

On The Development of Neutrosophic Picture Fuzzy Topological Spaces and Continuity

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ABSTRACT

This paper presents a systematic development of **Neutrosophic Picture Fuzzy Topology (NPFT)** as a generalized topological structure obtained by integrating neutrosophic logic with picture fuzzy set theory. Picture fuzzy sets extend intuitionistic fuzzy sets by admitting positive, neutral, and negative membership degrees, while neutrosophic logic independently characterizes truth, indeterminacy, and falsity, thereby enabling a richer representation of uncertainty.

Within this framework, neutrosophic picture fuzzy sets and neutrosophic picture fuzzy topological spaces are formally introduced. Fundamental topological notions, including open and closed sets, interior, and closure operators, are defined and their essential properties are examined. Mappings between neutrosophic picture fuzzy topological spaces are also studied. In particular, **Neutrosophic picture fuzzy continuous functions** are defined and several of their properties are established, such as preservation under composition and their relationships with interior and closure operators.

Representative examples are provided to illustrate the proposed concepts and validate the theoretical results. The resulting framework extends existing fuzzy and neutrosophic topological models and offers a coherent mathematical basis for addressing uncertainty and indeterminacy in non-classical settings.

Keywords: Neutrosophic Picture Fuzzy Set, Neutrosophic Picture Fuzzy Topological Spaces (NPFTS), Neutrosophic picture fuzzy open sets, Neutrosophic picture fuzzy closed sets, Neutrosophic picture fuzzy continuous functions.

1. INTRODUCTION

Classical set theory lacks the flexibility to model uncertainty and vagueness. To address this, Zadeh (1965) introduced Fuzzy sets, where elements have degrees of membership in $[0,1]$. Later, Atanassov (1986) extended this to Intuitionistic Fuzzy sets (IFS) by adding non-membership and hesitation degrees. To further generalize uncertainty modeling, Smarandache proposed Neutrosophic sets (NS), associating each element with independent degrees of truth, indeterminacy, and falsity. Simultaneously Picture fuzzy sets (PFS) were developed to capture opinions in terms of positive, neutral, and negative memberships along with refusal. However, neither Neutrosophic nor Picture fuzzy frameworks alone can fully represent uncertainty involving both indeterminacy and neutrality as separate components. To overcome this, we introduce

Neutrosophic Picture Fuzzy Topology (NPFT) - a novel hybrid model that unifies the expressive strengths of both NS and PFS.

This paper develops the basic structure of NPFT including definitions of Neutrosophic Picture Fuzzy sets and topology, open and closed sets, interior, closure and the concept of Neutrosophic Picture fuzzy Continuous functions is established along with several of their properties.

2. PRELIMINARIES

Definition 2.1:[1]

A Fuzzy set A in a universe of discourse X is a set of ordered pairs of the form

$A = \{(x, \mu_A(x)) : x \in X, \mu_A(x) \in [0,1]\}$ where $\mu_A: X \rightarrow [0,1]$ is the membership function of A and $\mu_A(x)$ is the membership degree of x in A .

Definition 2.2:[2]

An Intuitionistic Fuzzy set A in the universe of discourse X is of the form

$$A = \{(x, \mu_A(x), \nu_A(x)) : x \in X, \mu_A(x), \nu_A(x) \in [0,1], 0 \leq \mu_A(x) + \nu_A(x) \leq 1\}$$

where $\mu_A: X \rightarrow [0,1]$ is the membership function and $\nu_A: X \rightarrow [0,1]$ is the non membership function of A and $\mu_A(x), \nu_A(x)$ are the degrees of membership and non membership respectively for each x in A . Further $h(x) = 1 - \mu_A(x) - \nu_A(x)$ is said to be the degree of hesitation of x in A .

Definition 2.3:[5]

A single valued Neutrosophic set A in the universe X is of the form

$$A = \{(x, m(x), i(x), n(x)) : x \in X, m(x), i(x), n(x) \in [0,1], 0 \leq m(x) + i(x) + n(x) \leq 3\}$$

Where $m(x), i(x), n(x)$ are the degrees of membership, indeterminacy and non membership with respect to A , for all x in X .

Definition 2.4:[3]

A Picture fuzzy set A in the universe X is of the form

$$A = \{(x, \mu_A(x), \rho_A(x), \sigma_A(x)) : x \in X\}$$

where $\mu_A: X \rightarrow [0,1]$, $\rho_A: X \rightarrow [0,1]$ and $\sigma_A: X \rightarrow [0,1]$ are the positive, neutral and negative membership functions respectively and satisfying $\mu_A(x) + \rho_A(x) + \sigma_A(x) \leq 1$.

For all $x \in X, 1 - (\mu_A(x) + \rho_A(x) + \sigma_A(x))$ is the degree of refusal of x .

3. NEUTROSOPHIC PICTURE FUZZY TOPOLOGICAL SPACES

Definition 3.1: Neutrosophic Picture Fuzzy Set (NPFS)

Let $\mathbb{X}$ be a universe of discourse. A generalised Neutrosophic Picture Fuzzy Set $\mathcal{A}$ on $\mathbb{X}$ is defined as

$$\mathcal{A} = \{(x, \mathcal{T}_{\mathcal{A}}(x), \mathcal{I}_{\mathcal{A}}(x), \mathcal{F}_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{A}}(x)) : x \in \mathbb{X}\}$$

Where for each element of $x \in \mathbb{X}$,

$\mathcal{T}_{\mathcal{A}}(x) \in [0,1]$: Truth membership degree

$\mathcal{I}_{\mathcal{A}}(x) \in [0,1]$: Indeterminacy membership degree

$\mathcal{F}_{\mathcal{A}}(x) \in [0,1]$: False membership degree

$\mathcal{N}_{\mathcal{A}}(x) \in [0,1]$: Neutrality membership degree and

$$0 \leq \mathcal{T}_{\mathcal{A}}(x) + \mathcal{I}_{\mathcal{A}}(x) + \mathcal{F}_{\mathcal{A}}(x) + \mathcal{N}_{\mathcal{A}}(x) \leq 4.$$

For all $x \in \mathbb{X}$, $1 - (\mathcal{T}_{\mathcal{A}}(x) + \mathcal{I}_{\mathcal{A}}(x) + \mathcal{F}_{\mathcal{A}}(x) + \mathcal{N}_{\mathcal{A}}(x))$ is the degree of refusal of x .

