

Numerical Approximation of n^{th} Order Linear and Nonlinear Differential Equations Using Modified Fuzzy Transform

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Abstract - The use of differential equations has become an essential technique in modelling real-world situations with n number of solutions. Previously, to solve differential equations, numerical approaches based on the finite difference method and the approximation method were utilized. These approaches raised an issue of complexity in solving equations, as well as the issue of an unlimited number of iterations, which increased the time required to discover accurate solutions for systems. This study focused on the challenge of estimating the solvability of a linear and non-linear system for n^{th} order equations using the Fuzzy transform. The fuzzy transform is built as an ordinary algebra type, with its member functions modified using an enhanced Euler's technique and followed by a trapezoidal rule. The basic idea behind this proposed approach is that it splits the equations into equal intervals and uses direct fuzzy transform to estimate the value before using inverse fuzzy to get the exact solution. Theorems are used to determine the best approximation of the given approach and are therefore proven. Thus, in this research, the Modified Fuzzy Transform (MFT) has been presented and implemented in MATLAB to provide precise solutions to specified differential equations. Its efficiency was evaluated, and it was discovered that it was more effective with negligible error than prior approaches.

Keywords: Finite Difference method, Linear and Non-linear system, Approximation, Fuzzy transform, Euler's technique and Trapezoidal rule .

1. Introduction

Differential equations have great potential for modelling and comprehending real-world problems in science and engineering. Differential equations are not always analytically solvable, necessitating the use of numerical methods [1]. Many differential equations have no closed form solutions. Thus, many researchers are developing approximation methods for solving differential equations. Various types of transforms like Fourier, Laplace, integral, wavelet are useful approaches for building

approximation models and solving differential or integral-differential problems in classical mathematics [2]. The primary concept behind them is to convert the original model into a specific space where computation is easier. The translation back to the original space produces a close approximation of a model or solution. Similarly, Perfilieva [3] developed the fuzzy transform in, which is an approximation approach based on fuzzy sets. The fundamental notion behind F-transforms is a fuzzy partitioning of the world into fuzzy subsets. Perfilieva [4] introduced two forms of fundamental functions, triangular and sinusoidal shaped membership functions, in [5]. Similarly, further proposals for the core functions of fuzzy partition have been made by taking into account varied forms. They used standard approximation operators such as B-splines, Shepard kernels, Bernstein basis polynomials, and Favard-Szász-Mirakjan operators [6] to construct fuzzy transformations. In addition, characteristics of F-transforms and their enhancements have been reported in subsequent contributions, which continue the investigation of F-transform-based numerical techniques for ordinary differential equations (ODEs) [7]. An ordinary differential equation is a differential equation for a function of a single variable, such as $x(t)$, whereas a partial differential equation is a differential equation for a function of several variables [8]. The simplest ordinary differential equations can be integrated directly by finding anti-derivatives and also it is not always possible to find an analytical solution of the general first-order differential equation by integrating them directly. Thus, there was possibility to determine a unique numerical solution given an initial value by integrating them directly [9]. Further, linear first-order ordinary differential equation can be solved by use of an integrating factor. Previous methods show that ordinary differential equation having constant coefficients can be solved by learned solution method. However, these analyses of existence and uniqueness of a solution and solution methods for Boundary Value Problem (BVP) remained more sophisticated. The reason behind the complexity of solving BVP is that it requires solution of

a corresponding linear or non-linear system of equations which may cause additional technical efforts [10].

Euler-like method for solving Initial Value Problems [IVP] has been presented previously, this approach has been further used for second order IVPs [11]. But unfortunately it doesn't help to solve higher order derivatives in the systems. For both BVP and IVPs an analysis of F-transform based approach has been initiated, ordinary differential equation having constant coefficients were solved by learned solution method [12]. In similar manner, consider a second order ordinary differential equation that collects a partial pair with first nodes and is described in terms of a first order differential equation [13]. The second order equations were computed generally using the concept of superposition, which asserts a theory. For solving these equations, Wronskian technique [14] was presented previously to find two solutions for second order equations and attempt to create a general solution. While utilizing this technique, unfortunately Wronskian parameters could not be equal to zero at its initial conditions because there were two independent linear solutions [15]. The appropriate ersatz for the particular solution is running continuous times with respect to its homogeneous term t , whereas the inhomogeneous term in the differential equation provided is in the linear form and a solution of the corresponding homogeneous differential equation is obtained in the wronskian format [16]. Later, it has been found that the Laplace transform technique provides a simpler, more elegant solution for second order differential equations. At $t = 5$, the only challenge has been finding the initial conditions of the homogeneous ordinary differential equations using Laplace. To overcome the previous long-time assessment in a nonlinear differential equation a new method for approximation of functions and its application in numerical solution of BVPs has been presented in this paper. The objective of the method is to obtain the zero stability, consistency, convergence and order of the method by Taylor's series approximation [17-20]. In this paper, analysis and apply the F-transform method to linear and nonlinear n^{th} order BVPs using the optimal coefficients have been presented.

Major contributions of this paper can be given as

- I. A mathematical model that has an alternative approach for solving differential equations using fuzzy transform has been presented.
- II. A fuzzy based approximation method that solves ordinary differential equations that used improved Euler's method and trapezoidal rule.

- III. The approximation method proposed here is designed so as to overcome the drawbacks of complexity in solving equations for linear and non-linear case.
- IV. Proposed mathematical model has been used to solve the linear and non-linear boundary value condition problem for n^{th} order differential equations.

By considering all the above stated contributions a mathematical model has been presented in this paper. Further this paper has been organized as follows, section 2 presents the review of existing techniques. Section 3 briefly explains the proposed mathematical model and section 4 presents the implementation results. Section 5 concludes the whole paper.

