On Hamilton-connectedness and detour concepts of Generalized Sierpiński Graphs

Y. Sherlin Nisha, V. Rajeswari

Department of Mathematics, Sri Sairam Institute of Technology,
Chennai- 600044,India.

Department of Mathematics, Sri Sairam Institute of Technology,
Chennai - 600044,India.

Abstract

The structure of a chemical compound can be represented by a molecular graph, a labeled graph whose vertex and edge labels specify the atom and bond types, respectively. Topological indices are graph invariants defined to analyze the drug molecular structures and pharmacy characteristics, such as melting point, boiling point, and other biochemical activities. Of the numerous topological indices, the detour index has proven to be among the most useful at estimating many biochemical properties of molecules. In this paper we contribute to the literature of computational chemistry by providing exact expressions for the detour index of joins of Hamilton-connected (*HC*) graphs. This improves upon existing results by loosening the require ment of a molecular graph being Hamilton-connected and only requirement certain sub graphs to be Hamilton-connected. In this paper we prove that the generalized Sierpinski graph of any Hamilton-connected graph is Hamilton connected. Hence we find the Detour index of the same.

Keywords: Detour index, Sierpiński graphs, elongated path, Hamiltonian properties, Hamilton-connected.

AMS

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Introduction:

Let $G=(V,E)$ be a non-empty graph of order $n\geq 2$, and l is a positive integer, We denote by V^l is the set of words of length l on alphabet V. The letters of a word u of length l are de- noted by $u_1,u_2,u_3...u_l$. The concatenation of two words u and v is denoted by uv. Klavzar and U.Miilutinovic introduced in [31] the graph $S(K_n,l)$, $l\geq 1$, whose vertex set V^l, where $\{u,v\}$ is an edge if and only if there exists $i\in \{1,2,3 \ldots . l\}$ such that

(i) $u_j = v_j$, if $j<i$

(ii) $u_l = v_i$

(iii) $u_j = v_i v_i$ and $v_j = u_i$ if $j > i$

As noted[50], in a compact form, the edge sets can be described

As $\left\{\left\{wu_i u^{d-1}{}_j, wu_j u^{d-1}{}_i\right\}: u_i, u_j \in V, i \neq j\ d\epsilon[t], w \in V^{l-d}\right\}$ The graph $S(K_3, l)$ is isomorphic to the graph of the Tower of Hanoi with l disks [31]. Later, those graphs have been called Sierpinski graphs in [51] and they were studied by now from numerous points of view.For instance, the authors of [32] studied identifying codes, locating- dominating codes, and total-dominating codes in Sierpinski graphs. In [33] the authors propose an algorithm, which makes use of three automata and the fact that there are at most two internally vertex disjoint shortest paths between any two vertices, to determine all shortest paths in Sierpinski graphs. The authors of [31] proved that for any $n \geq 1$ and $l \geq 1$,the Sierpinski graph $S(K_n, l)$ has a unique 1-perfect code (or efficient dominating set) if l is even, and $S(K_n, l)$ has exactly n 1-perfect codes if l is odd. The Hamming dimension of a graph G was introduced in [32] as the largest dimension of a Hamming graph into which G embeds as an irredudant induced sub graph. That paper gives an upper bound for the Hamming dimension of the Sierpinski graphs $S(K_n, l)$ for $n \geq 3$. It also shows that the Hamming dimension of $S(K_3, l)$ grows as 3^{l-3}. The idea of almost extreme vertex of $S(K_n, l)$ was introduced in [34]as a vertex that is either adjacent to an extreme vertex of $S(K_n, l)$ or is incident to an edge between two subgraphs of $S(K_n, l)$ isomorphic to $S(K_n, l-1)$. The authors of [32] deduced explicit formulas for the distance in $S(K_n, l)$ between an arbitrary vertex and an almost-extreme vertex. Also they gave a formula of them etric dimension of a Sierpinski graph, which was independently obtained by Parreau in her Ph.D. thesis. The eccentricity of an arbitrary vertex of Sierpinski graphs was studied in [34] where the main result gives an expression for the average eccentricity of $S(K_n, l)$. For a general background on Sierpinski graphs, the reader is invited to read the comprehensive survey [35] and references there in.

Figure 1 shows a graph G and the genenalized Sierpinski graph $S(K_4, l)$.

Notice that if {u,v} is an edge of $S(G, l)$. , then there is an edge {x,y} of G and a word w such that

$u = wxyyyy \ldots .y\ and\ v = wyxxxxx \ldots .x$ In general,$S(G, t)$. Can be contructed recursively from G with