Example:

Let $\mathbb{X} = \{a, b\}$. A Neutrosophic Picture Fuzzy Set (NPFS) be
 $\mathcal{A} = \{(a, 0.7, 0.1, 0.1, 0.05), (b, 0.4, 0.3, 0.2, 0.1)\}$.

Definition 3.2:

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two Neutrosophic Picture Fuzzy subsets of $\mathbb{X}$ such that

$$\mathcal{A}_1 = \left\{ \left(x, \mathcal{T}_{\mathcal{A}_1}(x), \mathcal{I}_{\mathcal{A}_1}(x), \mathcal{F}_{\mathcal{A}_1}(x), \mathcal{N}_{\mathcal{A}_1}(x) \right) : x \in \mathbb{X} \right\}$$

$$\mathcal{A}_2 = \left\{ \left(x, \mathcal{T}_{\mathcal{A}_2}(x), \mathcal{I}_{\mathcal{A}_2}(x), \mathcal{F}_{\mathcal{A}_2}(x), \mathcal{N}_{\mathcal{A}_2}(x) \right) : x \in \mathbb{X} \right\}$$

(i) Inclusion:

If $\mathcal{T}_{\mathcal{A}_1}(x) \leq \mathcal{T}_{\mathcal{A}_2}(x)$, $\mathcal{I}_{\mathcal{A}_1}(x) \geq \mathcal{I}_{\mathcal{A}_2}(x)$, $\mathcal{F}_{\mathcal{A}_1}(x) \geq \mathcal{F}_{\mathcal{A}_2}(x)$, $\mathcal{N}_{\mathcal{A}_1}(x) \leq \mathcal{N}_{\mathcal{A}_2}(x)$ for all $x \in \mathbb{X}$, then $\mathcal{A}_1$ is Neutrosophic picture fuzzy subset of $\mathcal{A}_2$ and denoted by $\mathcal{A}_1 \subseteq \mathcal{A}_2$.

(ii) Equality:

If $\mathcal{A}_1 \subseteq \mathcal{A}_2$ and $\mathcal{A}_2 \subseteq \mathcal{A}_1$ then $\mathcal{A}_1 = \mathcal{A}_2$.

(iii) Union:

Neutrosophic Picture Fuzzy Union of $\mathcal{A}_1$ and $\mathcal{A}_2$ denoted by $\mathcal{A}_1 \cup \mathcal{A}_2$ and defined by

$$\mathcal{A}_1 \cup \mathcal{A}_2 = \left\{ \left(x, \max(\mathcal{T}_{\mathcal{A}_1}(x), \mathcal{T}_{\mathcal{A}_2}(x)), \min(\mathcal{I}_{\mathcal{A}_1}(x), \mathcal{I}_{\mathcal{A}_2}(x)), \min(\mathcal{F}_{\mathcal{A}_1}(x), \mathcal{F}_{\mathcal{A}_2}(x)), \max(\mathcal{N}_{\mathcal{A}_1}(x), \mathcal{N}_{\mathcal{A}_2}(x)) \right) : x \in \mathbb{X} \right\}$$

(iv) Intersection:

Neutrosophic Picture Fuzzy Intersection of $\mathcal{A}_1$ and $\mathcal{A}_2$ denoted by $\mathcal{A}_1 \cap \mathcal{A}_2$ and defined by

$$\mathcal{A}_1 \cap \mathcal{A}_2 = \left\{ \left(x, \min(\mathcal{T}_{\mathcal{A}_1}(x), \mathcal{T}_{\mathcal{A}_2}(x)), \max(\mathcal{I}_{\mathcal{A}_1}(x), \mathcal{I}_{\mathcal{A}_2}(x)), \max(\mathcal{F}_{\mathcal{A}_1}(x), \mathcal{F}_{\mathcal{A}_2}(x)), \min(\mathcal{N}_{\mathcal{A}_1}(x), \mathcal{N}_{\mathcal{A}_2}(x)) \right) : x \in \mathbb{X} \right\}$$

(v)Complement:

Neutrosophic picture fuzzy complement of $\mathcal{A}$ is denoted by $\mathcal{A}^c$ and defined by

$$\mathcal{A}^c = \{(x, \mathcal{F}_{\mathcal{A}}(x), 1 - \mathcal{I}_{\mathcal{A}}(x), \mathcal{T}_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{A}}(x)) : x \in \mathbb{X}\}$$

(vi)Universal Set:

Neutrosophic

picture fuzzy universal set is defined by $I_{\mathbb{X}} = \{(x, 1, 0, 0, 0) : x \in \mathbb{X}\}$

(vii)Empty Set:

Neutrosophic picture fuzzy Empty set is defined by $O_{\mathbb{X}} = \{(x, 0, 0, 1, 0) : x \in \mathbb{X}\}$

Example:

Let $\mathbb{X} = \{x\}$ and $\mathcal{A}_1, \mathcal{A}_2$ be two Neutrosophic Picture Fuzzy subsets of $\mathbb{X}$ such that

$$\mathcal{A}_1 = \{(x, 0.3, 0.1, 0.2, 0.05)\}$$

$$\mathcal{A}_2 = \{(x, 0.4, 0.2, 0.3, 0.1)\}$$

then we have that $\mathcal{A}_1 \subseteq \mathcal{A}_2$ and

$$\mathcal{A}_1 \cup \mathcal{A}_2 =$$

$$\{(x, \max(0.3, 0.4), \min(0.1, 0.2), \min(0.2, 0.3), \min(0.05, 0.1))\}$$

$$= \{(x, 0.4, 0.1, 0.2, 0.05)\}$$

$$\mathcal{A}_1 \cap \mathcal{A}_2 = \{(x, \min(0.3, 0.4), \max(0.1, 0.2), \max(0.2, 0.3), \min(0.05, 0.1))\}$$

$$= \{(x, 0.3, 0.2, 0.3, 0.05)\}$$

$$\mathcal{A}_1^c = \{(x, 0.2, 0.9, 0.3, 0.05)\} \text{ and } \mathcal{A}_2^c = \{(x, 0.3, 0.8, 0.4, 0.1)\}.$$

