

Prime-Graph with Boolean Dynamics on Directed Networks: Periodic Orbits, Attractor Structure, and Mean-Field graphical Analysis

Dattatray Gangadhar Wani¹, Sangita Madhukar Wani² Naushad Abbas Musalman³, Sanjay Popatrao Patil⁴ Suyog Bhavsing Patil⁵ Kiran Shankar Nath⁶

¹ Assistant Teacher, Madhyamik Vidyalaya Pimparkhed, Tal. Dindori, Dist. Nashik, Maharashtra 422202, India dawa4647@gmail.com

² Assistant Teacher, Madhyamik Vidyalaya Pimparkhed, Tal. Dindori, Dist. Nashik, Maharashtra 422202, India smwani1975@gmail.com

³ Assistant Teacher, Zilla Parishad Primary School, Parmori, Tal. Dindori, Dist. Nashik, Maharashtra 422202, India naushadmusalman@gmail.com

⁴ Assistant Teacher, Punyashlok Ahilyadevi Holkar Madhyamik Vidyalaya, Anakai, Tal. Yeola, Dist. Nashik, Maharashtra, India sanjayppatil55@gmail.com

⁵ Assistant Teacher, Madhyamik Ashram School, Trimbakeshwar, Dist. Nashik, Maharashtra, India ,suyogpatil1977@gmail.com

⁶ Assistant Teacher, Shri Bhairavnath Madhyamik and Uchh Madhyamik Vidyalaya, Khed, Tal. Igatpuri, Dist. Nashik, Maharashtra, India kirannath6785@gmail.com

Abstract

We introduce a class of synchronous Boolean dynamical systems on finite loop-less directed graphs in which the evolution of each vertex is governed by a prime-count activation rule. Specifically, a vertex assumes the active state at the next iteration if and only if the number of its active incoming neighbors is a prime integer. This family of systems, referred to as Pulse Graphs, provides a new framework for investigating nonlinear discrete dynamics generated by arithmetic constraints on local interactions. The dynamics on complete directed graphs admit a complete analytical description. We show that every trajectory converges after at most three synchronous iterations to either a fixed point or a complementary two-cycle, derive exact formulas for the number of attractors, and obtain the activation probability

under independently generated random initial configurations. For sparse random directed graphs, a Poisson mean-field approximation yields a fold bifurcation at

$$k_* \approx 7.86963, \quad \mu_* \approx 3.36241,$$

which marks the emergence of bistability in the system. For arbitrary network size, we prove that the longest attainable attractor grows exponentially and satisfies

$$\frac{2 \cdot 2^{n-5} - 1}{3} \leq L(n) \leq \frac{2 \cdot 2^n - 1}{7} \quad n \geq 7.$$

Let $L(n)$ denote the maximum period of an attractor that can occur on a Pulse Graph with (n) vertices. An exhaustive search establishes the exact values

$$L(3) = L(5) = L(7) = 3, \quad L(4) = 3, \quad L(6) = 9.$$

The lower estimate is achieved by constructing a maximal-length affine feedback register whose update mechanism is implemented entirely through prime-count logic. For every $n \geq 6$, the resulting realization remains loopless, has maximum in-degree five, and contains only $(O(n))$ directed edges. These results establish Pulse Graphs as a mathematically tractable class of Boolean networks capable of exhibiting exponentially long periodic behavior together with analytically characterizable global dynamics.

1 Introduction

Boolean dynamical systems constitute an important class of discrete-time models for investigating the evolution of interacting binary variables over networked structures. In these systems, each vertex assumes one of the two states in $\{0, 1\}$, and its future state is determined by a local transition function acting on the states of neighboring vertices. Since their introduction in the pioneering work of Kauffman [Kauffman, 1969], Boolean networks have become fundamental tools in the analysis of gene regulation, distributed computation, neural systems, and complex network dynamics. The qualitative behavior of such systems is determined jointly by the topology of the underlying directed graph, the choice of local Boolean functions, and the adopted update mechanism.

The present work investigates a different class of local transition rules based on elementary number theory. Rather than comparing the number of active neighbors with a prescribed threshold, a vertex updates according to the arithmetic property of its in-coming activity count. More precisely, the next state is active whenever the number of active in-neighbors is a prime number and inactive otherwise. Because primality is neither monotone nor interval-preserving, the resulting update rule behaves fundamentally differently from conventional threshold mechanisms. For instance, activation occurs for three active inputs but disappears for four, illustrating that increasing the local input may reverse the output state. Such oscillatory behavior arising from prime counting naturally introduces nonlinear interactions into the network dynamics.

