Optimal Tax Policy of a Stage Structured Prey Predator Fishery: A Dynamic Reaction Model

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Abstract

In a fully dynamic model of an open-access fishery, the level of fishing effort expands or contracts according as the perceived rent (i.e., the net economic revenue to the fishermen) is positive or negative. A model reflecting this dynamic interaction between the perceived rent and the effort in a fishery is called a dynamic reaction model. In this paper, we study a dynamic reaction model, in which the prey species is subjected to harvesting in the presence of predator and it is assumed that an external agency regulates the fishery by imposing a suitable tax per unit biomass of landed fish. The fishing effort is taken as a dynamic variable depending on the capital invested in the fishery. The steady states and their stability are studied. The problem of the optimal tax policy is solved by Pontryagin's maximum principle keeping the ecological balance.

Keywords: Fisheries, Dynamic reaction, Prey-Predator system, harvesting, taxa- tion.

1 Introduction

Bioeconomic modeling of biological resources, such as fisheries and forestry, has gained significant importance in recent years due to concerns over their overexploitation to meet rising global demands. Colin Clark in his books [1, 2] has extensively discussed the techniques and challenges involved in the bioeconomic exploitation of these resources. Marine fisheries are inherently multi-species systems, and the exploitation of mixed-species fisheries has increasingly drawn the attention of researchers. While numerous models have been developed for single-species fisheries, studies on multi- species fisheries remain comparatively limited.

Mathematical modeling of problems concerning harvesting of multi-species fisheries has recently gained significant attention from researchers. Colin Clark [1, 3] Colin Clark has thoroughly explored the techniques and challenges associated with the bioeconomic exploitation of these resources. Constructing a realistic model of a multi- species community is inherently challenging, and even when successfully formulated, such models are often analytically intractable. Determining an optimal harvesting policy for mixed-species fisheries involves both theoretical and practical difficulties. Despite these challenges, efforts to study multi-species models with multiple state or control variables have been undertaken periodically [6, 12], [4], [5] [9].

Managing renewable resources is closely linked to addressing law enforcement challenges. With resource stocks declining and environmental conditions deteriorating, regulating the exploitation of biological resources has become has become increasingly critical. Marine fisheries, in particular, encounter substantial law enforcement hurdles in achieving sustainable management. Sutinen and Andersen [15] explored the economics of law enforcement in marine fisheries. Taxation, license fees, lease of property rights, seasonal harvesting etc. are usually considered as possible governing instruments in fishery regulation. Various issues associated with the choice of an optimal governing instrument and its enforcement in fishery were discussed by Anderson and Lee [8]. Optimal timing of harvest was adopted as a regulatory device by Kellogg et al. [12] in their study of the North Carolina Bay Scallop Fishery. Implementing a tax on each unit of biomass of landed fish is a potential regulatory tool for managing fisheries. Economists generally view taxation as a superior control policy due to its flexibility. However, political constraints often hinder its practical implementation. Despite this, theoreticians continue to study the implications of taxation as a regulatory measure. Harvesting problems with tax have been studied by [1, 2] , [3], [11], [13], [9] and many more.

This study explores a dynamic reaction model within a stage structured prey- predator fishery system in which the prey species is subjected to harvesting in the presence of predator species and taxation as the control instrument. The growth of the fish population obeys the logistic law [14]. A dynamic variable E which is a function of time is considered as harvesting effort and it is assumed to be proportional to the instantaneous amount of capital invested. Exploitation of the fishery is regulated by an agency by imposing a tax per unit biomass of landed fish. The gross rate at which capital is invested at any time is assumed to be proportional to perceived rent [1] at that time. This imposition of tax acts as a deterrent to the fishermen and helps to control harvesting of prey species and in turn, it helps the predator to grow. Capital theoretic approach is adopted to formulate the dynamic fishery model. The main aim of this paper is to find the proper taxation policy which would give the best possible benefit through harvesting to the society while preventing extinction of the predator. The existence of the possible steady states along with their local stability is discussed. The global stability of the interior equilibrium is also discussed. The various conditions are arising in the analysis give different ranges in which the tax must lie to get the desired results. Using the maximum principle optimal tax policy is discussed. Finally, it is illustrated that how the system works by taking numerical examples.

2 The problem formulation

The ecological set up of the formulation is as follows. There is a prey-predator system with predators having two stages namely juveniles and adults and their densities are denoted by N_2 and N_3 respectively. It is also assumed that only adult predators are capable of preying on the prey species and the juvenile predators live on their parents. The prey, whose density is denoted by N_1, is modeled by a logistic equation in the absence of the predator. One key feature of our model is intra specific competition in the consumer growth dynamics. This term describes either a self limitation of consumer or the influence of predation. This self limitation can occur if there is some other factor (besides food) which becomes limiting at high population densities.
Keeping these assumptions in view, the dynamics of the system may be governed by the following system of differential equation.

$$\begin{aligned} \frac{dN_1}{dt} &= r_1 N_1\left(1 - \frac{N_1}{K}\right) - \alpha N_1 N_3, \\ \frac{dN_2}{dt} &= \beta N_3 - r_2 N_2, \\ \frac{dN_3}{dt} &= -r_3 N_3 + m\alpha N_1 N_3 + \gamma N_2 - \delta N_3^2. \end{aligned} \tag{2.1}$$