Theorem 3.1:

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two NPFS, then we have $(\mathcal{A}_1 \cup \mathcal{A}_2)^c = \mathcal{A}_1^c \cap \mathcal{A}_2^c$ and $\mathcal{A}_1^c \cup \mathcal{A}_2^c =$

$$(\mathcal{A}_1 \cap \mathcal{A}_2)^c =$$

Proof:

By

definition,

$$\mathcal{A}_1 \cup \mathcal{A}_2 = \left\{ \left(x, \max(\mathcal{T}_{\mathcal{A}_1}(x), \mathcal{T}_{\mathcal{A}_2}(x)), \min(\mathcal{I}_{\mathcal{A}_1}(x), \mathcal{I}_{\mathcal{A}_2}(x)), \min(\mathcal{F}_{\mathcal{A}_1}(x), \mathcal{F}_{\mathcal{A}_2}(x)), \min(\mathcal{N}_{\mathcal{A}_1}(x), \mathcal{N}_{\mathcal{A}_2}(x)) \right) : x \in \mathbb{X} \right\}$$

$$(\mathcal{A}_1 \cup \mathcal{A}_2)^c = \left\{ \left(x, \min(\mathcal{F}_{\mathcal{A}_1}(x), \mathcal{F}_{\mathcal{A}_2}(x)), 1 - \min(\mathcal{I}_{\mathcal{A}_1}(x), \mathcal{I}_{\mathcal{A}_2}(x)), \max(\mathcal{T}_{\mathcal{A}_1}(x), \mathcal{T}_{\mathcal{A}_2}(x)), \min(\mathcal{N}_{\mathcal{A}_1}(x), \mathcal{N}_{\mathcal{A}_2}(x)) \right) : x \in \mathbb{X} \right\}$$

$$\mathcal{A}_1^c = \{(x, \mathcal{F}_{\mathcal{A}_1}(x), 1 - \mathcal{I}_{\mathcal{A}_1}(x), \mathcal{T}_{\mathcal{A}_1}(x), \mathcal{N}_{\mathcal{A}_1}(x)) : x \in \mathbb{X}\}$$

$$\mathcal{A}_2^c = \{(x, \mathcal{F}_{\mathcal{A}_2}(x), 1 - \mathcal{I}_{\mathcal{A}_2}(x), \mathcal{T}_{\mathcal{A}_2}(x), \mathcal{N}_{\mathcal{A}_2}(x)) : x \in \mathbb{X}\}$$

$$\mathcal{A}_1^c \cap \mathcal{A}_2^c = \left\{ \left(x, \min(\mathcal{F}_{\mathcal{A}_1}(x), \mathcal{F}_{\mathcal{A}_2}(x)), \max(1 - \mathcal{I}_{\mathcal{A}_1}(x), 1 - \mathcal{I}_{\mathcal{A}_2}(x)), \max(\mathcal{T}_{\mathcal{A}_1}(x), \mathcal{T}_{\mathcal{A}_2}(x)), \min(\mathcal{N}_{\mathcal{A}_1}(x), \mathcal{N}_{\mathcal{A}_2}(x)) : x \in \mathbb{X} \right) \right\}$$

We know that, $\max(1 - \mathcal{I}_{\mathcal{A}_1}(x), 1 - \mathcal{I}_{\mathcal{A}_2}(x)) = 1 - \min(\mathcal{I}_{\mathcal{A}_1}(x), \mathcal{I}_{\mathcal{A}_2}(x))$ then,

$$\mathcal{A}_1^c \cap \mathcal{A}_2^c = \left\{ \left(x, \min(\mathcal{F}_{\mathcal{A}_1}(x), \mathcal{F}_{\mathcal{A}_2}(x)), 1 - \min(\mathcal{I}_{\mathcal{A}_1}(x), \mathcal{I}_{\mathcal{A}_2}(x)), \max(\mathcal{T}_{\mathcal{A}_1}(x), \mathcal{T}_{\mathcal{A}_2}(x)), \min(\mathcal{N}_{\mathcal{A}_1}(x), \mathcal{N}_{\mathcal{A}_2}(x)) : x \in \mathbb{X} \right) \right\}$$

hence $(\mathcal{A}_1 \cup \mathcal{A}_2)^c = \mathcal{A}_1^c \cap \mathcal{A}_2^c$. Similarly we can prove $(\mathcal{A}_1 \cap \mathcal{A}_2)^c = \mathcal{A}_1^c \cup \mathcal{A}_2^c$.

Theorem 3.2:

For all Neutrosophic picture fuzzy subsets of $\mathbb{X}$,

$$(a) \mathcal{A}_1 \cup (\mathcal{A}_2 \cap \mathcal{A}_3) = (\mathcal{A}_1 \cup \mathcal{A}_2) \cap (\mathcal{A}_1 \cup \mathcal{A}_3)$$

$$(b) \mathcal{A}_1 \cap (\mathcal{A}_2 \cup \mathcal{A}_3) = (\mathcal{A}_1 \cap \mathcal{A}_2) \cup (\mathcal{A}_1 \cap \mathcal{A}_3)$$

Proof:

Let $x \in \mathbb{X}$, then

$$\mathcal{T}_{\mathcal{A}_1 \cup (\mathcal{A}_2 \cap \mathcal{A}_3)}(x) = \max(\mathcal{T}_{\mathcal{A}_1}(x), \mathcal{T}_{(\mathcal{A}_2 \cap \mathcal{A}_3)}(x)) = \max(\mathcal{T}_{\mathcal{A}_1}(x), \min(\mathcal{T}_{\mathcal{A}_2}(x), \mathcal{T}_{\mathcal{A}_3}(x)))$$

$$\begin{aligned} \mathcal{T}_{(\mathcal{A}_1 \cup \mathcal{A}_2) \cap (\mathcal{A}_1 \cup \mathcal{A}_3)}(x) &= \min(\mathcal{T}_{(\mathcal{A}_1 \cup \mathcal{A}_2)}(x), \mathcal{T}_{(\mathcal{A}_1 \cup \mathcal{A}_3)}(x)) \\ &= \min(\max(\mathcal{T}_{\mathcal{A}_1}(x), \mathcal{T}_{\mathcal{A}_2}(x)), \max(\mathcal{T}_{\mathcal{A}_1}(x), \mathcal{T}_{\mathcal{A}_3}(x))) \end{aligned}$$

Now there are three possibilities:

(i) If $\max(\mathcal{T}_{\mathcal{A}_1}(x), \mathcal{T}_{\mathcal{A}_2}(x), \mathcal{T}_{\mathcal{A}_3}(x)) = \mathcal{T}_{\mathcal{A}_1}(x)$ then, $\mathcal{T}_{\mathcal{A}_1 \cup (\mathcal{A}_2 \cap \mathcal{A}_3)}(x) = \mathcal{T}_{\mathcal{A}_1}(x)$

$$= \mathcal{T}_{(\mathcal{A}_1 \cup \mathcal{A}_2) \cap (\mathcal{A}_1 \cup \mathcal{A}_3)}(x)$$

(ii) If $\max(\mathcal{T}_{\mathcal{A}_1}(x), \mathcal{T}_{\mathcal{A}_2}(x), \mathcal{T}_{\mathcal{A}_3}(x)) = \mathcal{T}_{\mathcal{A}_2}(x)$ then,

$$\mathcal{T}_{\mathcal{A}_1 \cup (\mathcal{A}_2 \cap \mathcal{A}_3)}(x) = \max(\mathcal{T}_{\mathcal{A}_1}(x), \mathcal{T}_{\mathcal{A}_3}(x)) = \mathcal{T}_{(\mathcal{A}_1 \cup \mathcal{A}_2) \cap (\mathcal{A}_1 \cup \mathcal{A}_3)}(x)$$

(iii) If $\max(\mathcal{T}_{\mathcal{A}_1}(x), \mathcal{T}_{\mathcal{A}_2}(x), \mathcal{T}_{\mathcal{A}_3}(x)) = \mathcal{T}_{\mathcal{A}_3}(x)$ then,

$$\mathcal{T}_{\mathcal{A}_1 \cup (\mathcal{A}_2 \cap \mathcal{A}_3)}(x) = \max(\mathcal{T}_{\mathcal{A}_1}(x), \mathcal{T}_{\mathcal{A}_2}(x)) = \mathcal{T}_{(\mathcal{A}_1 \cup \mathcal{A}_2) \cap (\mathcal{A}_1 \cup \mathcal{A}_3)}(x)$$

Thus in all three cases we have,

$$\mathcal{T}_{\mathcal{A}_1 \cup (\mathcal{A}_2 \cap \mathcal{A}_3)}(x) = \mathcal{T}_{(\mathcal{A}_1 \cup \mathcal{A}_2) \cap (\mathcal{A}_1 \cup \mathcal{A}_3)}(x)$$

Similarly we can obtain that

$$\mathcal{I}_{\mathcal{A}_1 \cup (\mathcal{A}_2 \cap \mathcal{A}_3)}(x) = \mathcal{I}_{(\mathcal{A}_1 \cup \mathcal{A}_2) \cap (\mathcal{A}_1 \cup \mathcal{A}_3)}(x)$$

$$\mathcal{F}_{\mathcal{A}_1 \cup (\mathcal{A}_2 \cap \mathcal{A}_3)}(x) = \mathcal{F}_{(\mathcal{A}_1 \cup \mathcal{A}_2) \cap (\mathcal{A}_1 \cup \mathcal{A}_3)}(x)$$

$$\mathcal{N}_{\mathcal{A}_1 \cup (\mathcal{A}_2 \cap \mathcal{A}_3)}(x) = \mathcal{N}_{(\mathcal{A}_1 \cup \mathcal{A}_2) \cap (\mathcal{A}_1 \cup \mathcal{A}_3)}(x)$$

Hence (a) $\mathcal{A}_1 \cup (\mathcal{A}_2 \cap \mathcal{A}_3) = (\mathcal{A}_1 \cup \mathcal{A}_2) \cap (\mathcal{A}_1 \cup \mathcal{A}_3)$.

Similarly we can obtain that (b) $\mathcal{A}_1 \cap (\mathcal{A}_2 \cup \mathcal{A}_3) = (\mathcal{A}_1 \cap \mathcal{A}_2) \cup (\mathcal{A}_1 \cap \mathcal{A}_3)$.

Definition 3.3: Neutrosophic Picture Fuzzy Topological Spaces (NPFTS)

Let $\mathbb{X}$ denote a non empty crisp set. A family ζ of Neutrosophic Picture Fuzzy subsets of $\mathbb{X}$ is called Neutrosophic Picture Fuzzy Topology on $\mathbb{X}$ if,

- (i) $I_{\mathbb{X}}, O_{\mathbb{X}} \in \zeta$
- (ii) for any $\mathcal{A}_1, \mathcal{A}_2 \in \zeta$, we have $\mathcal{A}_1 \cap \mathcal{A}_2 \in \zeta$
- (iii) for any $\{\mathcal{A}_i\}_{i \in \Omega} \subseteq \zeta$, we have $\bigcup_{i \in \Omega} \mathcal{A}_i \in \zeta$, where Ω is an arbitrary index set.

If ζ forms a Neutrosophic Picture Fuzzy Topology on $\mathbb{X}$, then the pair $(\mathbb{X}, \zeta)$ denotes Neutrosophic Picture Fuzzy Topological Space (NPFTS).