2. Literature Review

Many methods have been presented previously for solving linear and nonlinear differential equations. Some of them have been reviewed in this section.

Khastan et al [1] have presented a numeric method based on fuzzy transform and finite difference method. Fuzzy transform based numerical approach, analyzed BVPs and generate solution based on finite difference method and approximated using fuzzy transform. However, these methods failed to analyze the stability of the system. Chen et al [2] have introduced an efficient algorithm that obtain an approximate solution for second-order ordinary differential equations. It was constructed directly by fuzzy transform from the original equation. However, this method focused mainly on coefficients of the equation and thus it remains inconvenient for second-order ordinary differential equations. Abu et al [3] have introduced a method using Fuzzy transform to solve initial value problems that enabled to get a continuous solution for IVPs. It used finite approximation for the derivative of IVPs that obtained solution for getting closer to the analytical solution for the number of partitions. However, this continuous solution raised the difficulty of computations for solving IVPs. Singh et al [4] have presented an approximate solution of nonlinear parabolic partial differential equation. It was a method for partial differential equation subject to Neumann boundary conditions is based on cubic B-spline collocation method. However, these solutions were not obtained without linearization, thus a numerical validity technique must be included to perform this technique. YAZGAN et al [5] have present a w-periodic solutions of nonlinear n^{th} order differential equation that has different delays. It used

some conditions to solve the periodicity of solutions that were obtained using coincidence degree theorem. However, these conditions were not sufficient for obtaining the solutions for differential equations.

Qin et al [6] have presented a method that solves high order non-linear ordinary differential equations using radionet. It used radionet technique for solving the higher order non-linear differential equations that was mainly based on perceptron network. However, this method failed to give analytical solution for higher order and non-linear ODE became complex. Shiralashetti et al [7] have presented Hermite wavelet based numerical method for the solution of two parameters singularly perturbed non-linear Benjamina-Bona-Mohany partial differential equation. It was based on the time discretization of Hermite wavelets series approximations with collocation technique. However, while applied in two parameters singularly perturbed non-linear Benjamina-Bona-Mohany partial differential equation this method struggled by giving different values for different cases. Zyarah et al [8] have presented a solution of the non-homogenous linear ordinary differential equation in a method of variation of parameters of k -th order. It solved linear ordinary differential equations of orders 2, 3 and expand to the order k . However, this method failed to change the constants back to original functions. Mohanty et al [9] have presented a five point Implicit Linear Multistep Method that interpolates and collocates a basic power series along with its derivatives to find approximate solution of fourth order ODEs. It failed to obtain zero stability along with its approximate solution at the sense of three numerical tests that are not obtained. Sahabandu et al [10] have presented a method of Directly Defining inverse Mapping (MDDiM) to solve a nonlinear ordinary differential equation with the freedom of choosing the inverse linear mapping directly. However it failed to perform proper matching on the convergence and stability analysis of the system.

To overcome the drawback of complexity and stability a nonlinear ordinary differential equation with the freedom of choosing the inverse linear mapping directly, instead of calculating the inverse of an operator has been required. By portioning the equations with respect to unknown derivatives, a transform that performs approximation and restores the original signal in space to minimize errors is required and thus a model that use fuzzy transform to solve the differential equations have been presented in this paper.

3. Modified Fuzzy Transform model

Differential equations are mostly used in different fields of science such as ecology, finance and medical types. In these kind of applications, many cases rise to find solution for non-linear and linear systems of differential equation that has been complex and challenging, thus it requires more advanced techniques to find solution. Likewise, a number of physical phenomena are modelled by systems of ordinary or partial equations, for usually obtaining exact solution of this numerical problem here Fuzzy transform has been used. Because earlier approaches failed to provide simple and efficient solutions in terms of both linear and nonlinear n th order systems, fuzzy transform has been utilized here to identify the estimated solution and provide a correct one. The suggested mathematical model begins by establishing approximation preliminaries. The n th order derivative with real parameters with respect to t has been chosen for a linear differential equation. In the beginning, variables were mapped to fuzzy values using two theories for function translation: the improved format of the Euler Theorem and the Trapezoidal rule. The constraints for n th order are produced by substituting the derivatives of the differential equation; these solutions are then partitioned as continuous fuzzy values function. This continuous fuzzy value equation is then approximated to yield the Fuzzy equation, which has been proposed to solve the differential equation in both the linear and non-linear cases.

3.1. Basic Concepts

By assuming an interval $[c, d] \subset \mathbb{R}$, where $\mathbb{R}$ is denoted as real numbers. A fuzzy set of $[c, d]$ has been predicted along with their membership functions that have been mapping from $[c, d]$ into $[0, 1]$. Definitions that are been used here too in the case of unknown derivatives have been given.

Definition 1

By assuming fixed nodes from $v_1, \dots, v_n$, where $v_1 < v_2 < \dots < v_n$ within the given interval $[c, d]$, as here it is considered that $v_1 = c$ and $v_n = d$. Let basic functions of fuzzy can be given as $B_1, \dots, B_n$ in the interval $[c, d]$. While they satisfy the conditions given below for values of $p = 1, \dots, n$ the values are then set to $v_0 = c$ and $v_{n+1} = d$

1. $B_p(v): [c, b] \rightarrow [0, 1]$ is continuous with $B_p(v) > 0$, if $x \in (v_{p-1}, v_{p+1})$ and $B_p(v) = 0$ if $x \notin (v_{p-1}, v_{p+1})$

2. $B_p(v), p = 2, \dots, n$ strictly increases on $[v_{p-1}, v_p]$ and $v_p(v), p = 1, \dots, n-1$ that decreases in a linear way $[v_p, v_{p+1}]$
3. For all $v \in [c, d], \sum_{p=1}^n B_p(v) = 1$