Here r_1 is the intrinsic growth rate of the prey, K is the carrying capacity of the prey, α is the predation parameter, m is the conversion factor, r_3 is the death rate of mature predator species, γ is the proportionality constant transformation of immature to mature predators, r_2 is the death rate of immature populations.
Defining $N_1 = \frac{K r_2 x_1}{r_1}$, $N_2 = \frac{\beta x_2}{m\alpha}$, $N_3 = \frac{r_2 x_3}{m\alpha}$ and $t = \frac{\tau}{r_2}$ We can rewrite (2.1) as

$$\begin{aligned} \frac{dx_1}{d\tau} &= a x_1 - x_1^2 - b x_1 x_3, \\ \frac{dx_2}{d\tau} &= x_3 - x_2, \\ \frac{dx_3}{d\tau} &= -c x_3 + d x_1 x_3 + e x_2 - f x_3^2, \end{aligned} \tag{2.2}$$

where $a = \frac{r_1}{r_2}$, $b = \frac{1}{m}$, $c = \frac{r_3}{r_2}$, $d = \frac{m\alpha K}{r_1}$, $e = \frac{\gamma\beta}{r_2^2}$ and $f = \frac{\delta}{m\alpha}$.
Harvesting has a strong impact on the dynamic evaluation of a population subjected

to it. Depending on the nature of applied harvesting strategy, the long run stationary density of population may be significantly smaller than the long-run stationary density of a population in the absence of harvesting. Harvesting can lead to the incorporation of a positive probability of extinction within a finite time frame. Thus studying population dynamics with harvesting is both interesting and crucial subject for understanding population behavior. In order to study the effect of harvesting on the system (2.2), we consider the following system:

$$\frac{dx_1}{d\tau} = ax_1 - x_1^2 - bx_1x_3 - h(t),$$
$$\frac{dx_2}{d\tau} = x_3 - x_2, \qquad (2.3)$$
$$\frac{dx_3}{d\tau} = -cx_3 + dx_1x_3 + ex_2 - fx_3^2,$$

where $h = qEx_1$ is the harvesting rate at any time t, which is based on the catch- per-unit-effort hypothesis (Clark, 1990), q is the catchability coefficient, E is the harvesting effort.

The harvesting agency does not adjust the effort due to the presence of the predator. Since the predator is unable to evolve a strategy for its survival, the regulating agency comes to rescue of the predator through a suitable tax policy. The regulating agency levies a tax τ (> 0) per unit biomass of the landed prey fish to control the exploitation. Any subsidy to the fisherman may be interpreted as a negative value of τ. The net economic revenue to the fisherman (perceived rent) is $[q(p - \tau)x_1 - c_1]E$, where p is the fixed price per unit of the prey species $(p > \tau)$. c_1 is the fixed cost of harvesting per unit effort. The regulatory agency and fisherman are actually two different components of the society at a large. Hence the revenues earned by them are the revenues accrued to the society through the fishery.

The Net economic revenue to the society is

$$[pqEx_1 - c_1E] = [q(p - \tau)x_1 - c_1]E + \tau qx_1E$$

which equals to the net economic revenue to the fisherman (perceived rent) plus the economic revenue to the regulatory agency. We now consider a dynamic reaction model by assuming that the level of effort in harvesting expands or contracts according to whether the perceived rent is positive or negative. The harvesting effort E is therefore, a dynamic variable governed by the differential equation:

$$\frac{dE}{dt} = \lambda[q(p - \tau)x_1 - c_1] \qquad (2.4)$$

where λ is a stiffness parameter measuring the intensity of reaction between the effort and the perceived rent. Thus using (2.4), the system (2.3) becomes:

$$\begin{aligned}
\frac{dx_1}{dt} &= ax_1 - x_1^2 - bx_1x_3 - qx_1E,\\
\frac{dx_2}{dt} &= x_3 - x_2,\\
\frac{dx_3}{dt} &= -cx_3 + dx_1x_3 + ex_2 - fx_3^2,\\
\frac{dE}{dt} &= \lambda[q(p-\tau)x_1 - c_1]E,
\end{aligned} \tag{2.5}$$

with initial condition $x_1(0) \geq 0$, $x_2(0) \geq 0$, $x_3(0) \geq 0$, $E(0) \geq 0$.

3 Dynamical behavior of the system

3.1 The steady states:

A steady state of dynamical system (2.5) is an equilibrium point (x_1, x_2, x_3, E) at which $\dot{x}_1 = 0$, $\dot{x}_2 = 0$, $\dot{x}_3 = 0$ and $\dot{E} = 0$. Now we have analyzed the existence of the steady states and their nature. Particularly from biological point of view we only concentrate on the interior or positive equilibrium of the model where all species co-exist. The possible steady states of this system are

(i) The system has the trivial steady state $P_1(0, 0, 0, 0)$.

(ii) The boundary steady state $P_2(a, 0, 0, 0)$.

(iii) The prey free steady state without harvesting effort $P_3(0, \frac{e-c}{f}, \frac{e-c}{f}, 0)$ which is feasible provided $e > c$.

(iv) Predator free equilibrium $P_4(\bar{x}_1, 0, 0, \bar{E})$, where $\bar{x}_1 = \frac{c_1}{q(p-\tau)}$, $\bar{E} = \frac{a - \frac{c_1}{q(p-\tau)}}{q}$. It is feasible if $a > \frac{c_1}{q(p-\tau)} > 0$.

(v) The harvesting effort free equilibrium $P_5(\hat{x}_1, \hat{x}_2, \hat{x}_3, 0)$, where $\hat{x}_1 = \frac{bc-be+af}{bd+f}$, $\hat{x}_2 = \frac{e-c+ad}{bd+f}$, $\hat{x}_3 = \frac{e-c+ad}{bd+f}$. It is feasible if $ad + e > c > \frac{be-af}{b}$.