Example:

Let $\mathbb{X} = \{a, b\}$ and consider the Neutrosophic Picture Fuzzy Set $\mathcal{A} = \{(a, 0.5, 0.2, 0.1, 0), (b, 0.4, 0.3, 0.2, 0)\}$

Here $I_{\mathbb{X}} = \{(a, 1, 0, 0, 0), (b, 1, 0, 0, 0)\}$ and $O_{\mathbb{X}} = \{(a, 0, 0, 1, 0), (b, 0, 0, 1, 0)\}$

Then the collection $\zeta = \{I_{\mathbb{X}}, O_{\mathbb{X}}, \mathcal{A}\}$ satisfies the conditions (i),(ii),(iii), so ζ is a Neutrosophic Picture Fuzzy Topology on $\mathbb{X}$. Then the pair $(\mathbb{X}, \zeta)$ is a Neutrosophic Picture Fuzzy Topological Space (NPFTS).

Definition 3.4:

A Neutrosophic Picture Fuzzy Set $\mathcal{A} \subseteq \mathbb{X}$ is called a Neutrosophic Picture Fuzzy Open Set if $\mathcal{A} \in \zeta$, where ζ is a Neutrosophic Picture Fuzzy Topology on $\mathbb{X}$.

Definition 3.5:

A Neutrosophic Picture Fuzzy Set $\mathcal{A} \subseteq \mathbb{X}$ is said to be Neutrosophic Picture Fuzzy Closed Set if its complement $\mathcal{A}^c$ is a Neutrosophic Picture Fuzzy Open Set. (i.e. $\mathcal{A}^c \in \zeta$)

Definition 3.6:

Let $(\mathbb{X}, \zeta)$ be a NPFTS over $\mathbb{X}$ and let $\mathcal{A} \subseteq \mathbb{X}$ be a NPFS, then the Neutrosophic Picture Fuzzy interior of $\mathcal{A}$ denoted by $\text{int}(\mathcal{A})$ is defined as the union of all Neutrosophic Picture Fuzzy open sets contained in $\mathcal{A}$.

$$\text{i.e.} \quad \text{int}(\mathcal{A}) = \bigcup \{U \in \zeta : U \subseteq \mathcal{A}\}$$

Clearly $\text{int}(\mathcal{A})$ is the largest Neutrosophic Picture Fuzzy open set contained in $\mathcal{A}$.

Definition 3.7:

Let $(\mathbb{X}, \zeta)$ be a NPFTS over $\mathbb{X}$ and let $\mathcal{A} \subseteq \mathbb{X}$ be a NPFS, then the Neutrosophic Picture Fuzzy closure of $\mathcal{A}$ denoted by $\text{cl}(\mathcal{A})$ is defined as the intersection of all Neutrosophic Picture Fuzzy closed sets containing $\mathcal{A}$.

i.e. $\text{cl}(\mathcal{A}) = \bigcap \{ C \subseteq \mathbb{X} : C \text{ is closed and } \mathcal{A} \subseteq C \}$

Clearly $\text{cl}(\mathcal{A})$ is the smallest Neutrosophic Picture Fuzzy closed set containing $\mathcal{A}$.

4. NEUTROSOPHIC PICTURE FUZZY CONTINUITY

Definition 4.1:

Let $h: X \rightarrow Y$ be a function and $A \in \Gamma_X$, $B \in \Gamma_Y$. The image of A under h is defined by

$$h(A) = \{y, T_{h(A)}(y), I_{h(A)}(y), F_{h(A)}(y), N_{h(A)}(y) : y \in Y\}$$

$$\text{where } T_{h(A)}(y) = \begin{cases} \sup\{T_A(z); z \in h^{-1}(y)\} & \text{if } h^{-1}(y) \neq \emptyset \\ 0 & \text{if } h^{-1}(y) = \emptyset \end{cases}$$

$$I_{h(A)}(y) = \begin{cases} \inf\{I_A(z); z \in h^{-1}(y)\} & \text{if } h^{-1}(y) \neq \emptyset \\ 1 & \text{if } h^{-1}(y) = \emptyset \end{cases}$$

$$F_{h(A)}(y) = \begin{cases} \inf\{F_A(z); z \in h^{-1}(y)\} & \text{if } h^{-1}(y) \neq \emptyset \\ 1 & \text{if } h^{-1}(y) = \emptyset \end{cases}$$

$$N_{h(A)}(y) = \begin{cases} \inf\{N_A(z); z \in h^{-1}(y)\} & \text{if } h^{-1}(y) \neq \emptyset \\ 0 & \text{if } h^{-1}(y) = \emptyset \end{cases}$$

Definition 4.2:

The pre image of B under h is defined by

$$h^{-1}(B) = \{x, T_{h^{-1}(B)}(x), I_{h^{-1}(B)}(x), F_{h^{-1}(B)}(x), N_{h^{-1}(B)}(x) : x \in X\}$$

$$\text{Where } T_{h^{-1}(B)}(x) = T_B(h(x))$$

$$I_{h^{-1}(B)}(x) = I_B(h(x))$$

$$F_{h^{-1}(B)}(x) = F_B(h(x))$$

$$N_{h^{-1}(B)}(x) = N_B(h(x))$$

Example:

Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$ and let $f: X \rightarrow Y$ be defined by

$$f(x_1) = y_1$$

$$f(x_2) = y_1$$

$$f(x_3) = y_2$$

Consider the NPFS $A = \{(x_1, 0.8, 0.1, 0.1, 0.0), (x_2, 0.5, 0.2, 0.3, 0.0), (x_3, 0.6, 0.3, 0.1, 0.0)\}$

Image:

$$\text{Then for } y_1, T_{f(A)}(y_1) = \max(0.8, 0.5) = 0.8$$

$$I_{f(A)}(y_1) = \min(0.1, 0.2) = 0.1$$

$$F_{f(A)}(y_1) = \min(0.1, 0.3) = 0.1$$

$$N_{f(A)}(y_1) = \min(0.0, 0.0) = 0.0$$

Then for y_2 , $T_{f(A)}(y_2) = 0.6, I_{f(A)}(y_2) = 0.3, F_{f(A)}(y_2) = 0.1, N_{f(A)}(y_2) = 0.0$

