \rightarrow I \rightarrow R$$

and for two-dose vaccination:

$$S \rightarrow V1 \rightarrow V2$$

The development of measles models can therefore be summarised as:

Classical SEIR $\rightarrow$ Vaccination Models $\rightarrow$ Stochastic Models $\rightarrow$ Fractional Models $\rightarrow$ Spatial and Network Models $\rightarrow$ Optimal Control Models.

2. DEVELOPMENT OF COMPARTMENTAL MEASLES MODELS

The earliest mathematical approaches to measles transmission generally used deterministic compartmental models. The population was divided into epidemiological classes such as susceptible, exposed, infectious and recovered individuals. These models provided the basic mathematical framework for studying measles transmission and introduced important concepts such as equilibrium points, the basic reproduction number and stability.

The classical transmission structure is:

$$S \rightarrow E \rightarrow I \rightarrow R$$

The mathematical significance of the SEIR framework is that it represents the latent period between infection and the development of infectiousness. This is particularly important for measles because exposed individuals do not immediately transmit the disease.

However, classical SEIR models (Berhe & Makinde, 2020) often assume homogeneous mixing and do not explicitly represent vaccination. Consequently, later studies extended the model by incorporating vaccination compartments.

3. VACCINATION-BASED MATHEMATICAL MODELS

Vaccination is the most important control strategy in measles transmission models. The basic SEIR model has therefore been extended to SVEIR-type models: (El Hajji & Albargi, 2022)

$$S \rightarrow V \rightarrow E \rightarrow I \rightarrow R$$

In these models, vaccination moves susceptible individuals into a protected compartment. The mathematical analysis generally focuses on how vaccination changes the reproduction number and the stability of the disease-free equilibrium.

A study conducted in Borno State, Nigeria, developed a deterministic SVEIR model using real measles data from 2015–2023. (Sbarra et al., 2021) The model was fitted to observed measles cases and used to estimate important epidemiological parameters, assess vaccine effectiveness and investigate vaccination strategies. The study demonstrated the importance of combining mathematical models with actual epidemiological observations.

From a mathematical point of view, this work connects:

Observed Data $\rightarrow$ Parameter Estimation $\rightarrow$ Mathematical Model $\rightarrow$ Numerical Prediction

However, the model does not fully represent the effects of waning immunity, age structure or spatial variation in vaccination coverage.

4. MATHEMATICAL MODELS INCORPORATING MULTIPLE VACCINATION DOSES

Some recent models have considered the effect of two-dose vaccination:

$$S \rightarrow V1 \rightarrow V2$$

where V1 represents individuals receiving the first dose and V2 represents individuals receiving the second dose. (Peter et al., 2023)

This is mathematically important because vaccination is not always a single event. The protection level may depend on the number of doses received.

The mathematical interpretation is:

First Dose $\rightarrow$ Partial Protection

Second Dose $\rightarrow$ Enhanced Protection

These models provide a more realistic representation of vaccination programmes than models containing only one vaccinated compartment. (Suffel et al., 2025)

However, many such models do not simultaneously consider the decline of immunity after vaccination. Therefore, a further extension may be represented as:

$$V2 \rightarrow W \rightarrow S$$

where W represents waning immunity.

5. MATHEMATICAL MODELLING OF WANING IMMUNITY

The assumption of permanent immunity is a major simplification in many mathematical models.

A more realistic structure is:

$$V \rightarrow W \rightarrow S$$

where immunity decreases with time.

Mathematically, vaccine protection may be represented by a time-dependent function $VE(t)$ rather than by a constant value.

The incorporation of waning immunity is important because it changes the long-term behaviour of the model. It may increase the susceptible population and influence the persistence of measles.

Thus:

Waning Immunity → Increase in Susceptible Population → Potential Increase in Transmission

This area remains important for further mathematical development.

6. STOCHASTIC MATHEMATICAL MODELS

Deterministic models describe the average behaviour of an epidemic. However, disease transmission is also affected by random events.

Stochastic models incorporate this randomness into the mathematical system.

The general form is:

$$dX = F(X)dt + G(X)dW(t)$$

where the second term represents random fluctuations.

Stochastic measles models are important because two populations with identical initial conditions may experience different epidemic outcomes due to random transmission events.

Mathematically, stochastic modelling is particularly relevant for small populations, outbreak extinction, random transmission events and uncertainty in epidemiological parameters.

However, stochastic models are computationally more complex than deterministic systems.

7. FRACTIONAL-ORDER MATHEMATICAL MODELS

Another important development in measles modelling is the use of fractional differential equations.

A fractional-order model may be written as:

$$CD_{t^{\alpha}} X(t) = F(X(t)), 0 < \alpha \leq 1 \text{ (Chatterjee et al., 2024)}$$

The advantage of fractional calculus is that it can incorporate memory effects into the model.