(vi) Lastly the interior steady state $P_6(x_1^*, x_2^*, x_3^*, E^*)$ where $x_1^* = \frac{c_1}{q(p-\tau)}$, $x_2^* = x_3^* = \frac{e-c+\frac{c_1}{q(p-\tau)}}{f}$ and $E^* = \frac{1}{q}[a - \frac{c_1}{q(p-\tau)} - b(\frac{e-c+\frac{c_1}{q(p-\tau)}}{f})]$. This interior steady state exists when $p - \frac{c_1 d}{q(c-e)} < \tau < p - \frac{c_1(f+bd)}{q(af+bc-be)}$.

3.2 Local Stability

The variational matrix of the system (2.5) is

$$V(x_1, x_2, x_3, E) = \begin{pmatrix} a - 2x_1 - bx_3 - qE & 0 & -bx_1 & -qx_1 \\ 0 & -1 & 1 & 0 \\ dx_3 & e & -c + dx_1 - 2fx_3 & 0 \\ \lambda q(p-\tau)E & 0 & 0 & \lambda[q(p-\tau)x_1 - c_1] \end{pmatrix}.$$

Theorem 3.2.1 The system (2.5) is unstable around the trivial equilibrium

P_1.

Proof. Now the variational matrix at $P_1(0, 0, 0, 0)$ is

$$V(0,0,0,0) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & e & -c & 0 \\ 0 & 0 & 0 & -\lambda c_1 \end{pmatrix}.$$

And since one of the eigen value of the variational matrix is a which is $> o$, so $P_1(0, 0, 0, 0)$ is unstable.

Theorem 3.2.2 The system (2.5) is locally asymptotically stable around the equilibrium point P_2.

Proof. The Variational matrix at $P_2(a, 0, 0, 0)$ is

$$V(a,0,0,0) = \begin{pmatrix} -a & 0 & -ab & -aq \\ 0 & -1 & 1 & 0 \\ 0 & e & -c+ad & 0 \\ 0 & 0 & 0 & -\lambda[aq(p-\tau)-e] \end{pmatrix}.$$

The characteristic equation is $(-a-\mu)(-\mu+\lambda(aq(p-\tau)-c_1))(c-ad-e+(1+c-ad)\mu+\mu^2)$. So $P_2(a, 0, 0, 0)$ is locally asymptotically stable if $\lambda aq(p-\tau)-c_1 < 0$, $1+c > ad$ and $c-e-ad > 0$.

Theorem 3.2.3 The system (2.5) is locally asymptotically stable around the equilibrium point P_3 if $a - \frac{b(e-c)}{f} < 0$ and $e > c$.

Proof. The Variational matrix at $P_3(0, \frac{e-c}{f}, \frac{e-c}{f}, 0)$ is

$$V\left(0, \frac{e-c}{f}, \frac{e-c}{f}, 0\right) = \begin{pmatrix} a - \frac{b(e-c)}{f} & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ \frac{d(e-c)}{f} & e & c-2e & 0 \\ 0 & 0 & 0 & -\lambda c_1 \end{pmatrix}.$$

From this Variational matrix we find that $P_3(0, \frac{e-c}{f}, \frac{e-c}{f}, 0)$ is locally asymptotically stable if $a - \frac{b(e-c)}{f} < 0$ and $e > c$.

Theorem 3.2.4 The system (2.5) is locally asymptotically stable around the predator free equilibrium P_4 if $m_1 > 0$, $m_1m_2 - m_3 > 0$ and $m_3(m_1m_2 - m_3) - m_4m_1^2 > 0$.

Proof. The Variational matrix at $P_4(\bar{x}_1, 0, 0, \bar{E})$ is

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$$V(\bar{x}_1, 0, 0, \bar{E}) = \begin{bmatrix} a - 2\bar{x}_1 - q\bar{E} & 0 & -b\bar{x}_1 & -q\bar{x}_1 \\ 0 & -1 & 1 & 0 \\ 0 & e & -c + d\bar{x}_1 & 0 \\ \lambda q(p-\tau)\bar{E} & 0 & 0 & \lambda[q(p-\tau)\bar{x}_1 - c_1] \end{bmatrix}.$$

Putting $\bar{E} = \frac{a-\bar{x}_1}{q}$ in the above variational matrix $V(\bar{x}_1, 0, 0, \bar{E})$ we get the characteristic equation at $P_4(\bar{x}_1, 0, 0, \bar{E})$ which is $\mu^4 + m_1\mu^3 + m_2\mu^2 + m_3\mu + m_4 = 0$ where $m_1 = 1 + c + (1 - d)\bar{x}_1$,
$m_2 = c - e + \bar{x}_1(1 + c - d + aq\lambda(p-\tau)) - \bar{x}_1^2(d + q\lambda(p-\tau))$,
$m_3 = \bar{x}_1(c - e) - d\bar{x}_1^2 + \lambda q\bar{x}_1(p-\tau)(a - \bar{x}_1)(1 + c - d\bar{x}_1)$, $m_4 = \lambda q\bar{x}_1(p-\tau)(a - \bar{x}_1)(c - e - d\bar{x}_1)$.

By Routh-Hurwith criteria if $m_1 > 0$, $m_1m_2 - m_3 > 0$ and $m_3(m_1m_2 - m_3) - m_4m_1^2 > 0$ we can conclude that the system is asymptotically stable at $P_4(\bar{x}_1, 0, 0, \bar{E})$.

Theorem 3.2.5 The system (2.5) is locally asymptotically stable around the equilibrium P_5 if $A_1 > 0$, $A_1A_2 - A_3 > 0$ and $A_3(A_1A_2 - A_3) - A_4A_1^2 > 0$.