From the perspective of cellular automata and Boolean networks, the proposed update mechanism belongs to the class of totalistic rules, since the transition function depends exclusively on the total number of active in-neighbors and is independent of their individual identities. Totalistic Boolean dynamics have attracted considerable attention as generalizations of classical cellular automata and graph dynamical systems because they

provide mathematically tractable models while exhibiting a wide range of collective behaviors [Marr and Hütt, 2009, Goles et al., 2021]. Replacing threshold conditions with arithmetic constraints considerably enlarges the family of admissible transition functions and raises new questions concerning the long-term evolution of the resulting systems.

The dynamics generated by prime activation contrast sharply with those of monotone threshold networks. For synchronously updated symmetric threshold systems, every periodic orbit has length at most two [Goles and Olivos, 1980]. Comparable conclusions were obtained for synchronous bi-threshold networks, although longer cycles become possible under asynchronous updating [Kuhlman et al., 2011]. Since primality does not define a monotone Boolean function, these classical results are no longer applicable. Consequently, the maximum attainable period of a synchronous prime-activated network becomes a natural theoretical problem that has not been previously investigated.

The objective of this paper is to determine the dynamical consequences of adopting prime numbers as the sole activation criterion. Our analysis combines extremal constructions with exact dynamical characterization. For arbitrary directed graphs, we demonstrate that prime-count activation is sufficiently expressive to realize Boolean feedback architectures supporting exponentially long periodic attractors. In contrast, for complete directed graphs, the same local rule admits a complete analytical description in which every trajectory rapidly converges to a highly restricted class of attractors. This striking difference illustrates that the complexity of Pulse Graphs emerges from the interplay between an arithmetic activation rule and the structural properties of the underlying interaction network.

Definition 1. Let $G = (V, E)$ be a finite directed graph without self-loops. A Pulse Graph is the discrete-time Boolean dynamical system associated with G , where each vertex $v \in V$ possesses a binary state

$$x_t(v) \in \{0, 1\}, \quad t = 0, 1, 2, \dots$$

For every vertex v , denote its set of incoming neighbors by

$N^-(v) = \{u \in V : (u, v) \in E\}$, and define the corresponding input count at time t as

$$k_t(v) = \sum_{u \in N^-(v)} x_t(u).$$

Hence, $k_t(v)$ represents the total number of active vertices having directed edges toward v . The network evolves synchronously according to the prime-count activation function

$$x_{t+1}(v) = \begin{cases} 1, & \text{if } k_t(v) \in P, \\ 0, & \text{if } k_t(v) \notin P, \end{cases}$$

where P denotes the set of all prime numbers.

A configuration x is called a fixed point whenever it is invariant under the above evolution rule. More generally, an attractor of period p is a collection of p distinct configurations that repeat cyclically under successive synchronous updates. In particular, the case $p = 1$ corresponds to a fixed point, whereas every attractor with $p > 1$ is referred to as a nontrivial periodic attractor.

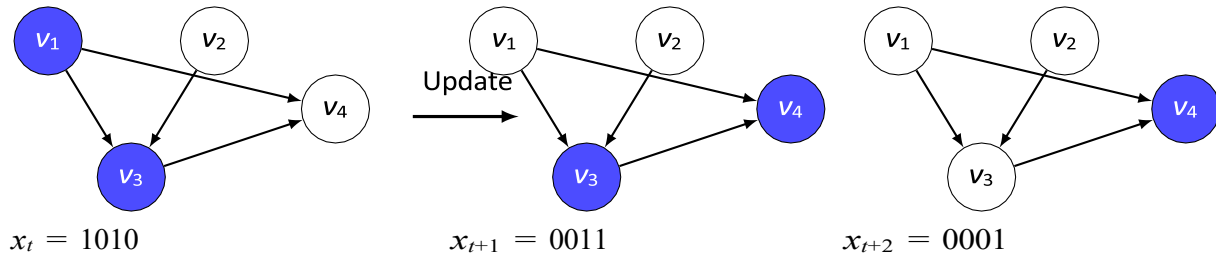


Figure 1: Illustration of synchronous evolution in a Pulse Graph. Blue vertices denote active states and white vertices denote inactive states. The network evolves from $x_t = 1010$ to $x_{t+1} = 0011$ and subsequently to $x_{t+2} = 0001$ according to the prime-neighbor activation rule.