In classical models, dX/dt describes the instantaneous rate of change. In fractional models, $D^{\alpha}X(t)$ allows the current behaviour to depend on previous states of the system. (Peter et al., 2023)

This makes fractional-order models useful for representing biological processes in which historical effects influence current disease dynamics.

The mathematical studies in this area generally investigate existence of solutions, uniqueness, stability, numerical approximation and optimal control.

8. NUMERICAL METHODS FOR FRACTIONAL MEASLES MODELS

The solution of nonlinear fractional differential equations is generally difficult to obtain analytically. Therefore, numerical methods are essential.

One reviewed study applied the Haar Wavelet Collocation Method to a fractional-order SEIR measles model.

The mathematical process was:

Fractional Model (Chatterjee et al., 2024) → Wavelet Approximation → Algebraic System → Numerical Solution

The importance of this work lies in its contribution to numerical mathematics and computational epidemiology.

However, from an epidemiological perspective, the model could be further improved by incorporating vaccination, waning immunity and real measles data.

Thus, the mathematical and epidemiological aspects should be integrated in future research.

9. GEOGRAPHICAL HETEROGENEITY IN MEASLES VACCINATION

A large-scale study of measles vaccination coverage demonstrated that vaccination coverage varies considerably between geographical regions.

This is mathematically important because many compartmental models assume a single uniform vaccination rate.

In reality:

$$v = v(x, y, t)$$

may depend on geographical location and time.

Therefore:

National Vaccination Coverage $\neq$ Uniform Local Vaccination Coverage

This geographical heterogeneity can produce local susceptible populations and increase the possibility of outbreaks.

The mathematical implication is that future models may require partial differential equations, metapopulation models, spatially structured ordinary differential equations and geographical vaccination functions.

This represents an important connection between mathematical modelling and public-health data.

10. EQUILIBRIUM AND STABILITY ANALYSIS

A major mathematical component of measles modelling is equilibrium analysis.

The disease-free equilibrium is represented by E_0 , where $I = 0$.

The endemic equilibrium is represented by E^* , where $I^* > 0$. (Owolanke et al., 2026)

The stability of these equilibria is commonly analysed using the Jacobian matrix.

The basic threshold is:

$$R_0 < 1$$

for disease elimination, while:

$$R_0 > 1$$

indicates the possibility of disease persistence.

The next-generation matrix is commonly used to derive:

$$R_0 = \rho(FV^{-1})$$

This mathematical analysis is common to almost all modern compartmental measles models.

However, the calculation of R_0 becomes more complicated when the model includes multiple vaccination classes, waning immunity, age structure, fractional derivatives or spatial heterogeneity.

11. SENSITIVITY ANALYSIS

Sensitivity analysis is used to identify the parameters that most strongly influence disease transmission.

The transmission rate β usually has a positive influence on infection, whereas the recovery rate γ generally has a negative influence.

The mathematical interpretation is:

$$\beta \uparrow \rightarrow I(t) \uparrow \quad \&$$

$$\gamma \uparrow \rightarrow I(t) \downarrow$$

Sensitivity analysis is useful because it helps determine which parameters should be targeted by control interventions.

It also identifies parameters that require more accurate estimation from epidemiological data.

12. OPTIMAL CONTROL OF MEASLES TRANSMISSION

Optimal control theory provides a mathematical method for determining the most effective control strategy.

A controlled epidemic model can be represented a

$$dX/dt = F(X, u)$$

where $u(t)$ represents the intervention.

Possible control variables include vaccination, booster vaccination, treatment, contact reduction and health education.

The objective is to minimize both disease burden and intervention cost:

$$\min J(u)$$

A general objective functional is:

$$J(u) = \int_0^T [AI(t) + B/2 u^2(t)] dt$$

Pontryagin's Maximum Principle is then used to derive the optimality system. (Ahmed et al., 2024)

The mathematical procedure includes:

State Equations $\rightarrow$ Hamiltonian $\rightarrow$ Adjoint Equations $\rightarrow$ Optimal Control (Berhe & Makinde, 2020)

This combines differential equations, control theory and numerical analysis.

13. COMPARATIVE LITERATURE REVIEW

The reviewed papers collectively demonstrate the progressive development of mathematical modelling for measles transmission and control. Although the studies differ in model structure, mathematical technique and data source, they share a common objective: to understand the mechanisms responsible for measles persistence and to identify effective strategies for reducing transmission. The common mathematical foundation is the compartmental representation of the population into epidemiological classes and the use of differential equations to describe transitions between these classes.