Proof. Putting the value of $x_2 = x_3 = \frac{a-x_1}{b}$ and $E = 0$ in $V(x_1, x_2, x_3, E)$ we get, the characteristic equation at $P_5(\hat{x}_1, \hat{x}_2, \hat{x}_3, 0)$ is $\mu^4 + A_1\mu^3 + A_2\mu^2 + A_3\mu + A_4 = 0$ where $A_1 = \frac{1}{b}(a_{11}x_1 + a_{12})$,
$A_2 = \frac{1}{b}(b_{11}x_1^2 + b_{12}x_1 + b_{13})$,
$A_3 = \frac{1}{b}(c_{11}x_1^3 + c_{12}x_1^2 + c_{13}x_1 + c_{14})$,
$A_4 = \frac{1}{b}(d_{11}x_1^3 + d_{12}x_1^2 + d_{13}x_1 + d_{14})$,
$a_{11} = -2f - b(-1 + d + q\lambda(p-\tau))$,
$a_{12} = b(1 + c + Eq + \lambda c_1) + 2af$,
$b_{11} = 2f(-1 + q\lambda(p-\tau)) + b(d(-2 + q\lambda(p-\tau)) + q\lambda(-p+\tau))$,
$b_{12} = -2f(1 + Eq + a(-1 + q\lambda(p-\tau))) + b(1 + d(-1 + a - Eq) + q\lambda(-p+\tau) + c(1 + q\lambda(-p-\tau))) + (b - bd - 2f)\lambda c_1$,
$b_{13} = 2af(1 + Eq) + b(c - e + (1 + e)Eq) + 2af\lambda c_1 + b(1 + c + qE)\lambda c_1$, $c_{11} = 2(bd + f)q\lambda(p-\tau)$,
$c_{12} = -2f(1 + (-1 + a)q\lambda(p-\tau)) - b(d(2 + (-1 + a)q\lambda(p-\tau)) + (1 + c)q\lambda(p-\tau)) - 2(bd + f)\lambda c_1$,
$c_{13} = 2f(a - Eq - aq\lambda(p-\tau)) + b(c + ad - e - dEq - cq\lambda(p-\tau) + (c - e)q\lambda\tau) + 2f(-1 + a - Eq)\lambda c_1 + b(1 + c + d(-1 + a - Eq))\lambda c_1$,
$c_{14} = Eq[b(c - e) + 2af] + \lambda c_1(2af(1 + Eq) + b(c - e + (1 + c)Eq))$,
$d_{11} = c_{11}$,
$d_{12} = \lambda[(ad - bc + be - 2af)q(p-\tau) - 2(bd + f)c_1]$,
$d_{13} = \lambda[2f(a - Eq)c_1 + b(c + ad - e - dEq)c_1]$, $d_{14} = Eq\lambda c_1[b(e - e) + 2af]$

By Routh-Hurwith criteria if $A_1 > 0$, $A_1A_2 - A_3 > 0$ and $A_3(A_1A_2 - A_3) - A_4A_1^2 > 0$ we can conclude that the system is asymptotically stable at $P_5(\hat{x}_1, \hat{x}_2, \hat{x}_3, 0)$.

Theorem 3.2.6 The system (2.5) is locally asymptotically stable around the equilibrium P_6 if $a_1 > 0$, $a_1a_2 - a_3 > 0$ and $a_3(a_1a_2 - a_3) - a_4a_1^2 > 0$.

Proof. The Variational matrix at $P_6(x_1^*, x_2^*, x_3^*, E^*)$ is

$$V(x_1, x_2, x_3, E) = \begin{bmatrix} a - 2x_1^* - bx_3^* - qE & 0 & -bx_1^* & qx_1^* \\ 0 & -1 & 1 & 0 \\ dx_3^* & e & -c + dx_1^* - 2fx_3^* & 0 \\ \lambda q(p-\tau)E^* & 0 & 0 & \lambda[q(p-\tau)x_1^* - c_1] \end{bmatrix}.$$

The characteristic equation is $\mu^4 + a_1\mu^3 + a_2\mu^2 + a_3\mu + a_4 = 0$ where $a_1 = 1 - c + 2e + (1 + d)x_1^*$

$a_2 = dx_1^{*2} + bdx_1^*x_3^* + \lambda q(p - \tau)x_1^*E^* + x_1^*(1 - c + d + 2e) + e - c$

$a_3 = dx_1^{*2} + x_1^*(-c + e) + bdx_1^*x_3^* + \lambda q(p - \tau)x_1^*E^*(1 - c + 2e + dx_1^*)$

$a_4 = \lambda q(p - \tau)x_1^*E^*(-c + e + dx_1^*)$

By Routh-Hurwith criteria if $a_1 > 0$, $a_1a_2 - a_3 > 0$ and $a_3(a_1a_2 - a_3) - a_4a_1^2 > 0$ we can conclude that the system is asymptotically stable at $P_6(x_1^*, x_2^*, x_3^*, E^*)$.

4 Hopf bifurcation at $P_6(x_1^*, x_2^*, x_3^*, E^*)$

We know that characteristic equation is

$$\mu^4 + a_1\mu^3 + a_2\mu^2 + a_3\mu + a_4 = 0 \tag{4.1}$$

where a_1, a_2, a_3, a_4 are interpreted above. Let us assume that $\mu = i\omega$ is a root of the equation (4.1), then we get,

$$(\omega^4 - a_2\omega^2 + a_4) + i(a_3\omega - a_1\omega^3) = 0 \tag{4.2}$$

Separating real and imaginary parts,

$$\omega^4 - a_2\omega^2 + a_4 = 0 \tag{4.3}$$

$$a_3\omega - a_1\omega^3 = 0 \tag{4.4}$$

Solving (4.3) and (4.4) we get,

$$a_3(a_1a_2 - a_3) - a_1^2a_4 = 0 \tag{4.5}$$

which is a quadratic equation of $\tau = \tau_H$.

