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ON THE UNSOLVED OF THE NUMBER THEORY : A REVIEW

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Abstract - Number theory, the branch of mathematics that explores the properties and relationships of integers, has long been a captivating field filled with unsolved puzzles. This review paper aims to provide a comprehensive overview of the prominent unsolved problems in number theory, shedding light on the challenges that continue to baffle mathematicians and inspire further research. By delving into the depths of these enigmas, we explore their historical significance, relevance in modern mathematics, and potential avenues for future breakthroughs. From the mysteries of prime numbers to the riddles of Diophantine equations, this review offers a glimpse into the fascinating realm of unsolved problems in number theory. This review will help the person to identify some the current gaps left in the research of number also how far has the research being cared out till and how it can it be taken forward .We have also provided python codes for specific problems which can help for there verification process.

***Key Words*:** Number theory, phyton,

1.INTRODUCTION

Number theory, the oldest and arguably most fundamental branch of mathematics, has witnessed countless breakthroughs throughout history. However, despite significant progress, numerous tantalizing questions remain unanswered. This section provides a brief introduction to the importance of number theory, highlighting its relevance in various fields and setting the stage for the exploration of unsolved problems.

2. Importance of number theory

2.1.1

Fundamental Concepts: Number theory focuses on the properties and relationships of integers, forming the basis of mathematics itself. It deals with fundamental concepts such as prime numbers, divisibility, modular arithmetic, and congruences. Understanding these concepts is crucial for various branches of mathematics, including algebra, analysis, and cryptography.

2.1.2 Encryption and Cryptography: One of the most widely used encryption algorithms, the RSA algorithm, relies heavily on number theory. RSA encryption involves the use of large prime numbers for key generation, encryption, and decryption processes. The security of the RSA algorithm assumes that it is computationally infeasible to factorize large numbers into their prime factors. This assumption is rooted in number theory, specifically the difficulty of prime factorization. (1).

2.1.3 Computational Efficiency: Number theory has contributed to the development of efficient algorithms for various computational tasks. Prime factorization, which lies at the heart of many number theory problems, is used in cryptography, data compression, and computer simulations. The study of number theory has led to the discovery of sophisticated algorithms, enhancing computational efficiency across diverse applications. (2)

2.1.4 Data Analysis: Number theory plays a role in data analysis by providing tools to explore and understand patterns in datasets. For example, prime numbers and their distribution have been used to study the distribution of gaps between consecutive events in various natural phenomena, such as earthquakes or stock market fluctuations. Number theoretic methods and algorithms help analyse the distribution and correlations within the data, providing insights into underlying patterns and potential predictive models. (3)

2.1.5 Pattern Recognition: Number theory aids in the development of pattern recognition algorithms and techniques. The study of prime numbers, divisibility rules, and modular arithmetic can help identify patterns and regularities within datasets. By applying number theoretic concepts, researchers can design algorithms that automatically detect and classify patterns in various domains, such as image recognition, signal processing, and machine learning.

2.2 Historical significance

2.2.1 Diophantus and Diophantine Equations: In the 3rd century CE, the Greek mathematician Diophantus pioneered the study of Diophantine equations, named after him. These equations involve finding integer solutions to polynomial equations. Diophantus' work marked a shift towards algebraic methods and laid the groundwork for later developments in number theory.

2.2.2 Islamic Golden Age: During the Islamic Golden Age (8th to 14th centuries), scholars made significant contributions to number theory. Persian mathematician Al-Khwarizmi introduced Hindu-Arabic numerals and the decimal positional system to the Islamic world. Al-Khwarizmi's book, "The Compendious Book on Calculation by Completion and Balancing," also explored algebraic methods and provided solutions to linear and quadratic equations.

2.2.3 Fermat and the Last Theorem: In the 17th century, Pierre de Fermat, a French lawyer and amateur mathematician, posed what would become one of the most famous problems in number theory: Fermat's Last Theorem. Fermat claimed to have a proof for the theorem but left only a tantalizing note without revealing it. The theorem states that there are no whole number solutions to the equation x^n + y^n = z^n for n > 2. Fermat's Last Theorem remained unsolved for over 350 years and captivated mathematicians until its proof by Andrew Wiles in 1994.