Hence the image $f(A) = \{(y_1, 0.8, 0.1, 0.1, 0.0), (y_2, 0.6, 0.3, 0.1, 0.0)\}$

Pre image:

Consider the NPFS $B = \{(y_1, 0.7, 0.2, 0.1, 0.0), (y_2, 0.6, 0.3, 0.1, 0.0)\}$

Then,

$$T_{f^{-1}(B)}(x) = T_B(f(x))$$

$$I_{f^{-1}(B)}(x) = I_B(f(x))$$

$$F_{f^{-1}(B)}(x) = F_B(f(x))$$

$$N_{f^{-1}(B)}(x) = N_B(f(x))$$

$$f^{-1}(B)(x_1) = B(f(x_1)) = B(y_1) = (0.7, 0.2, 0.1, 0.0)$$

$$f^{-1}(B)(x_2) = B(f(x_2)) = B(y_1) = (0.7, 0.2, 0.1, 0.0)$$

$$f^{-1}(B)(x_3) = B(f(x_3)) = B(y_2) = (0.6, 0.3, 0.1, 0.0)$$

Hence the pre image $f^{-1}(B)$ on X is

$$f^{-1}(B) = \{(x_1, 0.7, 0.2, 0.1, 0.0), (x_2, 0.7, 0.2, 0.1, 0.0), (x_3, 0.6, 0.3, 0.1, 0.0)\}$$

Theorem 4.1:

Let $h: \mathbb{X} \rightarrow \mathbb{Y}$ be a function, then

- (i) $h^{-1}(K^c) = [h^{-1}(K)]^c$ for all $K \in \Gamma_{\mathbb{Y}}$
- (ii) $[h(N)]^c \subseteq h(N^c)$ for all $N \in \Gamma_{\mathbb{X}}$
- (iii) For all $K_1, K_2 \in \Gamma_{\mathbb{Y}}$, $K_1 \subseteq K_2$ implies $h^{-1}(K_1) \subseteq h^{-1}(K_2)$
- (iv) For all $N_1, N_2 \in \Gamma_{\mathbb{X}}$, $N_1 \subseteq N_2$ implies $h(K_1) \subseteq h(K_2)$
- (v) $h(h^{-1}(K)) \subseteq K$ for all $K \in \Gamma_{\mathbb{Y}}$
- (vi) For all $N \in \Gamma_{\mathbb{X}}$, $N \subseteq h^{-1}(h(N))$, if h is injective.

Proof:

(i) Suppose $x \in \mathbb{X}$, then for any $K \in \Gamma_{\mathbb{Y}}$, we have,

$$T_{h^{-1}(K^c)}(x) = T_{K^c}(h(x)) = F_K(h(x)) = F_{h^{-1}(K)}(x) = T_{[h^{-1}(K)]^c}(x)$$

$$T_{h^{-1}(K^c)}(x) = T_{[h^{-1}(K)]^c}(x)$$

Similarly, we can show that

$$I_{h^{-1}(K^c)}(x) = I_{[h^{-1}(K)]^c}(x)$$

$$F_{h^{-1}(K^c)}(x) = F_{[h^{-1}(K)]^c}(x)$$

$$N_{h^{-1}(K^c)}(x) = N_{[h^{-1}(K)]^c}(x)$$

Hence $h^{-1}(K^c) = [h^{-1}(K)]^c$ for all $K \in \Gamma_{\mathbb{Y}}$.

(ii) Let $y \in \mathbb{Y}$ such that $h^{-1}(y) \neq \emptyset$ then for $N \in \Gamma_{\mathbb{X}}$, we have,

$$T_{[h(N)]^c}(y) = F_{h(N)}(y) = \inf\{F_N(z) : z \in h^{-1}(y)\}$$

$$\text{and } T_{h(N^c)}(y) = \sup\{T_{N^c}(z) : z \in h^{-1}(y)\} = \sup\{F_N(z) : z \in h^{-1}(y)\}$$

Therefore,

$$T_{[h(N)]^c}(y) \leq T_{h(N^c)}(y) \text{ for all } y \in \mathbb{Y} \text{ such that } h^{-1}(y) \neq \emptyset$$

Also clearly $T_{[h(N)]^c}(y) = T_{h(N^c)}(y) = 0$ for all $y \in \mathbb{Y}$ such that $h^{-1}(y) = \emptyset$

Similarly, we can show that

$$N_{[h(N)]^c}(y) = N_{h(N^c)}(y), F_{[h(N)]^c}(y) \geq F_{h(N^c)}(y) \text{ and}$$

$$I_{[h(N)]^c}(y) \geq I_{h(N^c)}(y) \text{ for all } y \in \mathbb{Y}$$

$$\text{Thus, } [h(N)]^c \subseteq h(N^c).$$

(iii) Let $K_1, K_2 \in \Gamma_{\mathbb{Y}}$ such that $K_1 \subseteq K_2$, then for $x \in \mathbb{X}$, we have

$$T_{h^{-1}(K_1)}(x) = T_{K_1}(h(x)) \leq T_{K_2}(h(x)) = T_{h^{-1}(K_2)}(x)$$

$$T_{h^{-1}(K_1)}(x) \leq T_{h^{-1}(K_2)}(x)$$

Similarly, we can show that,

$$N_{h^{-1}(K_1)}(x) \leq N_{h^{-1}(K_2)}(x), F_{h^{-1}(K_1)}(x) \geq F_{h^{-1}(K_2)}(x) \text{ and } I_{h^{-1}(K_1)}(x) \geq I_{h^{-1}(K_2)}(x)$$

$$\text{Thus, } h^{-1}(K_1) \subseteq h^{-1}(K_2)$$

(iv) Let $N_1, N_2 \in \Gamma_{\mathbb{X}}$ such that $N_1 \subseteq N_2$

Suppose $y \in \mathbb{Y}$ and $h^{-1}(y) \neq \emptyset$, then

$$T_{h(N_1)}(y) = \sup\{T_{N_1}(z) : z \in h^{-1}(y)\} \leq \sup\{T_{N_2}(z) : z \in h^{-1}(y)\} = T_{h(N_2)}(y)$$

$$T_{h(N_1)}(y) \leq T_{h(N_2)}(y) \text{ if } h^{-1}(y) \neq \emptyset$$

$$\text{clearly } T_{h(N_1)}(y) = T_{h(N_2)}(y) = 0 \text{ for } y \in \mathbb{Y} \text{ if } h^{-1}(y) = \emptyset$$