Definition 2

Each fuzzy functions are identified by its membership functions as given below

$$B_1(v) = \begin{cases} 1 - \frac{v-v_1}{h_1}, & v_1 \leq v \leq v_2 \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

$$B_p(v) = \begin{cases} \frac{(v-v_{p-1})}{h_{p-1}}, & v_{p-1} \leq v \leq v_p \\ 1 - \frac{(v-v_p)}{h_p}, & v_p \leq v \leq v_{p+1} \\ 0, & \text{Otherwise} \end{cases} \quad (2)$$

$$B_n(v) = \begin{cases} \frac{(v-v_{n-1})}{h_{n-1}}, & v_{n-1} \leq v \leq v_n \\ 0, & \text{otherwise} \end{cases} \quad (3)$$

These equations are termed as triangular basic functions, here $p = 2, 3, \dots, n-1, h_k = v_{p+1} - v_p$.

Definition 3

Let $f(x)$ be a n times continuously differential function, then $F_{n,1}(x), \dots, F_{n,n}(x)$ be the n^{th} order fuzzy transform of function $f(x)$ with respect to the above stated partition $B_{n,1}(x), \dots, B_{n,n}(x)$ for derivatives.

Then for every $k \leq \mathbb{R}$, as $n \rightarrow \infty$, then it can be given as

$$F_{n,i(n,x_0)}^k(x_0) \rightarrow f^{(p)}(v_0) \quad (3)$$

An integer N such that for every v_0 and for every $n \geq N$ then it can be given as

$$|F_{n,i(n,x_0)}(v_0) - f(v_0)| < \varepsilon \cdot \frac{1}{n^d} \quad (4)$$

$$|F'_{n,i(n,x_0)}(v_0) - f'(v_0)| \leq \varepsilon \cdot \frac{1}{n^{d-1}} \quad (5)$$

$$|F''_{n,i(n,x_0)}(v_0) - f^{(p)}(v_0)| \leq \varepsilon \cdot \frac{1}{n^{d-p}} \quad (6)$$

$$|F^{(k)}_{n,i(n,x_0)}(v_0) - f^{(p)}(v_0)| \leq \varepsilon \cdot \frac{1}{n^{d-p}} \quad (7)$$

$$|F_{n,i(n,x_0)}^{(d)}(v_0) - f^{(p)}(v_0)| \leq \varepsilon \quad (8)$$

In these equations (4) -(7), ε is denotes the Euler's value of error variable that has to be minimized to get accurate solution. The above equations state that for continuous fuzzy transform function that has with respect to each increase in integer values.

Definition 4

Given f is a continuous function on $[c, d]$ and $B_p(v), p = 1, \dots, n$ be a uniform fuzzy partition of $[c, d], n \geq 2$. A vector of real numbers $F(f) = (F_1, F_2, \dots, F_n)$ that can be given as

$$F_p[f] = \frac{\int_a^b f(v)B_p(v)dv}{\int_a^b B_p(v)dv} = \frac{\int_{v_{p-1}}^{v_{p+1}} f(v)B_p(v)dv}{\int_{v_{p-1}}^{v_{p+1}} B_p(v)dv}, p = 1, \dots, n \quad (9)$$

This equation is termed as direct fuzzy transform f with respect to $B_1, \dots, B_n$.

Remark 1

Elements of $F_1[f], \dots, F_n[f]$ are called components of fuzzy transform. While $B_1, \dots, B_n$ forms a uniform fuzzy partition then the expression can be given as

$$F_1[f] = \frac{2}{h} \int_{v_1}^{v_2} f(v)B_1(v)dv, \quad (10)$$

$$F_n[f] = \frac{2}{h} \int_{v_{n-1}}^{v_n} f(v)B_n(v)dv \quad (11)$$

$$F_k[f] = \frac{1}{h} \int_{v_{k-1}}^{v_{k+1}} f(v)B_p(v)dv, p = 2, \dots, n-1 \quad (12)$$

This uniform fuzzy partition is utilized for uniform intervals of differential equation that are given as equation form.

Remark 2

An important property of the direct fuzzy transform as well as inverse fuzzy is their linearity namely given $f, g \in C[c, d]$ and $\alpha, \beta \in \mathbb{R}$, if $h = \alpha f + \beta g$, then $F[h] = \alpha F[f] + \beta F[g]$ and $\hat{h} = \alpha \hat{f} + \beta \hat{g}$.

Definition 5

By considering that $x_1, \dots, x_n$ are equidistant with distance given as $h, h = (d - c)/(n - 1)$. A system of fuzzy sets $B_1, \dots, B_n[a, b] \rightarrow [0, 1]$ that has been generalized to be a uniform fuzzy partitions of $[c, d]$ that can be given as

$$B_p(t) = \begin{cases} B_p(t, a), & t \in [c, d] \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \lambda \beta K \left(\frac{t-t_k}{h} \right), & t \in [c, d] \\ 0, & \text{otherwise} \end{cases} \quad (13)$$

In this equation, $t_p = c + (p-1)h$, $\beta = 1/P(0)$, $P(0) \neq 0$, $\beta > 0$ and $\lambda > 0$. A function generated is given by K and basic function has been given as B_p , $p = 1, \dots, n$.