Proposition 2 (Elementary properties of fixed configurations). *Let $G = (V, E)$ be a Pulse Graph. Then the following statements hold.*

1. *Every vertex with zero in-degree becomes inactive after one update and remains inactive thereafter.*
2. *The configuration in which every vertex is active is a fixed point precisely when the in-degree of each vertex is a prime number.*
3. *For every integer $n \geq 3$, there exists a loopless Pulse Graph with n vertices for which the all-active configuration is invariant.*
4. *Among loopless Pulse Graphs, three is the minimum number of vertices for which the all-active state can occur as a fixed point.*

Proof. Suppose that $v \in V$ has no incoming edges. Then

$$N^-(v) = \emptyset,$$

which immediately implies

$$k_t(v) = 0$$

for every time step t . Since zero is not a prime number, the update rule yields

$$x_{t+1}(v) = 0,$$

establishing the first assertion.

Now consider the configuration in which

$$x(v) = 1 \quad \text{for every } v \in V.$$

Under this configuration, each incoming neighbor contributes one active input, so that

$$k(v) = |N^-(v)|.$$

Consequently, every vertex preserves its active state if and only if its in-degree is prime. This proves the second statement.

To verify the third claim, consider the directed cycle on n vertices in which every edge is replaced by two oppositely directed edges,

$$v_i \leftrightarrow v_{i+1}, \quad i = 1, \dots, n,$$

where the indices are taken modulo n . Every vertex receives exactly two incoming edges, and because 2 is prime, the all-active configuration is unchanged after each synchronous update.

Finally, if $n = 1$, the unique vertex has in-degree zero. When $n = 2$, the absence of self-loops implies that every vertex has in-degree at most one. Since neither 0 nor 1 belongs to the set of prime numbers, the all-active configuration cannot be a fixed point in either case. Therefore, $n = 3$ is the smallest network size for which such a fixed configuration is possible.

Remark 1. *The restriction on self-loops is essential. If loops are permitted, a Pulse Graph[□] with only two vertices can admit the all-active fixed configuration by assigning to each vertex both a self-loop and an edge from the other vertex. Consequently, every vertex receives exactly two active inputs, satisfying the prime activation condition.*

2 Dynamics of Small Pulse Graphs

Before investigating asymptotic properties for arbitrary network size, we examine the behavior of Pulse Graphs containing a small number of vertices. Since the underlying state space is finite and the evolution rule is deterministic, every trajectory must eventually repeat. This permits an exhaustive computational analysis of all loopless directed graphs for $n \leq 5$, allowing the exact determination of the longest attainable periodic orbit.

2.1 Existence of Periodic Orbits

Proposition 3. *Every trajectory of a finite Pulse Graph eventually becomes periodic.*

Proof. Assume that the graph contains n vertices. Each global configuration is represented by a binary vector in

$$\{0, 1\}^n,$$

so the total number of distinct configurations equals

$$2^n.$$

Let

$$x_0, x_1, x_2, \dots$$

be the sequence generated from an arbitrary initial configuration. Since more than 2^n configurations cannot all be distinct, the Pigeonhole Principle guarantees that two states in the sequence coincide.

Because the Pulse update defines a deterministic mapping from one configuration to the next, revisiting a previous state forces the subsequent evolution to repeat indefinitely. Hence every orbit ultimately enters a periodic cycle. $\square$

2.2 Illustrative Pulse Graphs

2.2.1 A Three-State Periodic Orbit on Four Vertices

To demonstrate the emergence of nontrivial dynamics, consider the loopless directed graph with vertex set

$$V = \{v_0, v_1, v_2, v_3\},$$

whose edge set is

$$E = \{v_0 \rightarrow v_1, v_1 \rightarrow v_2, v_2 \rightarrow v_0, v_2 \rightarrow v_3, v_3 \rightarrow v_1, v_0 \rightarrow v_3, v_3 \rightarrow v_2\}.$$

This network combines a directed three-cycle with additional feedback connections, producing a periodic orbit of length three under the synchronous prime-activation rule. The example illustrates that even a small Pulse Graph can exhibit sustained cyclic behavior once the interaction pattern provides sufficient feedback between vertices.