Thus, $T_{h(N_1)}(y) \leq T_{h(N_2)}(y)$ for all $y \in \mathbb{Y}$. Similarly,

we can show that,

$$N_{h(N_1)}(y) \leq N_{h(N_2)}(y), F_{h(N_1)}(y) \geq F_{h(N_2)}(y) \text{ and } I_{h(N_1)}(y) \geq I_{h(N_2)}(y) \text{ for all } y \in \mathbb{Y}.$$

$$\text{Hence } h(N_1) \subseteq h(N_2).$$

(v) Let $K \in \Gamma_{\mathbb{Y}}$ and $y \in \mathbb{Y}$ such that $h^{-1}(y) \neq \emptyset$ then

$$T_{h(h^{-1}(K))}(y) = \sup\{T_{h^{-1}(K)}(z) : z \in h^{-1}(y)\} = \sup\{T_K(h(z)) : z \in h^{-1}(y)\} = T_K(y)$$

Also if $h^{-1}(y) = \emptyset$, then $T_{h(h^{-1}(K))}(y) = 0 \leq T_K(y)$

Similarly, we can show that

$$N_{h(h^{-1}(K))}(y) \leq N_K(y), F_{h(h^{-1}(K))}(y) \geq F_K(y), \text{ and } I_{h(h^{-1}(K))}(y) \geq I_K(y), \text{ for all } y \in \mathbb{Y}.$$

Thus, $h(h^{-1}(K)) \subseteq K$ for all $K \in \Gamma_{\mathbb{Y}}$.

(vi) Let $h: \mathbb{X} \rightarrow \mathbb{Y}$ be an injective function and $N \in \Gamma_{\mathbb{X}}$, then for all $x \in \mathbb{X}$,

$$N_{h^{-1}(h(N))}(x) = N_{h(N)}(h(x)) = \inf \{N_N(z) : z \in h^{-1}(h(x))\}$$

$$N_{h^{-1}(h(N))}(x) = N_N(x) \quad (\text{Since } h \text{ is injective function})$$

Similarly, we can prove

$$T_{h^{-1}(h(N))}(x) \geq T_N(x), F_{h^{-1}(h(N))}(x) \leq F_N(x) \text{ and } I_{h^{-1}(h(N))}(x) \leq I_N(x)$$

Thus, $N \subseteq h^{-1}(h(N))$.

Definition 4.3:

Let $(\mathbb{X}, \zeta_{\mathbb{X}})$ and $(\mathbb{Y}, \zeta_{\mathbb{Y}})$ be two NPFTSs. A function $f: \mathbb{X} \rightarrow \mathbb{Y}$ is called Neutrosophic Picture Fuzzy continuous if for every Neutrosophic Picture Fuzzy open set $V \in \zeta_{\mathbb{Y}}$, the set $f^{-1}(V) \in \zeta_{\mathbb{X}}$.

Definition 4.4:

Let $(\mathbb{X}, \zeta)$ be a NPFTS and $N \in \Gamma_{\mathbb{X}}$. Then $A \in \Gamma_{\mathbb{X}}$ is called a Neutrosophic Picture fuzzy Neighbourhood of N if $N \subseteq S \subseteq A$ for some $S \in \zeta$.

Theorem 4.2:

Let $(\mathbb{X}, \zeta_1)$ and $(\mathbb{Y}, \zeta_2)$ be two NPFTS. Then $h: (\mathbb{X}, \zeta_1) \rightarrow (\mathbb{Y}, \zeta_2)$ is NPFC iff for every $T \in \zeta_2$, $h^{-1}(T) \in \zeta_1$ and h is injective.

Proof:

Suppose that h is NPFC, then h is injective. Let $T \in \zeta_2$. Let $N \subseteq h^{-1}(T)$ then $h(N) \subseteq h(h^{-1}(T))$. Since $T \in \zeta_2$, $h(N)$ has a neighbourhood B such that $B \subseteq T$ then $h^{-1}(B)$ is neighbourhood of N . Moreover $h^{-1}(B) \subseteq h^{-1}(T)$. Hence $h^{-1}(T) \in \zeta_1$. Conversely suppose that for every $T \in \zeta_2$, $h^{-1}(T) \in \zeta_1$. Let $N \in \Gamma_{\mathbb{X}}$ and B be a neighbourhood of $h(N)$. Then $h(N) \subseteq T' \subseteq B$ for some $T' \in \zeta_2$. By assumption $h^{-1}(T') \in \zeta_1$ and $N \subseteq (h^{-1}(h(N))) \subseteq h^{-1}(T') \subseteq h^{-1}(B)$. This means that $h^{-1}(B)$ is a neighbourhood of N which further implies that h is NPFC function.

Theorem 4.3:

Let $(\mathbb{X}, \zeta_{\mathbb{X}})$, $(\mathbb{Y}, \zeta_{\mathbb{Y}})$ and $(\mathbb{Z}, \zeta_{\mathbb{Z}})$ be NPFTS. If $f: (\mathbb{X}, \zeta_{\mathbb{X}}) \rightarrow (\mathbb{Y}, \zeta_{\mathbb{Y}})$ and $g: (\mathbb{Y}, \zeta_{\mathbb{Y}}) \rightarrow (\mathbb{Z}, \zeta_{\mathbb{Z}})$ are NPFC functions then the composition $g \circ f: (\mathbb{X}, \zeta_{\mathbb{X}}) \rightarrow (\mathbb{Z}, \zeta_{\mathbb{Z}})$ defined by $(g \circ f)(x) = g(f(x))$, $\forall x \in \mathbb{X}$ is also NPFC.