The literature begins with deterministic SEIR/SVEIR models, which provide the basic framework for describing infection, exposure, vaccination and recovery (Chatterjee et al., 2024). These models are extended through double-dose vaccination to represent the sequential nature of immunization. Other (Peter et al., 2023) studies introduce age structure and waning immunity to improve biological realism (El Hajji & Albargi, 2022). Stochastic (Xue et al., 2021) models incorporate random fluctuations in transmission, while fractional-order models introduce memory effects into the epidemic dynamics. Network and spatial approaches address the limitations of homogeneous mixing by representing heterogeneous contacts and geographical variation. Data-driven models connect mathematical systems with observed surveillance data through parameter estimation and calibration. Finally, optimal-control and cost-effectiveness studies translate mathematical analysis into intervention strategies and policy decisions.

A common mathematical workflow is therefore: model formulation $\rightarrow$ positivity and boundedness $\rightarrow$ disease-free equilibrium $\rightarrow$ endemic equilibrium $\rightarrow$ basic reproduction number R_0 $\rightarrow$ stability analysis $\rightarrow$ sensitivity analysis $\rightarrow$ numerical simulation $\rightarrow$ intervention optimization. This common structure allows results from apparently different studies to be compared within one mathematical framework.

Across the literature, vaccination is consistently identified as the central intervention for measles control. However, the effectiveness of vaccination depends on coverage, timing, vaccine effectiveness, geographical distribution and the persistence of immunity (Sbarra et al., 2021). The reviewed studies also consistently show that the transmission rate and vaccination-related parameters are among the most

Study/Approach	Mathematical Structure	Main Contribution	Main Limitation
Classical SEIR	ODE compartmental model	Basic transmission dynamics	Homogeneous mixing
SVEIR(Saleh et al., 2025)	Vaccination compartment	Vaccination impact	Often permanent immunity
Double-dose vaccination	$S \rightarrow V1 \rightarrow V2$	More realistic immunisation	Waning may be omitted
Stochastic model	SDE system	Random transmission effects	Computational complexity
Fractional model(Chatterjee et al., 2024)	Fractional differential equations	Memory effects	Parameter interpretation
Haar wavelet/collocation(Hoang & Ehrhardt, 2025)	Numerical approximation	Solution of complex fractional systems	Requires advanced computation
Age-structured model(Berhe & Makinde, 2020)	Age-dependent equations	Age-specific transmission and immunity	Requires demographic data
Network model(Xue et al., 2021)	Contact graph/network	Heterogeneous contacts	Requires contact information
Spatial model(Hoang & Ehrhardt, 2025)	PDE/metapopulation	Regional variation	Requires spatial data
Data-driven SVEIR	Calibrated ODE model	Real-data prediction	Reporting and data quality
Optimal control(Xue et al., 2021)	Controlled differential equations	Intervention optimisation	Depends on objective/cost assumptions
Cost-effectiveness(Owolanke et al., 2026)	Control + economic analysis(Sbarra et al., 2021)	Decision support	Requires reliable cost data

influential parameters in determining the reproduction number and epidemic size

The common conclusion of the reviewed literature is that measles transmission cannot always be adequately described by a simple homogeneous deterministic model. Realistic mathematical modelling increasingly requires consideration of multiple vaccination doses, immunity loss, heterogeneous

14. CRITICAL RESEARCH GAP

The reviewed literature demonstrates that significant progress has been made in mathematical measles modelling. However, the existing studies generally examine different mathematical components separately.

Some studies focus on vaccination, some on stochastic dynamics, some on fractional calculus, some on spatial heterogeneity, some on optimal control and some use real epidemiological data.

contacts, geographical differences, uncertainty, memory effects and real surveillance data. Therefore, the most important trend in the literature is the movement from simple theoretical models towards integrated, data-informed and computationally advanced mathematical systems.

However, there remains a need for a mathematically integrated framework combining:

Vaccination + Waning Immunity + Age Structure + Spatial Heterogeneity + Advanced Differential Equations + Real Data + Optimal Control

Such a model could provide a more realistic description of measles transmission.

15. PROPOSED MATHEMATICAL RESEARCH DIRECTION

Based on the reviewed literature, a future mathematical model may consider the following transmission structure:

$$S \rightarrow V1 \rightarrow V2 \rightarrow W \rightarrow E \rightarrow I \rightarrow R$$

where S represents susceptible individuals, V1 first-dose vaccinated individuals, V2 second-dose vaccinated individuals, W individuals with waning immunity, E exposed individuals, I infectious individuals and R recovered individuals.

The mathematical study of such a model may include:

6. Local and global stability.
7. Sensitivity analysis.
8. Numerical simulation.
9. Optimal control.
10. Model validation using real epidemiological data.

This framework provides a clear mathematical direction for the study of Computational Modelling and Optimal Control of Measles Epidemic in Human Population.

16. CONCLUSION

The literature on mathematical modelling of measles has developed from classical deterministic SEIR models to more advanced frameworks incorporating vaccination, multiple vaccine doses, waning immunity, stochastic effects, fractional calculus, spatial heterogeneity and optimal control.