Lemma 1. From the above definition the basic functions B_p , $p = 1, \dots, n$ that has a h -uniform partition of fuzzy that has shifted copies of $\lambda \beta K$ defined. Then, for each $p = 1, \dots, n$, $B_k(t_k) = \lambda$, $t_k \in [t_k - h, t_k + h]$

Proof

Using trapezoid formula with nodes t_{p-1}, t_p, t_{p+1} to the numerical calculation of integral that has been given as $p = 1, \dots, n$ and $0 < \lambda = 1 / \left(\int_{-1}^1 \beta P(t) dt \right)$. Thus, in the limits of a to b the value of integral of $f(v)$ can be given in general as

$$\int_c^d f(v) dv \approx (d-c) \cdot \frac{f(c)+f(d)}{2} \quad (14)$$

For the given interval before applying to the trapezoidal rule, the equation is integrated using by fractional Adams-Bashforth-Moulton method and given for $y(t_p + 1)$ is given as

$$y(t_p + 1) = y(t_p) + \int_{t_p}^{t_{p+1}} [f(z, y(z))] dz \quad (15)$$

This integral has been then rearranged as

$$\int_c^d [g(z)] dz \approx \frac{d-c}{2} [g(c) + g(d)] \quad (16)$$

In this stage by applying trapezoidal rule, then the result has been given as

$$y(t_{p+1}) = y(t_p) + \frac{h}{2} [f(t_p, y(t_p)) + f(t_{p+1}, y(t_{p+1}))] \quad (17)$$

In this equation, $y(t_p) = y_p$ and $y(t_{p+1}) = y_{p+1}$, then it can be given as

$$y_{p+1} = y_p + \frac{h}{2} [f(t_p, y_p) + f(t_{p+1}, y_{p+1})] \quad (18)$$

By taking a predictor y_{p+1}^r the first step of y_{p+1} is given as

$$\int_c^d [g(z)] dz \approx (d-c)g(c) \quad (19)$$

Then, by means of applying trapezoidal rule, the construction known as Euler formula or one-step Adams-Bashforth method given as

$$y_{p+1} = y_p + \frac{h}{2} [f(t_p, y_p) + f(t_{p+1}, y_{p+1}^r)] \quad (20)$$

After the construction of predictor under Adams-Moulton method it can be rewritten in the form that can be given as

$$\int_0^{t_{p+1}} [(t_{p+1} - z)^{\alpha-1} g(z)] dz = \sum_{j=0}^p b_{j,K} + g(t_j) \quad (21)$$

In this equation, $b_{j,k+1}$ can be given as

$$b_{j,p+1} = \frac{h^\alpha}{\alpha} [(p+1-j)^\alpha - (p-j)^\alpha] \quad (22)$$

In order to solve the equations numerically the following equation has been used. With the help of these equations an approximation model has been built for linear and non-linear case.

3.2. Steps for construction of Fuzzy partition

1. Specify number of n components and calculate the step $h = (t_n - t_1)/(n-1)$
2. Assign nodes $t_1 < \dots < t_n$, here $t_p = t_1 + h(p-1)$
3. By selecting basic membership functions as trapezoidal ones that helps to determine the course of $\hat{f}$ as that shows whether the system in linear or non-linear equation
4. By constructing a uniform fuzzy partition from $[t_1, t_n]$ by trapezoidal functions
5. Then the approximated values that have derivatives are replaced with respect to fuzzy partition from step 1.
6. Calculate the approximation based on x and y by inverse fuzzy transform that has been applied to $[X_1, \dots, X_n]$ and $[Y_1, \dots, Y_n]$

3.3. Numerical solution of linear and non-linear equations using fuzzy transform

An n^{th} order differential equation is given as linear then it can be given as

$$y^{(n)}(x) + a_1 y^{(n-1)}(x) + \dots + a_{n-1} y'(x) + a_n y(x) = 0 \quad (23)$$

And assumed that the variable y and all its derivatives in the equation are given a power of one. For all the coefficients $a_n(x), a_{n-1}(x), \dots, a_0(x)$, the function $g(x)$ are either constants or depend only on the independent variable (x) . If the conditions are not satisfied then the system is considered as non-linear. To solve this equation, a scheme of fuzzy transform with respect to the functions f and g on the interval $[t_1, t_n]$. Before applying the direct fuzzy transform to both parts of differential equation the values are replaces derivatives by Taylor expansion by their approximations $y(x+h) = y(x) + y'(x) + \dots + y^{(n-1)}(x) + y^n(x)$ (24)

By taking $y(x)$ to L.H.S and divide it by h then it can be given as

$$\frac{y(x+h)-y(x)}{h} = y'(x) + \frac{h}{2!} y''(x) + \dots + \frac{h}{n!} y^n(x) \quad (25)$$

$$y'(x) = \frac{y(x+h)-y(x)}{h} + O(h) \quad (26)$$

Thus, $O(h)$ has been a minimum value it can be eliminated for estimation

$$y'(x) = \frac{y^+(x)-y(x)}{h} \quad (27)$$

By applying direct fuzzy transform at both sides of the equation, the result can be given as

$$F_n[y'(x)] = \frac{1}{h} [Fn(y^+(x) - Fn(x))] \quad (28)$$

$$F_n[y'] = [y'_1 \dots y'_{n-1}] \quad (29)$$

$$F_n[x] = [x_1 \dots x_{n-1}] \quad (30)$$

$$F_n[x^+] = [x_1^+ \dots x_{n-1}^+] \quad (31)$$

$$X_k^+ = X_{k+1}, k = 2, \dots, n-2 \quad (32)$$

$$Y_k^+ = Y_{k+1}, k = 2, \dots, n-2 \quad (33)$$

$$\begin{cases} X_1^+ = X_2 \\ Y_1^+ = Y_2 \end{cases} \quad (34)$$

By applying lemma 1

$$Y_k^+ = y_{k+1} = y_k + \frac{h}{2} [f(t_k, y_k) + f(t_{k+1}, y_{k+1})] \quad (35)$$

$$\frac{2Y_k^+}{h} = (Y_{k+1} - Y_k) + O(h), k = 1, \dots, n-1 \quad (36)$$

At this stage the values are then converted to $(n-1) \times n$ matrix is given as

$$D = \frac{h}{2} \begin{pmatrix} -1 & 1 & 0 & \dots & 0 \\ \vdots & -1 & 1 & \ddots & \vdots \\ 0 & 0 & -1 & \dots & 1 \end{pmatrix} \quad (37)$$