Now differentiating the characteristic equation (4.1) w.r.t., τ we get,

$$\left.\frac{d\mu}{d\tau}\right|_{\mu=i\omega} = -\left.\frac{\dot{a}_1\mu^3+\dot{a}_2\mu^2+\dot{a}_3\mu}{4\mu^3+3a_1\mu^2+2a_2\mu+a_3}\right|_{\mu=i\omega}$$

$$= \frac{-\omega^2\dot{a}_2+i(\omega\dot{a}_3-\omega^3\dot{a}_1)\,(a_3-3a_1\omega^2)+i(2a_2\omega-4\omega^3)}{}$$

$$= \frac{4\dot{a}_1\omega^6+(3a_1\dot{a}_2-2\dot{a}_1a_2-4\dot{a}_3)\omega^4+(2a_2\dot{a}_3-\dot{a}_2a_3)\omega^2}{(a_3-3a_1\omega^2)^2+(2a_2\omega-4\omega^3)^2} + i\,\frac{(3a_1\dot{a}_1-4\dot{a}_2)\omega^5-3a_1\dot{a}_3\omega^4+(2a_2\dot{a}_2-\dot{a}_1a_3)\omega^3+a_3\dot{a}_3\omega}{(a_3-3a_1\omega^2)^2+(2a_2\omega-4\omega^3)^2}$$

Now monotonicity condition of the real part of the complex root one can easily establish the hopf bifurcation occur at $\tau = \tau_H$ if $\frac{d}{d\tau}(Re(\mu(\tau)))\big|_{\tau=\tau_H} \neq 0$.

Now $\frac{d}{dt}(Re(\mu(\tau)))|_{\tau=\tau_H}$

$= \frac{d}{dt}(Re(\mu(\tau)))|_{\mu=i\omega}$

$= \frac{4a_1'\omega^6+(3a_1a_2'-2a_1'a_2-4a_3')\omega^4+(2a_2a_3'-a_2'a_3)\omega^2}{(a_3-3a_1\omega a^2)^2+(2a_2\omega-4\omega^3)^2}$

So hopf bifurcation occurs at $\tau = \tau_H$ if

$4a_1'\omega^6 + (3a_1a_2' - 2a_1'a_2 - 4a_3')\omega^4 + (2a_2a_3' - a_2'a_3)\omega^2 \neq 0$ i.e., if $\frac{1}{a_3}[a_1^2a_3^2a_2' + (a_1a_2a_3 - a_3^2)(a_1a_3' - a_3a_1')] \neq 0$

5 Global Stability

Theorem 5.1 The system (2.5) is globally asymptotically stable around the interior equilibrium P_6 if $\beta_1 = \frac{\beta_3 d}{b} = \beta_4\lambda(p-\tau)$ and $\beta_2 = \frac{\beta_3 e x_2^*}{x_3^*}$.

Proof. In this section we have discussed about global stability of this system (2.5). To prove the global stability, we have defined a Lyapunov function

$$V(x_1^*, x_2^*, x_3^*, E^*) = \sum_{i=1}^{3} \beta_i (x_i - x_i^* - x_i^* log(\frac{x_i}{x_i^*})) + \beta_4(E - E^* - log(\frac{E}{E^*}))$$

where β_i, $i = 1, 2, 3, 4$ are suitable constants to be determined in the subsequent step.
Now differentiating $V(x_1^*, x_2^*, x_3^*, E^*)$ w.r.t., t, we get,

$$\frac{dV}{dt} = \beta_1\frac{\dot{x}_1}{x_1}(x_1 - x_1^*) + \beta_2\frac{\dot{x}_2}{x_2}(x_2 - x_2^*) + \beta_3\frac{\dot{x}_3}{x_3}(x_3 - x_3^*) + \beta_4\frac{\dot{E}}{E}(E - E^*)$$

$= -\beta_1(x_1 - x_1^*)^2 - \beta_3 f(x_3 - x_3^*)^2 + (\beta_3 d - \beta_1 b)(x_1 - x_1^*)(x_3 - x_3^*) + (\beta_4\lambda q(p - \tau) - \beta_1 q)(x_1 - x_1^*)(E - E^*) + (\frac{\beta_2}{x_3^*} + \frac{\beta_3 e}{x_2^*})(x_2 - x_2^*)(x_3 - x_3^*) - \frac{\beta_2 x_3}{x_2 x_2^*}(x_2 - x_2^*)^2 - \frac{\beta_3 e x_2}{x_3 x_3^*}(x_3 - x_3^*)^2$

$= -\beta_1(x_1 - x_1^*)^2 - \beta_3 f(x_3 - x_3^*)^2 + (\beta_3 d - \beta_1 b)(x_1 - x_1^*)(x_3 - x_3^*) + (\beta_4\lambda q(p - \tau) - \beta_1 q)(x_1 - x_1^*)(E - E^*) - (\sqrt{\frac{\beta_2 x_3}{x_2 x_2^*}}(x_2 - x_2^*) - \sqrt{\frac{\beta_3 e x_2}{x_3 x_3^*}}(x_3 - x_3^*))^2 + (\frac{\beta_2}{x_3^*} - \frac{\beta_3 e}{x_2^*})^2(x_2 - x_2^*)(x_3 - x_3^*)$

If we choose β_i, $i = 1, 2, 3, 4$ such that $\beta_1 = \frac{\beta_3 d}{b} = \beta_4\lambda(p-\tau)$ and $\beta_2 = \frac{\beta_3 e x_2^*}{x_3^*}$ then we get,

$$\frac{dV}{dt} = -\beta_1(x_1 - x_1^*)^2 - \beta_3 f(x_3 - x_3^*)^2 - (\sqrt{\frac{\beta_2 x_3}{x_2 x_2^*}}(x_2 - x_2^*) - \sqrt{\frac{\beta_3 e x_2}{x_3 x_3^*}}(x_3 - x_3^*))^2$$

Which implies that, $\frac{dV}{dt} \leq 0$.