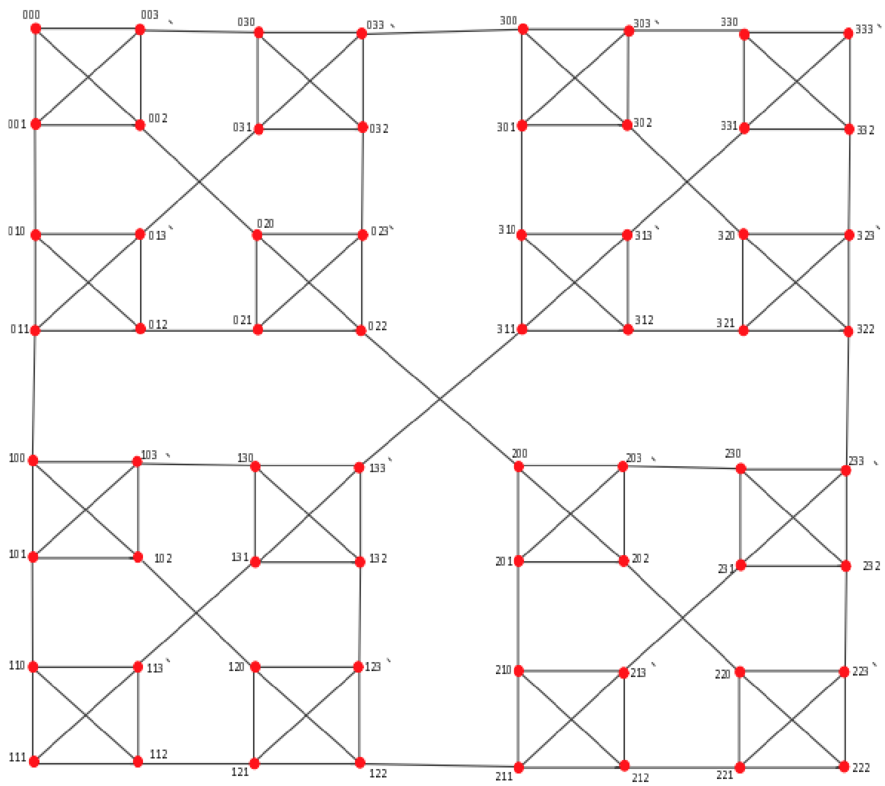

Figure 1: sierpinski graph $(K_4, 3)$

The following process: $S(G,1)$=G. and for $t \geq 2$, we copy n times $S(G,l-1)$ add the letter x at the beginning of each label of the vertices belonging to the copy of $S(G,l-1)$ corresponding to x. Then for every edge{x,y} of G, add an edge between vertex $wxyyyy \ldots .y$ *and* $wyxxxxx \ldots .x$ See, for instance,Figure2,vertices of the form $xx\ldots x$ are called extreme vertices of $S(G,l)$. Notice that for any graph G of order n and any integer $t \geq 2$, $S(G,l)$ has n extreme vertices and, if x has degree $d(x)$ in G, then the extreme vertex $xx\ldots x$ of $S(G,l)$also has degree $d(x)$. More over the degrees of two vertices $yxx\ldots x$ and $xyy\ldots y$, which connect two copies of $S(G,l-1)$, are equal to $d(x)$+1 and $d(y)$ + 1,respectively.

For any $w \in V^{l-1}$and $l \geq 2$, the subgraph $\langle V_w \rangle$ of $S(G,l)$, induced by $\langle V_w \rangle$ =wx : $x \in V$, is isomorphic to G. Notice that there exists only one vertex $u \in \langle V_w \rangle$ of the form $w'xx\ldots x$, where $w' \in V^r$ for some $r \leq t-2$. We will say that $w'xx\ldots x$ is the extreme vertex of $\langle V_w \rangle$, which is an extreme vertex in $S(G,l)$whenever $r = 0$ By definition of $S(G,l)$. Suppose that W is an interconnection network (network for short). A path (cycle) in W is called a Hamiltonian path (Hamiltonian cycle) if it contains every node of W is called a Hamiltonian if there is a Hamitonian cycle in W, and it is called Hamiltonian-connected

[63] if there is a Hamiltonian path between every two distinct nodes of W. Some topologies, such as the hierarchical cubic network [36], are Hamiltonian-connected.

Since no defaults and link faults may develop in a network, it is practically important to consider faulty networks. A network W is called $k-node$(k-link) Hamiltonian if it remains Hamiltonian after removing any$k$nodes (links) [36]. If W has node (link) connectivity k+2 and is $k-node$ (k-link) Hamiltonian, then it can tolerate a maximal number of node (link) faults while embedding a longest fault-free cycle. Some networks have been shown to be $k-node$ Hamiltonian and $k-link$ Hamiltonian. For example, the hierarchical cubic network with connectivity n+1 is $n-1$-link Hamiltonian [36]. The$n-dimensional$ twisted cube [36] is $n-2$-node Hamiltonian and $n-2$-link Hamiltonian.

Note that an $n-link$ Hamiltonian graph cannot be guaranteed to be $n-node$ Hamiltonian. For example, the $n-cube$ is$n-2$-link Hamiltonian, but not $n-2$-node Hamiltonian[38]. In [68], the WK-recursive network with connectivity $d-1$ was shown to be$d$ 3-link Hamiltonian. It is not possible to reuse their approach of replacing link failures with node failures because a faulty node will cause $d-1$ links to fail and because their approach can handle at most $d-3$ faulty links.