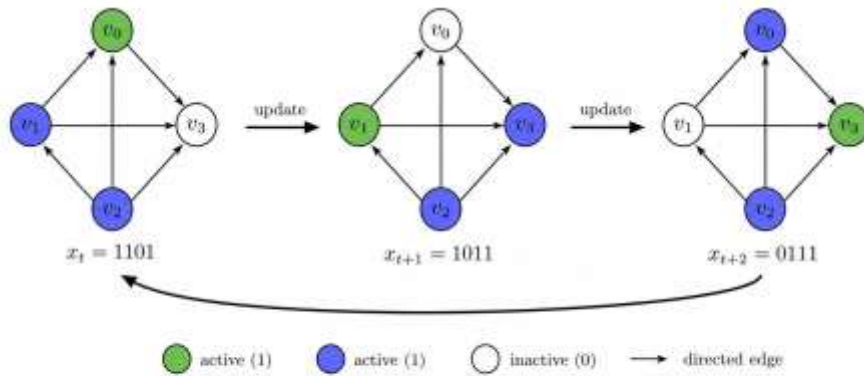


Figure 1. A period-3 attractor on a 4-vertex Pulse Graph. Green and blue vertices are active (value 1), while white vertices are inactive (value 0). The network evolves as

$$1101 \rightarrow 1011 \rightarrow 0111 \rightarrow 1101.$$

Figure 2: The full period-3 attractor on the 4-vertex Pulse Graph.

The five-vertex example exhibits the following cyclic evolution: $00011 \rightarrow 01100 \rightarrow 10011 \rightarrow 01101 \rightarrow$

10111

$\rightarrow 01111 \rightarrow 11111 \rightarrow 11110 \rightarrow 10110 \rightarrow 00011$.

Since each configuration in the above sequence appears exactly once before the initial state reoccurs, the resulting attractor has minimal period

9.

The corresponding in-degrees of the vertices

$$(v_0, v_1, v_2, v_3, v_4)$$

are

$(2, 2, 3, 3, 4)$.

Consequently, when the network reaches the configuration

$$11111,$$

the first four vertices continue to satisfy the prime activation criterion, whereas the fifth vertex receives four active inputs and therefore becomes inactive. Hence,

$$11111 \rightarrow 11110.$$

This transition highlights one of the distinguishing features of Pulse Graphs. Unlike monotone threshold dynamics, activating additional neighboring vertices does not necessarily preserve the active state. In the present example, increasing the number of active inputs from three to four causes the update value to switch from

1 to 0, because 3 is prime while 4 is composite.

Such behavior is impossible in classical monotone threshold networks, where increasing the number of active inputs cannot deactivate an already active vertex [Goles et al., 2021]. The prime-count update rule therefore introduces an intrinsically nonmonotone mechanism that significantly enriches the dynamical behavior.

2.3 Complete Enumeration for Small Networks

Definition 4. For each positive integer n , let

$$L(n)$$

denote the largest attractor period among all loopless labeled Pulse Graphs with exactly n vertices.

Proposition 5 (Maximum periods for small Pulse Graphs). The exact values of $L(n)$ for networks containing at most five vertices are given by

n	Loopless labeled directed graphs	$L(n)$
1	1	1
2	4	1
3	64	1
4	4096	3
5	1 048 576	9

Proof. For each value of $n \leq 5$, every labeled loopless directed graph was generated. Since each graph possesses

$$2^{n(n-1)}$$

possible edge configurations and

$$2^n$$

possible initial Boolean states, every trajectory was computed under synchronous Pulse updates until a repeated configuration appeared. The period of the resulting attractor was then determined.

Finally, the largest observed attractor period over the complete collection of graphs yields the corresponding value of $L(n)$ reported in the table. $\square$

The computational results presented above show that periodic behavior first arises when the network contains four vertices. Consequently, $n = 4$ represents the smallest system size capable of supporting an attractor with period greater than one.