Therefore, the equality has been rewritten as

$$Fn[y'] = DFn[y] \quad (38)$$

Where,

$$F_n[y'] = [y'_1 \dots y'_{n-1}] \quad (39)$$

And

$$F_n[x^+] = [x_1^+ \dots x_{n-1}^+] \quad (40)$$

By considering the first row of initial value

$$D^* = \frac{h}{2} \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ \vdots & 1 & 0 & \ddots & \vdots \\ 0 & -1 & 1 & \dots & 0 \\ 0 & 0 & -1 & \dots & 1 \end{pmatrix} \quad (41)$$

Thus, D^* is given matrix in non-singular form, depending on initial conditions and the matrix an expansion of $F_n[f]$.

$$F_n^*[f] = \left[\frac{x_1}{h}, F_1, \dots, F_{n-1} \right]^T \quad (42)$$

Solution of this equation can be given as

$$F_n[x] = (D^*)^{-1} \cdot F_n^*[f] \quad (43)$$

By computing inverse matrix D^* , then it can be given as

$$(D^*)^{-1} = h \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ \vdots & 1 & 0 & \ddots & \vdots \\ 0 & 1 & 1 & \dots & 0 \\ 0 & 0 & 1 & \dots & 1 \end{pmatrix} \quad (44)$$

By this matrix this can be given as

$$X_{k+1} = X_k + hF_k, X_1 = (\alpha | Y_{k+1} = Y_k + hG_k),$$

$$Y_1 = \beta, k = 1, \dots, n-1 \quad (45)$$

However, it cannot be applied directly by using the function $f(t, x, y)$ or $g(t, x, y)$, because it uses unknown functions x and y . These unknown derivatives are then replaced by fuzzy components given below

$$\widehat{F}_k[f] = \frac{\int_{t_1}^{t_n} f(t, X_k, Y_k) A_k(t) dt}{\int_{t_1}^{t_n} A_k(t) dt} \quad (46)$$

Components of fuzzy transform for x and y that has been approximated by

$$X_{k+1} = X_k + h\widehat{F}_k \quad (47)$$

Finally, by using inverse fuzzy transform the equation can be given as

$$x_n(t) = \sum_{k=1}^n X_k A_k(t) \quad (48)$$

$$y_n(t) = \sum_{k=1}^n Y_k A_k(t) \quad (49)$$

where $A_k(t), k = 1, 2, \dots, n$ are given basic functions. The proposed method is similar to the well-known Euler method and under similar assumptions. It has the same degree of accuracy. In the next theorem, we obtain an error estimate in the context of a fuzzy partition and error analysis.

Example 1.

Let us consider a linear boundary value problem of five order

$$y^{(5)}(x) = y(x) - 15e^x - 10xe^x, 0 \leq x \leq 1 \quad (50)$$

$$y(0) = 0, y'(0) = 0, y(1) = 0, y'(1) = -e \quad (51)$$

Let there be a solution that

$$y(x) = x(1-x)e^x \quad (52)$$

Now by using the obtained solution of proposed mathematical model we get

$$y_{p+1} = y_p + \frac{h}{2} [f(t_p, y_p) + f(t_{p+1}, y_{p+1})] \quad (53)$$

This gives

$$y_5 = y_4 + \frac{h}{2} [f(t_4, y_4) + f(t_5, y_5)] \quad (54)$$

Let the initial conditions be taken as

$$y(0) = 0, u(0) = B \quad (55)$$

By using simplification, we get

$$y^{(5)}(x) = y(x) - 15e^x - 10xe^x, 0 \leq x \leq 1 \quad (56)$$

$$y'(x) + \frac{h}{2!} y''(x) + \dots + \frac{h}{(n-1)!} y^{(n-1)}(x) + \frac{h}{n!} y^n(x) \quad (57)$$

$$y^{(5)}(x) = x + \frac{Ax^3}{3!} + \frac{1}{3!} \int_0^x (x-t)^3 u_k(t) dt \quad (58)$$

$$u^{(5)}(x) = B + 5(1-e^x) - 10xe^x + \int_0^x y_k(t) dt \quad (59)$$

With initial approximations

$$u_0(x) = B, y_0(x) = 0 \quad (60)$$

Consequently, with the help of proposed model we obtain an approximation

$$y_1 = x + \frac{1}{6} Ax^3 + \frac{1}{24} Bx^4 \quad (61)$$

$$u_1 = B - 5e^x - 10xe^x + 5 \quad (62)$$

$$y_2 = -24x + \frac{1}{6} Ax^3 + \frac{1}{24} Bx^4 + \frac{5}{24} x^4 + 35e^x - 10xe^x - 35 - \frac{5}{6} x^3 - \frac{15}{2} x^2 \quad (63)$$

$$u_2 = B + \frac{1}{2} x^2 + \frac{1}{24} Ax^4 + \frac{1}{120} Bx^5 - 5e^x - 10xe^x + 5 \quad (64)$$

$$y_3 = -24x + \frac{1}{6} Ax^3 + \frac{1}{24} Bx^4 + \frac{5}{24} x^4 - 10xe^x + 35e^x + \frac{1}{362880} Bx^9 + \frac{1}{720} x^6 + \frac{1}{40320} Ax^8 - 35 - \frac{5}{6} x^3 - \frac{15}{2} x^2 \quad (65)$$