Each node of $S(W_k, l)$. Is labeled as a $l-digit$ radix k number. Node $u_{l-1}u_{l-2}...u_1u_0$ is adjacent to1. $u_{l-1}u_{l-2}...u_1v$, where $v_l = u_0$ and adjacent to 2. $u_{l-1}u_{l-2} \ldots\ldots u_{j+1}v(v_0)$if $u_j = u_{j-1} = u_{j-1}u_j = u_{j-2}$ $=u_{j-2}=...u_0$, where $v_1=u_{j-1}v_0=u_j$ and $(b_0)^j$ denotes j consecutive v_0s. The links of 1 are called substituting links and are labeled 0. The link 2 is called a j flipping link and is labelled

j. For example, the 2-flipping link connects node 033 and node 300 in $S(W_5, 3)$. In addition, if $u_{l-1}=u_{l-2}= ...u_0$, then an open link labeled l is incident with $u_{l-1}u_{l-2}...u_1u_0$. The open link is reserved for further expansion; hence, it so the rend node is unspecified.

Note that $S(W_k, l)$ is a k-node Wheel graph augmented with l open links. Each node of $S(W_k, l)$ is incident with $l-1$ substituting links and one flipping link (open link). The substituting links are those within basic building blocks, and the j-flipping links are those connecting two embedded $S(W_k, l)$

In order to agree whether a given network is robust, a way to quantitatively measure network robustness is needed. Naturally robustness is all about back-up possibilities, or alternative paths, but it is a challenge to capture these concepts in a mathematical formula. During the past few decades a lot of robustness measures have been proposed [39]. Scientists from different back grounds like mathematics, chemistry, physics, network science and biological science have accepted the network robustness research as one of the upcoming research area. As a result, a lot of different methods to seizure the robustness properties of a network have been undertaken. All of these approached are based on the analysis of the under lying graph consisting of vertices connected by edges of a network [3,6,9,12,24,39]. One such category is the distance-based descriptors.

Hamilton-connected Graphs

A graph *G* is Hamilton-connected (*HC*-graphs) if every two vertices of *G* are connected by a Hamilton path. All *HC* graphs are Hamiltonian and all Hamiltonian graphs need not be Hamilton-connected. All complete graphs are Hamilton-connected and Hamiltonian (with the trivial exception of the singleton graph), and all bipartite graphs are not Hamilton-connected. Many researcher have done wide research work on Hamiltonian property of graphs in[14,16,28,32,34,38].

Theorem 1. *Let $U = u_{l-1}u_{l-2}\dots u_1u_0$ and $V = v_{l-1}v_{l-2}\dots v_1v_0$ be two distinct nodes in $S(W_k, l)$, where $d \geq 4$*

1. There is an $U - V$ Hamiltonian path for $S(W_k, l)$.

2. Given two distinct constants $a, b \in 0, 1, 2\dots k-1$ with $\{c, e\} \neq u_{l-1}, v_{l-1}$ there exists an $U - X$ path $V - Y$ path such that they are disjoint and contain all the nodes of $S(W_k, l)$ where $\{X, Y\} = \{(a)^l, b^l\}$ (Since$\{a, b\} \neq u_{l-1}, v_{l-1}\}$), we have $\{X, Y\} \neq \{U, V\}$, that is, it is possible that $U = X$ or $V = Y$ but not both. Note that if both $U = X$ and $V = Y$, then the $U - X$ path degenerates to a node U and the $V - Y$ path degenerates to a node V. As a result the $U - X$ path and the $V - Y$ path cannot contain all the nodes of $S(W_k, l)$

Proof. We proceed by by induction on l. Clearly the theorem holds for $l = 1$. Assume it holds for $l = t \geq 1$. The situation in the case of $l = t + 1$ isdiscussed below. In the rest of the proof, we will use $\rightarrow$ to denote a $l - flipping$ link in $S(W_k, l + 1)$. First we will prove part 1

Case 1: $u_l = v_l$, that is, U and V are not in the same subnetwork of level l. Let $a_0 = u_0$ and $\{b_0, b_1, \dots b_{k-1}\} = \{0, 1, \dots k-1\}$. Thus $b_0S(W_k, l), b_1S(W_k, l)\dots b_{k-1}S(W_k, l)$ denotes k subnetworks of level l. By assumption, there exists nodes in $b_iS(W_k, l)$ between two arbitrary distinct nodes in $b_iS(W_k, l)$ for each $i \in \{0, 1, 2\dots k-1\}$. Let $\Rightarrow$ H denote this path. An $U - V$ Hamiltonian path for $S(W_k, l + 1)$ constructed as follows (see Fig. 2):

$U \Rightarrow \mathrm{H}\ a_0a_1^l \rightarrow a_1a_0^l \Rightarrow \mathrm{H}\ a_1(a_2)^l \rightarrow \dots \rightarrow b_{k-2}b_{k-3}^l \Rightarrow \mathrm{H}\ b_{k-2}b_{k-1}^l \rightarrow b_{k-1}b_{k-2}^l \xrightarrow{\mathrm{H}} V$

Conclusion

In this paper we present certain results pertaining to the join graph operation on Hamilton-connected graphs and Sierpinski graphs, we give the exact expression of detour index of certain class of graphs. The detour index of Sierpinski graphs are under investigation.

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