3 Lower Bounds for Long Periodic Orbits

Theorem 6 (Evolution of the active set). *Let $S_t \subseteq V$ be the set of active vertices at time t , and let $m = |S_t|$.*

Then the Pulse update is completely determined by the primality of m and $m - 1$:

$$S_{t+1} = \begin{cases} V, & \text{if both } m \text{ and } m - 1 \text{ are prime,} \\ S_t, & \text{if } m - 1 \text{ is prime but } m \text{ is not,} \\ V \setminus S_t, & \text{if } m \text{ is prime but } m - 1 \text{ is not,} \\ \emptyset, & \text{otherwise.} \end{cases}$$

Proof. Each active vertex has exactly $m - 1$ active in-neighbours, whereas each inactive vertex has exactly m . Consequently, active vertices remain active precisely when $m - 1$ is prime, while inactive vertices switch on precisely when m is prime. Combining these two observations gives the four possible updates. $\square$

Theorem 7 (Periodic behaviour). *Every trajectory of $K^\rightarrow$ converges either to a fixed point or to a complementary two-cycle*

$$S \longleftrightarrow V \setminus S.$$

Proof. The previous theorem shows that every state can only evolve into one of

$$\emptyset, \quad S, \quad V \setminus S, \quad V.$$

Hence no periodic orbit longer than two is possible.

The fixed points are:

1. the empty configuration;
2. the all-active configuration whenever $n - 1$ is prime;
3. every proper subset $S \subsetneq V$ satisfying

$$|S| - 1 \text{ is prime,} \quad |S| \text{ is not prime.}$$

A complementary pair

$$S \leftrightarrow V \setminus S$$

forms a two-cycle exactly when both subset sizes are prime and neither equals 3. The exception occurs because $3 - 1 = 2$ is also prime, preventing the required state reversal. $\square$

Theorem 8 (Linear lower bound for maximal attractor periods). *For every integer $n \geq 5$,*

$$L(n) \geq n - 3.$$

Consequently, the family of Pulse Graphs admits periodic attractors of arbitrarily large length. In particular, there does not exist a universal constant M such that every finite Pulse Graph satisfies

$$L(n) \leq M.$$

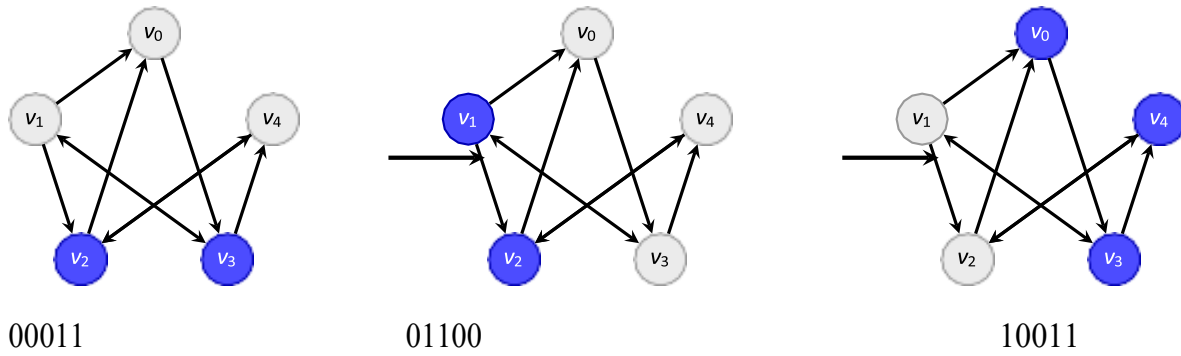


Figure 3: First three configurations of a period-9 orbit in a five-vertex Pulse Graph. Blue circles indicate active vertices, while light-gray circles denote inactive vertices. Bidirectional arrows represent pairs of opposite directed edges. The remaining configurations complete the periodic cycle before returning to the initial state.

Proof. Fix an integer $n \geq 5$ and define

$\ell = n - 3$, which represents the number of signal vertices.

Construct a Pulse Graph consisting of two components. The first component contains three control vertices c_0, c_1, c_2 ,

with directed edges in both directions between every pair. Consequently, each control vertex has exactly two incoming neighbors.

The second component is formed by the directed cycle

$$s_0 \rightarrow s_1 \rightarrow \cdots \rightarrow s_{\ell-1} \rightarrow s_0.$$

In addition, introduce the directed edges

$$c_0 \rightarrow s_i, \quad 0 \leq i < \ell,$$

so that every signal vertex receives one additional input from the control vertex c_0 .

Consider the initial configuration in which the three control vertices are active together with the single signal vertex s_0 , while all remaining signal vertices are inactive.

Since every control vertex receives exactly two active inputs from the other two control vertices, and 2 is prime, their states remain unchanged throughout the evolution. Hence the control subsystem acts as a permanent source of activation.