2.2.4 20th Century Advances: The 20th century witnessed remarkable advancements in number theory. The prime number theorem, which gives an estimate of the distribution of prime numbers, was proved independently by Jacques Hadamard and Charles Jean de la Vallée-Poussin in 1896. The development of abstract algebra, particularly the study of algebraic number fields and Galois theory, further enriched number theory. Mathematicians like Emmy Noether, Ernst Zermelo, and André Weil contributed to the understanding of algebraic structures and their connections to number theory.

2.2.5 Computational Advances: The advent of computers in the 20th century revolutionized number theory. Computer-assisted calculations and algorithms allowed for the exploration of large numbers, factorizations, and the discovery of new prime numbers. The development of primality testing algorithms, such as the Miller-Rabin test and the AKS primality test, enabled efficient verification of prime numbers.

2.2.6 Current Research and Unsolved Problems: Number theory continues to be an active area of research with numerous unsolved problems. Prominent among these are the Riemann Hypothesis, the Birch and Swinnerton-Dyer Conjecture, the Goldbach Conjecture, and the Twin Prime Conjecture. Researchers continue to explore prime numbers, elliptic curves, modular forms, and other intricate topics in pursuit of new insights and solutions to these longstanding problems.

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2. The Riemann Hypothesis:

The Riemann Hypothesis stands as one of the most celebrated unsolved problems in number theory. Named after the German mathematician Bernhard Riemann, it deals with the distribution of prime numbers and holds the potential to revolutionize the understanding of their patterns.

2.1) Introduction :- In his address to the Paris International Congress of Mathematicians in 1900, Hilbert mentioned the Riemann Hypothesis as one of 23 tasks for twentieth-century mathematicians to solve.. Now we realise it is up to mathematicians of the twenty-first century! Even though the Riemann Hypothesis (RH) has been around for more than 140 years, working on it right now is arguably more fascinating than at any other moment in its existence. Numerous branches of mathematics and physics have recently come together to produce an explosion of new research.

2.2) The problem : - The Riemann Hypothesis states that all non-trivial zeros of the Riemann zeta function have a real part of 1/2.

2.3) Why it is difficult to solve Riemann hypothesis :-

Because it incorporates complicated mathematical ideas and necessitates a thorough knowledge of number theory and advanced analysis, the Riemann Hypothesis is challenging to solve. It has not yet been established whether or not it is possible to verify the theory for all potential values without costly computing calculations.

2.4) Solution and attempts:- The paper by Tribikram pati (5) describes Riemann hypothesis and suggest a possible way by taking the hypothesis equivalent to the statement given

6. **Further Remarks** (I) It is known [8] that the Riemann Hypothesis is equivalent to the statement that, for any $\epsilon > 0$, and all sufficiently large $x > 0$, the relative error in the Prime Number Theorem:

$$\pi(x) \sim \mathrm{li}\,x, \text{ where } \mathrm{li}\,x = \lim_{\epsilon \to 0} \left[\int_0^{1-\epsilon} \frac{dt}{\log t} + \int_{1+\epsilon}^{x} \frac{dt}{\log t} \right],$$

is less that $x^{-\frac{1}{2}+\epsilon}$.

#photo taken from (5)

It is still advised to refer the full paper (5) for complete understanding of the solution .

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In (7) Maref wolf describes some physical problems related to the hypothesis such as the Polya–Hilbert conjecture, the links with random matrix theory, relation with the Lee–Yang theorem on the zeros of the partition function and phase transitions, random walks, billiard . In (8) Dave platt and Tim trudgian varies the hypothesis till numbers as large as (3*(10^12)).