$$u_3 = B - 12x^2 + \frac{1}{24} Ax^4 + \frac{1}{120} Bx^5 + \frac{1}{24} x^5 + 40e^x - 20xe^x - 35x - \frac{5}{24} x^4 - \frac{5}{2} x^3 - 40 \quad (66)$$

$$y_4 = -99x + \frac{1}{6} Ax^3 + \frac{1}{24} Bx^4 - \frac{5}{3} x^4 - 20xe^x + 120e^x - \frac{7}{24} x^5 + \frac{1}{362880} Bx^9 - \frac{1}{30} x^6 - \frac{1}{336} x^7 - \frac{1}{8064} x^8 + \frac{1}{72576} x^9 + \frac{1}{40320} Ax^8 - 120 - 40x^2 - 10x^3 \quad (67)$$

$$u_4 = B - 12x^2 + \frac{1}{24} Ax^4 + \frac{1}{120} Bx^5 + \frac{1}{24} x^5 - 20xe^x + 40e^x + \frac{1}{362880} Bx^{10} + \frac{1}{5040} x^7 + \frac{1}{362880} Ax^9 - 35x - \frac{5}{24} x^4 - \frac{5}{2} x^3 - 40 \quad (68)$$

$$y_5 = -99x + \frac{1}{6} Ax^3 + \frac{1}{6227020800} Ax^{13} + \frac{1}{24} Bx^4 + \frac{1}{87178291200} Bx^{14} - \frac{5}{3} x^4 + \frac{1}{39916800} x^{11} - 20xe^x + 120e^x - \frac{7}{24} x^5 + \frac{1}{362880} Bx^9 - \frac{1}{30} x^6 - \frac{1}{336} x^7 -$$

$$\frac{1}{8064}x^8 + \frac{1}{72576}x^9 + \frac{1}{725760}x^{10} + \frac{1}{362880}Ax^9 - 120x - \frac{40}{3}x^3 - \frac{5}{2}x^4 - 135 \quad (69)$$

Thus, obtain the solution is given as

$$y(x) = x + \frac{1}{6}Ax^3 + \frac{1}{24}Bx^4 - \frac{1}{8}x^5 - \frac{1}{30}x^6 - \frac{1}{144}x^7 + \left(\frac{1}{40320}A - \frac{1}{896}\right)x^8 + \left(-\frac{11}{72576} + \frac{1}{362880}B\right)x^9 - \frac{1}{45360}x^{10} - \frac{1}{403200}x^{11} - \frac{1}{3991680}x^{12} + \left(-\frac{1}{44478720} + \frac{A}{6227020800}\right)x^{13} + \left(-\frac{1}{544864320} + \frac{B}{87178291200}\right)x^{14} + O(x^{15}) \quad (70)$$

By applying boundary conditions $y(1) = 0$ and $y'(1) = -e$

$$A = -2.999999998 \text{ and } B = -8.000000004 \quad (71)$$

Final solution can be then given as

$$y = x - 0.49999999967x^3 - 0.33333333350x^4 - \frac{x^5}{8} - \frac{x^6}{30} - \frac{1}{144}x^7 - 0.001190476190x^8 - 0.00017361111112x^9 - \frac{x^{10}}{45360} - \frac{x^{11}}{403200} - \frac{x^{12}}{3991680} - 0.00000002296443268x^{13} - 0.0000000001927085261x^{14} + O(x^{15}) \quad (72)$$

Table 1. Error values

Example 1		
x	y(x)	
	Existing Technique1[32]	MFT
0.00	0.000000	0.000000
0.10	-3E - 11	-3E - 12
0.20	-2.4E - 10	-2.4E - 11
0.30	-4.4E - 10	-7E - 10
0.40	-8E - 10	-1.7E - 10
0.50	-1.2E - 9	-1.3E - 10
0.60	-2.2E - 9	-7.4E - 10

0.70	-1.9E - 9	-1E - 9
0.80	-1.7E - 9	-1.4E - 9
0.90	-1.3E - 9	-1.3E - 9
1.00	-1E - 9	-1E - 9

Example 2

In the next example let us consider a non-linear eighth order problem

$$y^{(viii)}(x) = e^{-x}y^2(x), 0 \leq x \leq 1 \quad (73)$$

With given boundary conditions as

$$y^{(i)}(0) = 1, i = 0, 1, \dots, 5 \quad y(1) = y'(1) = e \quad (74)$$

The exact solution is

$$y(x) = e^x \quad (75)$$

Now with the help of proposed model

$$y'(x) = \frac{y(x+h)-y(x)}{h} + O(h) \quad (76)$$

$$u'(x) = e^{-x}y^2 \quad (77)$$

With given initial conditions as

$$y(0) = 1, u(0) = B \quad (78)$$

Where

$$F(u) = \frac{1}{120} \int_0^x (x-t)^5 u(t) dt \quad (79)$$

Iterations can be given as

$$y_{k+1}(x) = \sum_{i=0}^5 \frac{x^i}{i!} + \frac{Ax^6}{6!} + \frac{1}{6!} \int_0^x (x-t)^6 u_k(t) dt \quad (80)$$

$$u_{k+1}(x) = B + \int_0^x e^{-t} y_k^2(t) dt \quad (81)$$

With initial approximations

$$y_0(x) = 1; u_0(x) = B \quad (82)$$

Again this approximation given a result of

$$y_1 = 1 + x + \frac{1}{2}x^2 + \frac{1}{5}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 A + \frac{1}{5040}Bx^7 \quad (83)$$

$$u_1 = B - e^{-x} + 1 \quad (84)$$

The solution can be given as

$$y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{Ax^6}{720} + \frac{Bx^7}{5040} + \frac{x^8}{40320} + \frac{x^9}{362880} + \frac{x^{10}}{3628800} + \frac{x^{11}}{39916800} + \frac{x^{12}}{479001600} + \frac{x^{13}}{6227020800} + \left(-\frac{1}{87178291200} + \frac{A}{43589145600}\right)x^{14} + \left(-\frac{1}{1307674368000} + \frac{B}{653837184000}\right)x^{15} + O(x^{16}) \quad (85)$$