Hence the system is globally asymptotically stable at $P_6(x_1^*, x_2^*, x_3^*, E^*)$.

6 Optimal tax policy

In this section, we apply Pontryagin's Maximum Principle to derive an optimal harvesting policy so that the regulatory agency is assured to achieve its goal of maximizing the total discounted net revenue generated by the fishery. Formally, this objective involves maximizing the present value J of a continuous stream of revenues over time, represented as

$$J = \int_0^\infty e^{-\delta t}(pqx_1 - c_1)E\,dt \tag{6.1}$$

where δ is the instantaneous rate of annual discount. Our objective is to determine a tax policy $\tau = \tau(t)$ that maximize J subject to the state equations of the system (2.5) and to the control constraints $\tau_{min} \le \tau(t) \le \tau_{max}$.

For this we have formed Hamiltonian for the control problem is given by

$$H = e^{-\delta t}(pqx_1 - c_1)E + \lambda_1(ax_1 - x_1^2 - bx_1x_3 - qx_1E) + \lambda_2(x_3 - x_2) + \lambda_3(-cx_3 + dx_1x_3 + ex_2 - fx^2) + \lambda_4(\lambda_1[q(p - \tau)x_1 - c_1])$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are adjoint variables. Since H is linear in control variable $\tau(t)$, so the optimal control will be the combination of bang-bang control and singular control. The optimal control $\tau(t)$ that maximizes H must satisfy $H_\tau = 0$ such that $\tau_{min} \le \tau(t) \le \tau_{max}$. Which implies that $\lambda_4 = 0$.
Now solving the adjoint equations

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = -[e^{-\delta t}pqE + \lambda_1(a - 2x_1 - bx_3 - qE) + \lambda_3 dx_3], \tag{6.2}$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = -[-\lambda_2 + e\lambda_3]. \tag{6.3}$$

$$\dot{\lambda}_3 = -\frac{\partial H}{\partial x_3} = -[-\lambda_1 bx_1 + \lambda_2 + \lambda_3(-c + dx_1 - 2fx_3)], \tag{6.4}$$

$$\dot{\lambda}_4 = -\frac{\partial H}{\partial E} = -[e^{-\delta t}(pqx_1 - c_1) - \lambda_1 qx_1] = 0. \tag{6.5}$$

From (6.5) we get,

$$\lambda_1 = e^{-\delta t}\left(p - \frac{c_1}{qx_1}\right). \tag{6.6}$$

Putting the value of λ_1 in (6.4) and solving (6.3) and (6.4) we get,

$$\lambda_3 = \mu_1 e^{-\alpha_1 t} + \mu_2 e^{-\alpha_2 t} + \frac{Re^{-\delta t}}{\delta^2 + A\delta + B},$$

where μ_1 and μ_2 are arbitrary constants,
α_1 and α_2 are the roots of the auxiliary equation $m^2 - Am + B = 0$,
$R = -\frac{1}{q}[b(1 + \delta)(pqx_1 - c_1)]$,
$A = 1 + c - dx_1 + 2fx_3$,
$B = c - e - dx_1 + 2fx_3$.

Since the shadow price $\lambda_1(t)e^{\delta t}$ is bounded at $t \to \infty$ if $\mu_1 = \mu_2 = 0$. The transver- sality condition at $t \to \infty$ requires that the shadow price $\lambda_i(t)e^{\delta t}$, $(i = 1, 2, 3)$ remain bounded. So we get,

$$\lambda_3 = \frac{Re^{-\delta t}}{\delta^2 + A\delta + B} \qquad (6.7)$$

Now using (6.6) and (6.7) in (6.2) we get,

$$e^{-\delta t}pqE + \frac{e^{-\delta t}(pqx_1 - c_1)}{qx_1}(a - 2x_1 - bx_3 - qE) = \frac{\delta e^{-\delta t}(pqx_1 - c_1)}{qx_1} - \frac{e^{-\delta t}bd(1+\delta)(pqx_1-c_1)x_3}{q[\delta^2+(1+c-dx_1+2fx_3)\delta+(c-e-dx_1+2fx_3)]}$$

Now for the equilibrium solution we have,

$$x_1^* = \frac{c_1}{q(p-\tau)} = \frac{c_1}{qT}, \qquad (6.8)$$

$$x_2^* = x_3^* = \frac{e - c + \frac{c_1 d}{q(p-\tau)}}{f} = \frac{e - c + \frac{c_1 d}{qT}}{f}, \qquad (6.9)$$

$$E^* = \frac{1}{q}\left(a - \frac{c_1}{q(p-\tau)} - b\left(\frac{e - c + \frac{c_1 d}{q(p-\tau)}}{f}\right)\right) = \frac{1}{q}\left(a - \frac{c_1}{qT} - b\left(\frac{e - c + \frac{c_1 d}{qT}}{f}\right)\right) \qquad (6.10)$$

where $T = p - \tau$.