Remarks – the advances made in Riemann hypothesis is incurable if we prove Riemann hypothesis it will be have profound implications for the distribution of prime numbers and the understanding of the behavior of the Riemann zeta function also it would potentially unlock new avenues for solving other long-standing mathematical problems and contribute to a deeper understanding of the mathematical universe.

3. Goldbach's Conjecture:
Another prominent unresolved question in number theory is Goldbach's Conjecture. Proposed by the German mathematician Christian Goldbach in 1742, it states that every even integer greater than 2 can be expressed as the sum of two prime numbers.
3.1) Introduction :- The Goldbach conjecture came from a letter from Goldbach to Euler it sated that
2n = p + p2
2n+1 = p + p2 + p3
Where p , p2 , p3 are prime numbers . In 20th century many matematicians became curious about this conjecture . it was one of the 23 problems in mathematics which d helbert mentioned in 1900 . Hardy said that the goldbach conjecture is one of the most difficult problems mathematics has ever faced however we have seen high advances in the field of Goldbach conjecture in the past 100 years or so
3.2) The problem : -
The "Goldbach Conjecture" states that every even integer greater than 2 can be expressed as the sum of two prime numbers.

3.3) Why it is difficult to solve The Goldbach Conjecture:-
The Goldbach Conjecture is difficult to prove because it involves the behaviour and properties of prime numbers, which are inherently complex and still not fully understood. We do not have a correct therom till now which states is whole soul prove of prime numbers like a number is prime if and only if it follows this equation something like this missing from mathematics The conjecture requires demonstrating that every even integer greater than 2 can always be expressed as the sum of two prime numbers, but finding a proof that holds true for all possible cases has proven to be a challenging task. Despite extensive computational searches and partial results, a general proof or counterexample to the Goldbach Conjecture remains elusive.

3.4) Solution and attempts:- The paper by H.A. HELFGOTT (10) describes and suggest a possible way by taking the hypothesis equivalent to the statement given

The proof given here works for all $n \geq C = 10^{27}$. (It is typical of analytic proofs to work for all n larger than a constant; see §1.2.1.) Verifying the main theorem for $n < 10^{27}$ is really a minor computational task; it was already done for all $n \leq 8.875 \cdot 10^{30}$ in [HP]. (Appendix C provides an alternative approach.) This finishes the proof of the main theorem for all n.

We are able to set major arcs to be few and narrow because the minor-arc estimates in [Helb] are very strong; we are forced to take them to be few and narrow because of the kind of L-function bounds we will rely upon. ("Major arcs" are small intervals around rationals of small denominator; "minor arcs" are everything else. See the definitions at the beginning of §1.3.)

As has been the case since Hardy and Littlewood [HL23], the approach is based on Fourier analysis, and, more particularly, on a study of exponential sums $\sum_p e(\alpha p)\eta(p/x)$, where η is a weight of our choice (a "smoothing function", or simply a "smoothing"). Such exponential sums are estimated in [Hela] and [Helb]

#photo taken from (10)
It is still advised to refer the full paper (10) for complete understanding of the solution .
. In (11) JORG RICHSTEIN varies the hypothesis till numbers as large as (4*(10^14)). ALSO IN (12) H. A. HELFGOTT AND DAVID J. PLATT VARIES THE CONJUCURE UPTO 8, 875, 694, 145, 621, 773, 516, 800, 000, 000, 000 IT IS ALSO NOTABLE THAT UNDER (9) WANG YUAG WE SEE THE CIRCLE METHOD , SIEVE METHOD BEING APPLIED IN ORDER TO SOLVE THE FAMOUS GOLDBERG CONJECTURE