For each signal vertex s_i , the incoming inputs consist of the fixed contribution from c_0 together with the state of the predecessor s_{i-1} on the directed cycle. Therefore,

$$k_t(s_i) = 1 + x_t(s_{i-1}),$$

where the indices are interpreted modulo ℓ .

Because

$$1 \notin P \quad \text{and} \quad 2 \in P,$$

the update rule simplifies to

$$x_{t+1}(s_i) = x_t(s_{i-1}).$$

Thus, at every synchronous update, the active signal advances by one position around the directed cycle.

Starting from the configuration

$$10 \cdots 0,$$

the signal completes one full revolution after exactly ℓ updates before returning to its initial position. Consequently, the constructed Pulse Graph possesses a periodic orbit of length

$$\ell = n - 3.$$

Therefore,

$$L(n) \geq \ell = n - 3,$$

which establishes the stated lower bound. Since ℓ increases without bound as n grows, the maximal attractor period cannot be bounded by any universal constant. $\square$

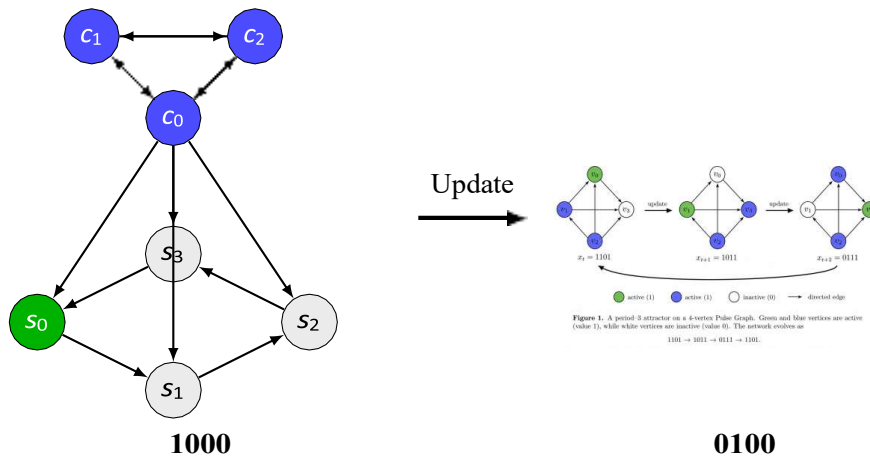


Figure 4: Rotation of the active signal in the register construction for $\ell = 4$. The three control vertices remain permanently active, while the single active signal advances by one position at each synchronous update.

3.1 Boolean Operations Induced by Prime Activation

The prime-count update rule is sufficiently expressive to implement a variety of elementary Boolean operations by introducing a suitable number of permanently active auxiliary vertices. Consequently, logical computation can be embedded directly into the dynamics of Pulse Graphs without modifying the underlying update mechanism.

Proposition 9 (Boolean gates generated by prime activation). *Assume that a vertex has two Boolean inputs*

$$x, y \in \{0, 1\},$$

together with q additional incoming edges from vertices whose states remain permanently equal to 1. The corresponding update function is

$$F_q(x, y) = \begin{cases} 1, & \text{if } q + x + y \in P, \\ 0, & \text{otherwise,} \end{cases}$$

where P denotes the set of prime numbers.

For $q = 0, 1, \dots, 4$, the resulting Boolean functions are summarized below:

q	$F_q(0, 0)$	$F_q(0, 1) = F_q(1, 0)$	$F_q(1, 1)$	Equivalent Boolean gate
0	0	0	1	$x \wedge y$
1	0	1	1	$x \vee y$
2	1	1	0	NAND(x, y)
3	1	0	1	XNOR(x, y)
4	0	1	0	$x \oplus y$

Furthermore, if only one variable input is present, then one permanently active input realizes the identity function, whereas three permanently active inputs produce logical negation.

Proof. The value of the update function depends exclusively on the integer

$$q + x + y.$$

Since

$$x + y \in \{0, 1, 2\},$$

evaluating the primality of these three possible sums immediately determines the entries of the table for each choice of q .

Now suppose that the vertex has a single variable input x . If one auxiliary vertex is permanently active, then the possible input counts are

$$1 + x \in \{1, 2\}.$$

Because $1 \notin P$ and $2 \in P$, the output coincides with x .