Proof:

Let $\mathcal{A}$ be a NPF open set in $\mathbb{Z}$.

Since g is NPFC, the inverse image of $\mathcal{V}$ under g is a NPF open set in $\mathbb{Y}$. (i.e)

$$g^{-1}(\mathcal{A}) \in \zeta_{\mathbb{Y}}$$

Since f is NPFC, the inverse image of $g^{-1}(\mathcal{A})$ under f is NPF open set in $\mathbb{X}$.

Hence $f^{-1}g^{-1}(\mathcal{A}) \in \zeta_{\mathbb{X}}$

But $f^{-1}g^{-1}(\mathcal{A}) = (g \circ f)^{-1}(\mathcal{A})$
 $(g \circ f)^{-1}(\mathcal{A}) \in \zeta_{\mathbb{X}}$

Hence the composition $g \circ f$ is NPFC.

Theorem 4.4:

Let $(\mathbb{X}, \zeta_{\mathbb{X}})$ and $(\mathbb{Y}, \zeta_{\mathbb{Y}})$ be two NPFTS and $f: (\mathbb{X}, \zeta_{\mathbb{X}}) \rightarrow (\mathbb{Y}, \zeta_{\mathbb{Y}})$ be a function. Then f is NPFC function iff for every NPFS $\mathcal{B}$ in $\mathbb{Y}$, $f^{-1}(\text{int}(\mathcal{B})) \subseteq \text{int}(f^{-1}(\mathcal{B}))$

Proof:

Necessity:

Let $\mathcal{B}$ be NPFS in $\mathbb{Y}$ and f be a NPFC function. We know that

$$\text{int}(\mathcal{B}) \subseteq \mathcal{B}$$

$$\Rightarrow f^{-1}(\text{int}(\mathcal{B})) \subseteq f^{-1}(\mathcal{B})$$

$$\Rightarrow \text{int}(f^{-1}(\text{int}(\mathcal{B}))) \subseteq \text{int}(f^{-1}(\mathcal{B}))$$

Since $\text{int}(\mathcal{B}) \in \zeta_{\mathbb{Y}}$ and $f^{-1}(\text{int}(\mathcal{B})) \in \mathbb{X}$ so that $\text{int}(f^{-1}(\text{int}(\mathcal{B}))) = f^{-1}(\text{int}(\mathcal{B}))$

$$f^{-1}(\text{int}(\mathcal{B})) \subseteq \text{int}(f^{-1}(\mathcal{B}))$$

Sufficiency:

Let $f^{-1}(\text{int}(\mathcal{B})) \subseteq \text{int}(f^{-1}(\mathcal{B}))$ for every NPFS $\mathcal{B}$ in $\mathbb{Y}$ and $\mathcal{C} \in \zeta_{\mathbb{X}}$.

Then $f^{-1}(\mathcal{C}) \subseteq \text{int}(f^{-1}(\mathcal{C}))$ and $f^{-1}(\mathcal{C}) = \text{int}(f^{-1}(\mathcal{C}))$ so that $f^{-1}(\mathcal{C}) \in \zeta_{\mathbb{X}}$.

Hence f is NPFC function.

Theorem 4.5:

Let $(\mathbb{X}, \zeta_{\mathbb{X}})$ and $(\mathbb{Y}, \zeta_{\mathbb{Y}})$ be two NPFTS and $f: (\mathbb{X}, \zeta_{\mathbb{X}}) \rightarrow (\mathbb{Y}, \zeta_{\mathbb{Y}})$ be a function. Then f is NPFC function iff for every NPFS $\mathcal{A}$ in $\mathbb{X}$, $f(\text{cl}(\mathcal{A})) \subseteq \text{cl}(f(\mathcal{A}))$.

Proof:

Necessity:

Let $\mathcal{A}$ be any NPF set in $\mathbb{X}$ and f be a NPFC function. We know that

$$\mathcal{A} \subseteq \text{cl}(\mathcal{A})$$

$$\Rightarrow f(\mathcal{A}) \subseteq \text{cl}(f(\mathcal{A}))$$

Then $\mathcal{A} \subseteq f^{-1}(f(\mathcal{A})) \subseteq f^{-1}(\text{cl}(f(\mathcal{A})))$ and $\text{cl}(\mathcal{A}) \subseteq \text{cl}(f^{-1}(\text{cl}(f(\mathcal{A}))))$

Because f is NPFC function and $\text{cl}(f(\mathcal{A}))$ is Neutrosophic closed set, we know

$$\text{cl}(f^{-1}(\text{cl}(f(\mathcal{A})))) = f^{-1}(\text{cl}(f(\mathcal{A}))). \text{ Hence } f(\text{cl}(\mathcal{A})) \subseteq \text{cl}(f(\mathcal{A}))$$

Sufficiency:

Let $f(\text{cl}(\mathcal{A})) \subseteq \text{cl}(f(\mathcal{A}))$ for every NPFS $\mathcal{A}$ in $\mathbb{X}$. Let $\mathcal{B}$ be a Neutrosophic closed set in $\mathbb{Y}$.

Then, $\text{cl}(f^{-1}(\mathcal{B})) \subseteq \text{cl}(\mathcal{B}) = \mathcal{B}$ and so $f^{-1}(\mathcal{B})$ is closed in $\mathbb{X}$. Then f is NPFC function.

CONCLUSION

In this paper, we introduced Neutrosophic Picture Fuzzy set, Neutrosophic Picture Fuzzy Topological spaces and investigated Neutrosophic Picture Fuzzy open and closed sets, Neutrosophic Picture Fuzzy interior, closure and Neutrosophic Picture Fuzzy continuous function.

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