By applying boundary condition at $x = 1$ gives a solution of $A = 0.99999999997$ and $B = 1.0000000026$

$$y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + 0.001388888888x^6 + 0.0001984126990x^7 + \frac{x^8}{40320} + \frac{x^9}{362880} + \frac{x^{10}}{3628800} + \frac{x^{11}}{39916800} + \frac{x^{12}}{479001600} + \frac{x^{13}}{6227020800} + 1.147074559 \times 10^{-11}x^{14} + 7.647163778 \times 10^{-13}x^{15} + O(x^{16}) \quad (86)$$

Table 2. Error values

Example 2		
x	$y(x)$	
	Existing Technique1[31]	MFT
0.25	$4.91E - 5$	$7.023E - 17$
0.50	$7.039E - 5$	$-3E - 15$
0.75	$4.99E - 5$	$5.77E - 11$
0.85	-	$4.55E - 15$
0.95	-	$3.71E - 14$
1.00	-	$1.60E - 14$

From these two examples it has been clearly visible that the proposed technique of fuzzy transform has been better for approximation of differential equations of linear and non-linear case. This error values are found to be minimized between normal solution and exact solution obtained. Previous technique has been calculated to give high error value that has been not suitable for system applications.

4. Result

In this paper, a Fuzzy transform based approximation model to predict the solution for linear and nonlinear system has been presented. Preliminaries are initialized for fuzzy transform as triangular function and trapezoidal functions. Trapezoidal functions are modified based on improved Euler method; this definition has been then proved. This equation is then estimated using direct fuzzy transform and approximated for accurate solution that was nearer to analytic solution. An inverse direct fuzzy transform has been executed to bring it back to original data. The proposed model of MFT has been implemented in MATLAB with the following system specification of Windows 8 OS with Intel Core i5 processor and 8GB RAM. Results are given for MFT at same n values but with different fuzzy mapping functions, exact plot of the approximated value has been presented.

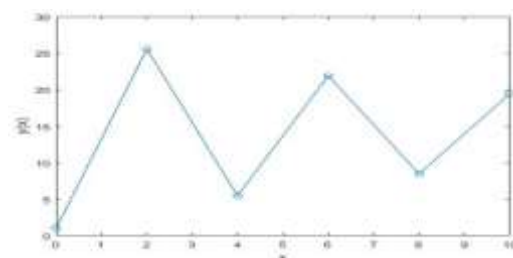


Figure 1. Approximated solution of fuzzy trapezoidal function for $n = 5$

F-transform of the given example has been calculated and obtained using the above equations, these values are plotted to show the basic fuzzy functions that has been generated for mapping purpose. Figure 1, shows the approximated basic trapezoidal function that has been echoed towards Euler's theorem and given as fuzzy membership function for mapping.

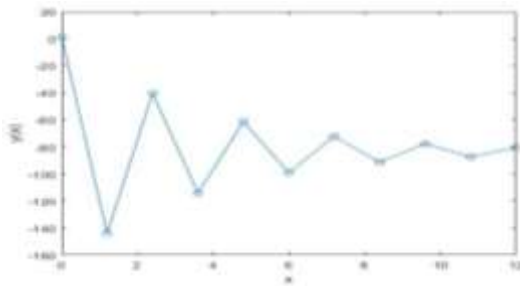


Figure 2. Approximated solution for fuzzy sinusoidal function of $n = 5$

Using the aforementioned equations, the F-transform of the provided example has been computed and produced; these values are displayed to demonstrate the fundamental fuzzy functions that have been constructed for mapping purposes. Figure 2 depicts an approximation of the basic fuzzy sinusoidal function, which has been presented as a fuzzy membership function for mapping.

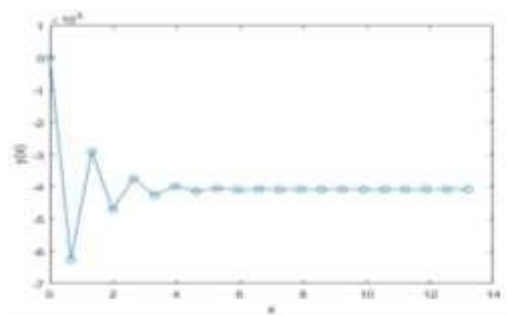


Figure 3. Approximated solution for fuzzy triangular function of $n = 5$

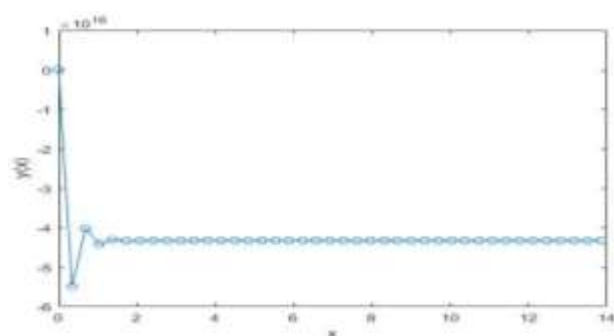


Figure 4. Approximated solution for fuzzy triangular function of $n = 8$

In this figure 3 and 4, a weighted triangular approximation of a fuzzy number to approximately represent a given fuzzy number with respect to distance d defined by Equation by its calculation expression and presented graphically. From this figure it has been clearly visible that the membership functions are linear towards end.