Similarly, with three permanently active inputs,

$$3 + x \in \{3, 4\}.$$

Since 3 is prime whereas 4 is composite, the resulting output is

$$\neg x.$$

This completes the proof. □

Remark 2. A complete bidirected graph on three vertices provides a simple mechanism for maintaining a constant logical signal. When initialized in the all-active configuration, each control vertex receives exactly two active incoming neighbors and therefore remains permanently active under the prime-count update rule. These persistent control vertices can serve as auxiliary inputs for the Boolean constructions established in Proposition 9. In particular, the availability of the NAND operation implies that Pulse Graphs are capable of realizing any feedforward Boolean circuit with fixed external inputs.

The register construction presented earlier establishes that the maximum attractor period grows without bound. The next result strengthens this observation considerably by demonstrating that prime-count dynamics can generate attractors whose lengths increase exponentially with the number of vertices.

3.2 Exponentially Long Periodic Orbits

Linear feedback shift registers generated from primitive polynomials have long been used to produce sequences of maximal period [Zierler, 1959]. More recently, analogous finite-field techniques have been employed to construct bounded-degree automata networks exhibiting one fixed point together with a cycle containing every remaining state [Aracena et al., 2026]. In the present setting, however, the local update function is fixed by the prime-count activation rule, so the desired feedback mechanism must be implemented entirely within this common dynamical framework.

The rotating-register construction yields attractors of linear length. By incorporating affine feedback realized through the prime-count logic gates, one obtains a substantially stronger result: a single orbit traverses every nonzero register configuration.

Theorem 10 (Exponential attractor periods). *For every $n \geq 5$,*

$$2^{n-3} - 1 \leq L(n) \leq 2^n - 1.$$

Consequently,

$$L(n) = \lambda(2^n).$$

Moreover, whenever $n \geq 6$, the lower bound is achieved by a loopless Pulse Graph whose maximum in-degree is at most five and whose total number of directed edges does not exceed

$$5n - 15.$$

Proof. The upper bound follows immediately from the finiteness of the state space. Every Pulse Graph on n vertices possesses exactly 2^n configurations. Since the all-zero configuration maps to itself, it forms a fixed point for every graph. Therefore no periodic orbit can contain more than the remaining

$$2^n - 1$$

states, implying

$$L(n) \leq 2^n - 1.$$

To establish the lower bound, first observe that the exhaustive enumeration of Section 3 gives

$$L(5) = 9,$$

and hence

$$9 > 2^{5-3} - 1.$$

Assume therefore that and

$$n \geq 6, m = n - 3.$$

define

The construction is carried out over the finite field

$$\mathbb{F}_2.$$

A theorem of Cohen [Cohen, 1990] guarantees that, for every

$$m \geq 3,$$

there exists a primitive element

$$\alpha \in \mathbb{F}_{2^m}$$

satisfying

$$\text{Tr}_{\mathbb{F}_{2^m}/\mathbb{F}_2}(\alpha) = 0.$$

Let

$$p(z) = z^m + a_{m-1}z^{m-1} + \cdots + a_1z + 1$$

denote the minimal polynomial of α . Since α is primitive, $p(z)$ is a primitive polynomial of degree m . Furthermore, over a field of characteristic two, the coefficient of z^{m-1} equals the field trace. Consequently,

$$a_{m-1} = 0,$$

which ensures that the resulting register construction contains no self-loop at its final stage.

Next define the companion transformation

$$A : \mathbb{F}_2^m \rightarrow \mathbb{F}_2^m$$

by

$$(Ax)_0 = x_{m-1},$$

and

$$(Ax)_i = x_{i-1} + a_i x_{m-1}, \quad 1 \leq i < m.$$

The characteristic polynomial of A is precisely $p(z)$. Because $p(z)$ is primitive, the linear transformation has multiplicative order

$$2^m - 1.$$

Equivalently, after identifying

$$\mathbb{F}_2^m \cong \mathbb{F}_{2^m},$$

the map A corresponds to multiplication by the primitive element α . Hence the zero vector remains fixed, whereas every nonzero vector belongs to a single cycle of length

$$2^m - 1.$$

Example 11 (A three-stage register). *When*

$$m = 3,$$

the primitive polynomial

$$p(z) = z^3 + z + 1$$

has coefficients

$$a_1 = 1, \quad a_2 = 0.$$

The corresponding linear feedback register satisfies

$$y'_0 = y_2, \quad y'_1 = y_0 + y_2, \quad y'_2 = y_1,$$

and generates the maximal cycle

$$100 \rightarrow 010 \rightarrow 001 \rightarrow 110 \rightarrow 011 \rightarrow 111 \rightarrow 101 \rightarrow 100.$$

Within a Pulse Graph, the XOR feedback is replaced by the affine XNOR gate, producing

$$x'_0 = x_2, \quad x'_1 = 1 + x_0 + x_2, \quad x'_2 = x_1.$$

This affine transformation is conjugate to the original linear register and therefore preserves the maximal period.