Remarks – The chances and advances made in solution of Goldberg conjecture are tumendrous but this still remains as the one of most famous and difffcult problems to solve in number theory also it is quite simple for one to prove that a even number is always a the sum of 2 odd numbers which can be easily done by a even number is 2q where q belongs to natural numbers now
2q = 2q +1 -1
q = 2q + 1 – (q +1)
2q = 2(2q + 1 – (q +1)) +1 -1
2q = 4q +2 -1 – 2(q+1)+1
2q = (4q+1) – (2(q+1) + 1)
Both of which are odd but this is simple we need to prove that the 2 numbers thus formed both are primes which indeed is a challenging task
The verification of Goldberg conjecture can also be done by using the python code given in the photo below

```
In [*]: import math

def is_prime(n):
    if n <= 1:
        return False
    for i in range(2, int(math.sqrt(n)) + 1):
        if n % i == 0:
            return False
    return True

def find_prime_pairs():
    q = 1
    while True:
        target = 2 * q
        pairs = []
        for i in range(2, target // 2 + 1):
            j = target - i
            if is_prime(i) and is_prime(j):
                pairs.append((i, j))
        if pairs:
            print(f"Pairs for 2q = {target}: {pairs}")
        q += 1

find_prime_pairs()
```

4. The Legendre's conjecture:
The Legendre's conjecture, proposed by the French mathematician Adrien-Marie Legendre in 1798, states that for any positive integer n, there exists at least one prime number between n^2 and (n + 1)^2. In other words, there is always a

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prime number in the interval [n^2, (n + 1)^2]. Despite being a seemingly simple statement, the conjecture remains unproven to this day. Many attempts have been made to prove or disprove Legendre's conjecture, but it still stands as an open problem in number theory, captivating mathematicians as they seek to unravel the mysteries of prime numbers and their distribution.
4.1) Introduction :- The Legendre's Conjecture, also known as the Legendre's Prime Number Conjecture, states that for any positive integer n, there exists at least one prime number between n^2 and (n+1)^2. Proposed by the French mathematician Adrien-Marie Legendre in 1798, this conjecture suggests that no matter how large the gap between consecutive perfect squares, there will always be a prime number within that range. While this conjecture remains unproven, it holds significant importance in number theory and has sparked numerous studies and research efforts aimed at understanding the distribution of prime numbers and the potential existence of primes in specific intervals.
4.2) The problem : - The conjecture states that there is always at least one prime number between any two consecutive perfect squares.
4.3) Why it is difficult to solve The Legendre's conjecture:-
Legendre's Conjecture is considered difficult to solve because it deals with the distribution of prime numbers, which is a topic that remains challenging and mysterious in mathematics. Here are a few reasons why the conjecture is difficult to prove or disprove:
1. Complexity of Prime Numbers: Prime numbers have fascinated mathematicians for centuries, and their distribution is not yet fully understood. There is no known formula or method that can generate prime numbers with certainty. As a result, analyzing the gaps between primes and proving their existence within specific ranges, such as between consecutive perfect squares, presents significant challenges.
2. Lack of a Generalized Prime Number Theorem: The Prime Number Theorem, proved by Jacques Hadamard and Charles Jean de la Vallée-Poussin in the late 19th century, gives an estimate of the number of primes up to a given value. However, it does not provide detailed information about the distribution of primes between specific intervals, like the gap between consecutive perfect squares. Without a generalized prime number theorem, proving Legendre's Conjecture becomes more difficult.

4.4) Solution and attempts:- The paper by Samuel Bonaya buya (13) describes Legendre's conjecture and suggest a possible way by taking the hypothesis equivalent to the statement given

Methodology

Carl Friedrich Gauss on investigating the properties of primes came up with a prime counting function that approximate the number of primes between the numbers 0 and x given by:

$$\pi(x) = \frac{x}{\ln x} \qquad 1$$

For the purpose of seeking for a proof of Legendre conjecture we will investigate the minimum number of primes between 1 and x.

The investigation will begin by introducing the suspected function:

$$\theta(x) = \frac{x}{\sqrt[4]{(2x)}} = \frac{x^{3/4}}{\sqrt[4]{2}} \qquad 3$$

#photo taken from (13)
It is still advised to refer the full paper (13) for complete understanding of the solution .