Now using (6.8), (6.9) and (6.10) we get the following equation for T:

$$H_0T^3 + H_1T^2 + H_2T + H_3 = 0 \qquad (6.11)$$

where

$H_0 = A_3A_6$,

$H_1 = A_4A_6 + A_3A_7 - A_0$, $H_2 = A_4A_7 - A_3A_5 - A_1$, $H_3 = -A_2 - A_4A_5$,

$A_0 = c_1bd(1+\delta)(c-e)$,

$A_1 = \frac{1}{q}(bc_1^2d^2p(1+\delta))$,

$A_2 = c_1bd(1+\delta)(p(e-c) + \frac{c_1d}{q})$, $A_3 \underset{q}{=} \delta^2 + \delta(1 - c + 2e) - c + e$, $A_4 = \frac{1}{q}c_1d(1+\delta)$,

$A_5 = pc_1f(2 + \frac{bd}{f})$,

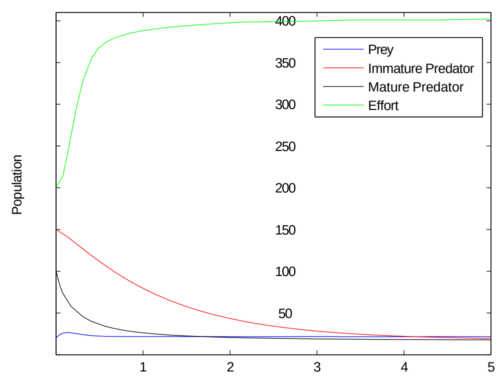
$A_6 = fq\delta$,

$A_7 = c_1f + (b(c-e) + af)pq$,

Let $T^* = (p - \tau^*)$ be a solution (if it exists) of (6.11). Using this value of T^* in (6.8), (6.9) and (6.10) we obtain the optimal equilibrium solution $(x_{1\delta}, x_{2\delta}, x_{3\delta}, E_\delta)$. We have established the existence of an optimal equilibrium solution that satisfies the necessary conditions of the Pontryagin's maximum principle. It is extremely difficult to find an optimal approach path consisting of a combination bang-bang control (i.e., $\tau = \tau_{min}$ or $\tau = \tau_{max}$) and non-equilibrium singular controls (i.e., $\tau_{min} < \tau(t) < \tau_{max}$). This difficulty was faced by Clark [3] even in the study of a simple model of two ecologically independent fish populations. The present model is much more complicated than the said model of Clark [3]. Due to these difficulties, we have considered an optimal equilibrium solution only.

7 Numerical Simulation

Let us assume the ecological parameters as $p = 15$, $c_1 = 10$, $c = 0.08$, $d = 0.05$, $e = 0.05$, $q = 0.05$, $f = 0.06$, $a = 43$, $b = 0.07$, $\lambda = 1$, $\delta = 0.01$. Then in order to ensure existence of the interior steady states $(x_1^*, x_2^*, x_3^*, E^*)$, we have to select tax τ, such that $3.11189 < \tau < 14.6663$. Again using these parameter values the roots of the cubic equation (6.11) found to be -86240.1, 9.23819 and 396.586. Therefore from the relation $\tau = p - T$ we get three values of τ as -86225.1, 5.76811, and 381.586. Along these three optimal values of τ, only 5.76811 lies in the interval (3.11189, 14.6663). Hence in this case the optimal value of the tax $\tau_\delta = 5.76811$ and the corresponding stable equilibrium is $P^*(21.664, 17.5534, 17.5534, 402.145)$.



0
0
Time

Fig. 1. Solution curves corresponding to the optimal tax $\tau = 5.76811$.

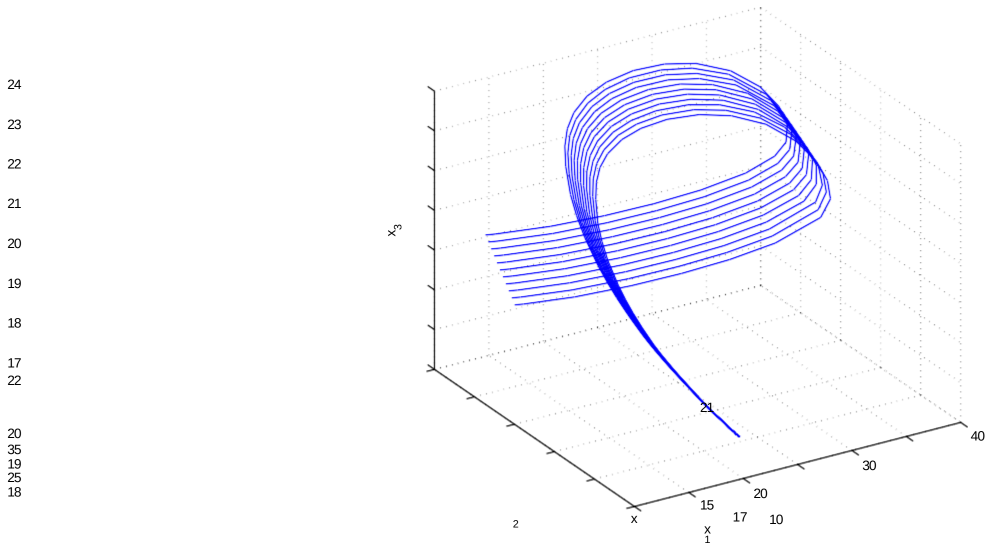


Fig. 2. Phase Space trajectories corresponding to the optimal tax τ = 5.76811 beginning with different levels of x_1, x_2 and x_3.