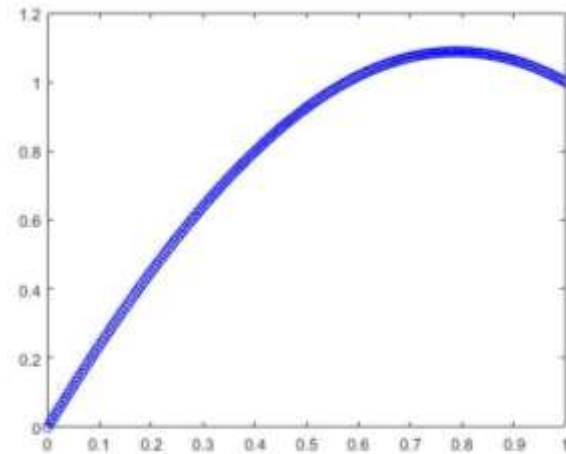


Figure 5. Approximated solution for linear system for $n = 5$

From example 1, the values and their responses are given to Matlab and the result are plotted in the graph. By varying the x value with respect to the time sequences the error values are tabulated and the respective errors are found to be minimized for the proposed technique. MFT has been found to be more effective for the approximation of finding best solutions for differential equations. From the graph it can be stated clearly that, better results are obtained using MFT at linear case. Here the dots represent the exact solution values and the line represent the normal solution that has been obtained, variations are found to be minimum for the exact boundary value equations.

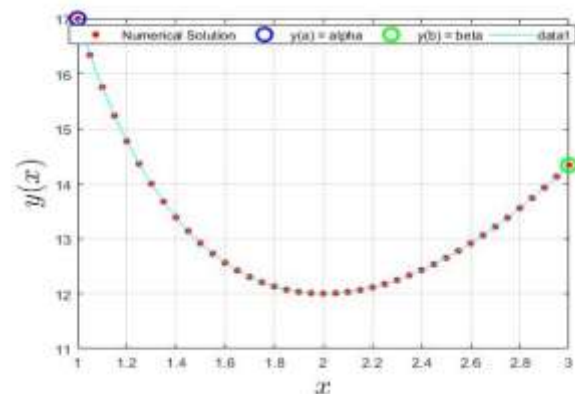


Figure 6. Approximated solution for nonlinear system for $n=8$

The numbers and answers from Example 2 are fed into Matlab, and the results are shown in the graph. The error values are tabulated by altering the x value with regard to the time sequences, and the relevant errors are determined to be minimized for the proposed approach. MFT has been proven to be more successful at approximating optimal solutions for differential equations. From figure 6, it has been understandable that the numerical solution

has been plotted as red dots and its respective boundary values are plotted so as to show that the system has been more efficient towards non-linear variables.

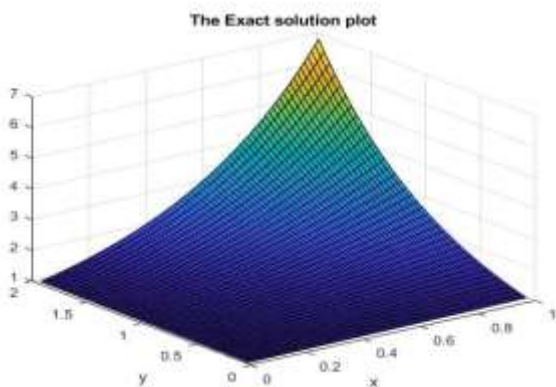


Figure 7. Exact solution for boundary value problem for linear system

Figure 7 and figure 8 represents the graph that shows that MFT has been efficient to predict the exact solution of the boundary value problem. The exact solution has been obtained between x in the interval of 0 to 1 and y interval of 0 to 2. Exact solution has been obtained at the range of 4,5,6 and 7.

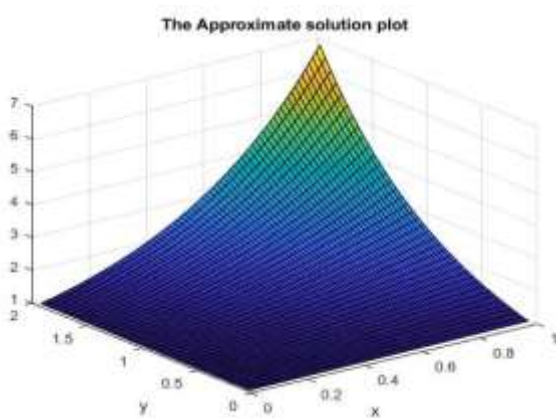


Figure 8. Approximate solution for boundary value problem for linear system

Similarly in the figure 8, the approximate solution that has been obtained at the range of 4,5,6 and 7. Thus the relative error has been minimized to an extent and the exact solution has been approximated towards end, this was due to the application of direct fuzzy after the partition of equations to uniform interval and then the applied fuzzy to membership function has been more effectively depicted to give the nearer exact solution.

5. Conclusion

The aim of this work is to provide an alternate way for solving Differential Equations of n th order. This study will focus on a numerical algorithm that will employ the Fuzzy transform as a technique to estimate the solution of Ordinary Differential Equations for both linear and nonlinear case. The numerical approach will be thoroughly explored and used to several instances with uniform and non-uniform partitions. Then direct and inverse MFT has been applied to obtain the exact solution. The results demonstrate that the MFT is an effective technique for estimating the solution of Differential Equations.

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