In (14) Raja Rama Gandhi describes Solution to the conjecture using prime counting function and the Bertrand's postulate . In (15)Baker Herman and pintz proved that there is always a prime number in the interval ((x) – (x)^(21/40) ,x) for large x

Remarks – the advances made in proff of legendre conjecture are less there are still very less papers related to this topic the legendre conjecture still legendre conjecture can be verified using python coding by the code given below

```python
import math

def is_prime(n):
    if n <= 1:
        return False
    if n == 2:
        return True
    if n % 2 == 0:
        return False
    sqrt_n = int(math.sqrt(n)) + 1
    for divisor in range(3, sqrt_n, 2):
        if n % divisor == 0:
            return False
    return True

def find_primes_between_squares():
    q = 1
    while True:
        lower_bound = q**2
        upper_bound = (q+1)**2
        primes = []
        for num in range(lower_bound , upper_bound):
            if is_prime(num):
                primes.append(num)
        yield primes
        q += 1

# Example usage
prime_generator = find_primes_between_squares()
while True:
    primes = next(prime_generator)
    q = int(math.sqrt(primes[0]))
    print("Prime numbers between", q**2 , "and", (q+1)**2 , ":", primes)
```

5. Twin Prime Conjecture:
The Twin Prime Conjecture postulates the existence of infinitely many pairs of prime numbers that differ by 2, such as (3, 5) and (11, 13). This section examines the historical background, progress, and recent developments in the pursuit of a proof for this conjecture, shedding light on the current understanding of twin primes and potential avenues for future research.
5.1) Introduction :- The Twin Prime Conjecture is a hypothesis in number theory that suggests an infinite number of twin primes exist. Twin primes are pairs of prime numbers that differ by two, such as (3, 5), (11, 13), or (17, 19). Proposed by the ancient Greek mathematician Euclid over two thousand years ago, the conjecture has intrigued mathematicians for centuries. Despite its simplicity, the proof of this conjecture remains elusive. Many notable mathematicians, including Pierre de Fermat and Alphonse de Polignac, have contributed to its study, but a rigorous demonstration of the infinitude of twin primes is yet to be established. Although computer-based searches have uncovered increasingly larger twin primes, the conjecture's complete resolution remains an open question, representing one of the unsolved problems in mathematics that continue to captivate and challenge researchers to this day..

5.2) The problem : - The Twin Prime Conjecture states that there are infinitely many pairs of prime numbers that differ by two.

5.3) Why it is difficult to solve The Twin Prime Conjecture:- The Twin Prime Conjecture is considered difficult to solve because it deals with the distribution of prime numbers, which is a topic that remains challenging and mysterious in mathematics. Here are a few reasons why the conjecture is difficult to prove or disprove:

1. Complexity of Prime Numbers: Prime numbers have fascinated mathematicians for centuries, and their distribution is not yet fully understood. There is no known formula or method that can generate prime numbers with certainty. As a result, analyzing the gaps between primes and proving their existence within specific ranges, such as between consecutive perfect squares, presents significant challenges.
2. Lack of a Generalized Prime Number Theorem: The Prime Number Theorem, proved by Jacques Hadamard and Charles Jean de la Vallée-Poussin in the late 19th century, gives an estimate of the number of primes up to a given value. However, it does not provide detailed information about the distribution of primes between specific intervals, like the gap between consecutive perfect squares. Without a generalized prime number theorem, The Twin Prime Conjecture becomes more difficult.

5.4) Solution and attempts:- The paper by JAMES MAYNARD (16) describes TWIN PRIME CONJECTURE and suggest a possible way by the useage of

A) numerical evidence
B) cramer's method
C) Circle method heuristic
D) Sieve methods

It is still advised to refer the full paper (16) for complete understanding of the solution .