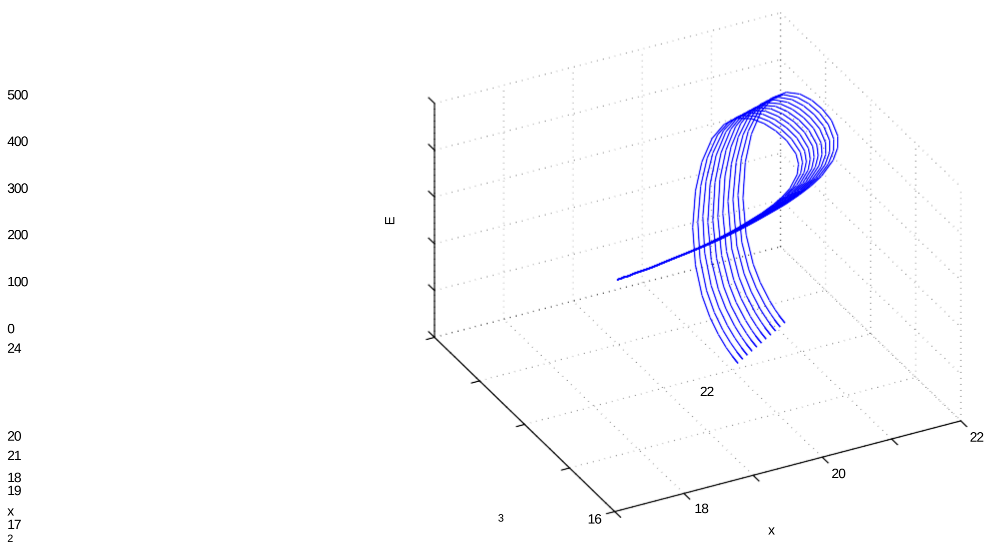


Fig. 3.Phase Space trajectories corresponding to the optimal tax τ = 5.76811 beginning with different levels of x_2, x_3 and E.

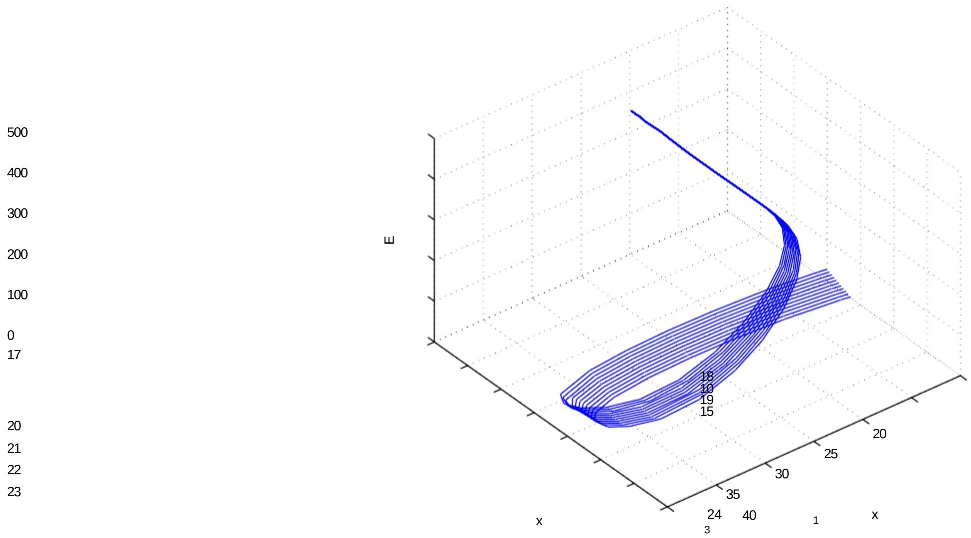


Fig. 4. Phase Space trajectories corresponding to the optimal tax $\tau = 5.76811$ beginning with different levels of x_1, x_3 and E.

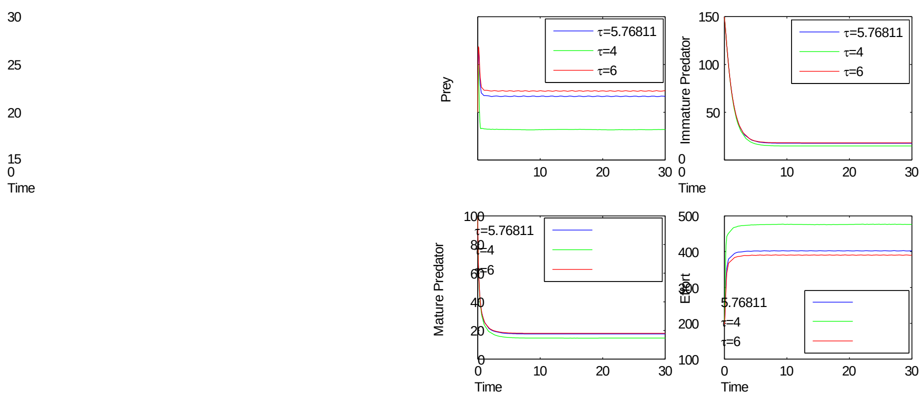


Fig. 5. Population curve against different values of τ.

8 Conclusion

In this paper, we derive and analyze the optimal tax policy within a stage-structured dynamic reaction population model, where prey is harvested in the presence of a predator. The model assumes that an external agency regulates the fishery by imposing a suitable tax per unit biomass of landed fish. The model also includes a fully dynamic interaction between fishing effort and perceived economic rent. Optimal equilibrium solution of the system is the equilibrium solution for which the present value of all future revenues from the fishing maximized which is discussed and illustrated by numerical simulation. In the Fig. 1, the solution curves are shown at optimal tax $\tau^* = 5.76811$. From the Fig. 2-4, we can conclude that the system (2.5) is globally asymptotically stable. The Fig. 5 shows that how population changes with respect to different values of tax. The existence of its steady states and their stability are then studied using eigen value analysis. With the tax rate as the regulatory instrument, it is essential to determine the optimal tax trajectories that maximize the net payoff from the fishery. We have done this with the help of Pontryagin's Maximum Principle. The numerical computations are carried out to justify the analytical result.

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