1. Formulation of nonlinear differential equations.
2. Positivity and boundedness of solutions.
3. Disease-free equilibrium.
4. Endemic equilibrium.
5. Basic reproduction number.

From a mathematical perspective, the most important components of a measles model are:

Model Formulation → Equilibrium Analysis → R_0 → Stability → Sensitivity Analysis → Numerical Solution → Optimal Control

The existing literature provides a strong foundation, but a significant research gap remains in developing an integrated model that combines vaccination, waning immunity, spatial heterogeneity, advanced mathematical dynamics and optimal control.

Therefore, future research should focus on developing mathematically rigorous and computationally efficient measles models that can provide both theoretical insights and practical guidance for epidemic control.

Mathematical Approach	Main Contribution	Limitation
SEIR	Basic disease transmission	Homogeneous mixing
SVEIR	Vaccination	Often permanent immunity
Two-dose model	Multiple vaccination stages	Limited waning immunity
Stochastic model	Random transmission	Computational complexity
Fractional model	Memory effects	Difficult parameter interpretation
Haar wavelet method	Numerical solution	Requires advanced computation
Spatial model	Geographical variation	Requires detailed data
Optimal control	Intervention optimisation	Depends on cost assumptions

REFERENCES

- [1] Ahmed, S., Jahan, S., Shah, K., & Abdeljawad, T. (2024). On mathematical modelling of measles disease via collocation approach. *AIMS Public Health*, 11(2), 628–653. <https://doi.org/10.3934/publichealth.2024032>
- [2] Berhe, H. W., & Makinde, O. D. (2020). Computational modelling and optimal control of measles epidemic in human population. *BioSystems*, 190. <https://doi.org/10.1016/j.biosystems.2020.104102>
- [3] Chatterjee, A. N., Sharma, S. K., & Al Basir, F. (2024). A Compartmental Approach to Modeling the Measles Disease: A Fractional Order Optimal Control Model. *Fractal and Fractional*, 8(8). <https://doi.org/10.3390/fractalfract8080446>
- [4] El Hajji, M., & Albargi, A. H. (2022). A mathematical investigation of an “SVEIR” epidemic model for the measles transmission. *Mathematical Biosciences and Engineering*, 19(3), 2853–2875. <https://doi.org/10.3934/mbe.2022131>
- [5] Hoang, M. T., & Ehrhardt, M. (2025). Differential equation models for infectious diseases: Mathematical modeling, qualitative analysis, numerical methods and applications. *SeMA Journal*. <https://doi.org/10.1007/s40324-025-00404-9>

- [6] Owolanke, A. O., Ogunbamike, O. K., Peter, O. J., & Agbaje, J. O. (2026). A mathematical model for measles transmission with optimal interventions and cost-effectiveness analysis. *Discover Public Health*, 23(1). <https://doi.org/10.1186/s12982-026-01932-z>
- [7] Peter, O. J., Fahrani, N. D., Fatmawati, Windarto, & Chukwu, C. W. (2023). A fractional derivative modeling study for measles infection with double dose vaccination. *Healthcare Analytics*, 4. <https://doi.org/10.1016/j.health.2023.100231>
- [8] Saleh, A. S., Kareem, L., Bakare, E., Charles, O., Eshikhena, G., Adegbite, G., Akoma, D., Ojielo, N., Gayawan, E., Gadanya, M., Ismaila, L., Ogunbode, O., & Kaduru, C. (2025). *Computational Modelling of Measles Transmission Dynamics Among Children Under 5 Years in Borno State, Nigeria: A Compartmental SVEIR Model*. <https://doi.org/10.21203/rs.3.rs-6946934/v1>
- [9] Sbarra, A. N., Rolfe, S., Nguyen, J. Q., Earl, L., Galles, N. C., Marks, A., Abbas, K. M., Abbasi-Kangevari, M., Abastabar, H., Abd-Allah, F., Abdelalim, A., Abdollahi, M., Abegaz, K. H., Abiy, H. A. A., Abolhassani, H., Abreu, L. G., Abrigo, M. R. M., Abushouk, A. I., Accrombessi, M. M. K., ... Mosser, J. F. (2021). Mapping routine measles vaccination in low- and middle-income countries. *Nature*, 589(7842), 415–419. <https://doi.org/10.1038/s41586-020-03043-4>
- [10] Suffel, A. M., Warren-Gash, C., McDonald, H. I., Kucharski, A., & Robert, A. (2025). Impact of vaccination timing and coverage on measles near elimination dynamics: a mathematical modelling analysis. *Nature Communications*, 16(1). <https://doi.org/10.1038/s41467-025-63710-w>
- [11] Xue, Y., Ruan, X., & Xiao, Y. (2021). Measles dynamics on network models with optimal control strategies. *Advances in Difference Equations*, 2021(1). <https://doi.org/10.1186/s13662-021-03306-y>