In (17) William dunham describes some physical problems related to the conjecture also talks about the origin of the twin prime conjecture . In (18) Antal Balog, Alina-Carmen Cojocaru, and Chantal Daviddescibe the twin prime conjecture for elipticcal curves and (19) a polynomial analog of twin prime conjecture is provided by pollack

Remarks – the advances made in proff of Twin Prime Conjecture are less there are still very less papers related to this topic the Twin Prime Conjecture still Twin Prime Conjecture can be verified using python coding by the code given below

```
def is_prime(n):
    if n < 2:
        return False
    for i in range(2, int(n ** 0.5) + 1):
        if n % i == 0:
            return False
    return True

def find_twin_primes():
    twin_primes = []
    n = 2
    while True:
        if is_prime(n) and is_prime(n + 2):
            twin_primes.append((n, n + 2))
        n += 1
    return twin_primes

twin_prime_pairs = find_twin_primes()

print("Twin primes found:")
for pair in twin_prime_pairs:
    print(pair)
```

3. CONCLUSIONS

This review paper offers an in-depth exploration of some of the most intriguing unsolved problems in number theory. By examining the historical context, mathematical significance, and ongoing research efforts related to these enigmas, we hope to inspire further investigation and progress in this captivating field. As mathematicians continue their quest to unlock the secrets of number theory, these unsolved mysteries stand as testaments to the power of human curiosity and the boundless potential of mathematical exploration.

REFERENCES

(1)} @inproceedings{Ramesh2018ImportanceON,
title={Importance of Number Theory in Cryptography},
author={D. Ramesh and Rajeshwari P. Patil},
year={2018}

(2) @article{Saridis2003ReviewON,
title={Review of number theory for computing},
author={Prodromos Saridis},
journal={SIGACT News},
year={2003},
volume={34},
pages={4-5}
}

(3) Hua, L-K., and Yuan Wang. Applications of number theory to numerical analysis. Springer Science & Business Media, 2012.

(4) Ore, O., 1988. Number theory and its history. Courier Corporation.

(5)

arXiv:math/0703367 [math.NT]

(or arXiv:math/0703367v2 [math.NT] for this version)
https://doi.org/10.48550/arXiv.math/0703367

(6) Connes, Alain. "An essay on the Riemann Hypothesis." Open problems in mathematics (2016): 225-257.
(7) Wolf, Marek. "Will a physicist prove the Riemann hypothesis?." Reports on Progress in Physics 83, no. 3 (2020): 036001.
(8) Platt, Dave, and Tim Trudgian. "The Riemann hypothesis is true up to 3· 10 12." Bulletin of the London Mathematical Society 53.3 (2021): 792-797.
(9) Wang, Y. (2002). The Goldbach Conjecture (Vol. 4). World scientific.
(10) Helfgott, H. A. (2013). The ternary Goldbach conjecture is true. arXiv preprint arXiv:1312.7748.
(11) Richstein, Jörg. "Verifying the Goldbach conjecture up to 4· 10^{14}." Mathematics of computation 70.236 (2001): 1745-1749.
(12) Helfgott, Harald A., and David J. Platt. "Numerical verification of the ternary Goldbach conjecture up to 8.875· 1030." Experimental Mathematics 22.4 (2013): 406-409.
(13) Buya, S. B. (2018). A simple proof of Legendre's conjecture. Journal of applied science and innovations.
(14) Gandhi, R.R. and Guedes, E., 2013. An elementary proof of Legendre's conjecture. Asia Journal of Mathematics and Physics, 2013, pp.1-7.
(15) Baker, R. C.; Harman, G.; Pintz, J. (2001), "The difference between consecutive primes, II" (PDF), Proceedings of the London Mathematical Society, 83 (3): 532–562, doi:10.1112/plms/83.3.532, S2CID 8964027
(16) Maynard, James. "The twin prime conjecture." Japanese Journal of Mathematics 14 (2019): 175-206.
(17) Dunham, William. "A Note on the Origin of the Twin Prime Conjecture." Notices of the International Consortium of Chinese Mathematicians 1, no. 1 (2013): 63-65.
(18) Balog, A., Cojocaru, A.C. and David, C., 2011. Average twin prime conjecture for elliptic curves. American journal of mathematics, 133(5), pp.1179-1229.
(19) Pollack P. A polynomial analogue of the twin prime conjecture. Proceedings of the American Mathematical Society. 2008;136(11):3775-84.