Figure 5 illustrates the graph-theoretic realization of the affine feedback register employed in the remainder of the proof.

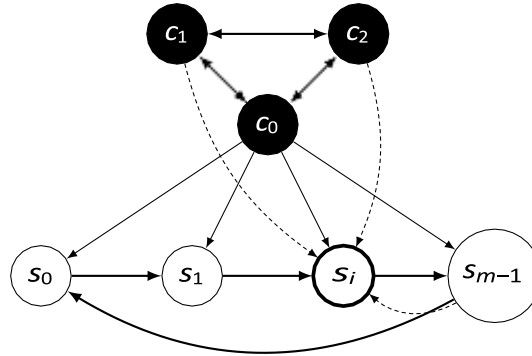


Figure 5: Compact schematic of the affine feedback register. The control triangle maintains permanent activation. Each signal vertex receives a bias from c_0 , while the tapped stage additionally receives feedback implementing the affine XNOR operation.

The exponential construction retains the permanent control module introduced in the rotating-register model. The only additional ingredient is the introduction of feedback at selected register stages, allowing the prime-count update rule to realize affine XNOR operations.

To construct the required network, consider three control vertices

$$c_0, c_1, c_2,$$

together with the signal vertices

$$s_0, s_1, \dots, s_{m-1}.$$

Each pair of control vertices is connected by directed edges in both directions, and the initial configuration activates all three controls. Consequently, every control vertex always receives exactly two active incoming neighbors. Since 2 is prime, the control triangle remains permanently active throughout the dynamics.

The first register stage s_0 receives two incoming edges, namely from c_0 and s_{m-1} .

Therefore,

$$x_{t+1}(s_0) = x_t(s_{m-1}).$$

For each register stage satisfying

$$1 \leq i < m,$$

the construction depends on the coefficient a_i .

If

$$a_i = 0,$$

the vertex s_i receives inputs only from c_0 and the preceding register stage s_{i-1} . Hence,

$$x_{t+1}(s_i) = x_t(s_{i-1}).$$

When

$$a_i = 1,$$

three additional incoming edges are introduced from the permanently active vertices c_1, c_2 and from the feedback vertex s_{m-1} . In this case the total number of active inputs is

$$3 + x_t(s_{i-1}) + x_t(s_{m-1}).$$

Since 3 and 5 are prime whereas 4 is composite, the update rule becomes

$$x_{t+1}(s_i) = 1 + x_t(s_{i-1}) + x_t(s_{m-1}) \quad \text{in } \mathbb{F}_2.$$

Thus every tapped register stage performs an affine XNOR operation. Because Co-hen's trace condition guarantees

$$a_{m-1} = 0,$$

the last register vertex is never tapped, eliminating the need for a self-loop. Collecting the signal variables into the vector

$$x = (x(s_0), \dots, x(s_{m-1}))^T,$$

the complete signal dynamics can be written as

$$x_{t+1} = T(x_t) = Ax_t + b,$$

where

$$b = (0, a_1, \dots, a_{m-1})^T.$$

Since the polynomial p is primitive and

$$m \geq 3,$$

the value 1 cannot be a root of p . Consequently,

$$\det(I - A) = p(1) \neq 0,$$

which shows that the matrix $I - A$ is nonsingular. Hence there exists a unique vector

$$z \in \mathbb{F}^m$$

satisfying

$$(I - A)z = b.$$

Introduce the translated variables

$$y_t = x_t + z.$$

Using the identity

$$Az + b = z,$$

we obtain

$$\begin{aligned} y_{t+1} &= x_{t+1} + z \\ &= A(y_t + z) + b + z \\ &= Ay_t. \end{aligned}$$

Therefore the affine transformation T is conjugate, via translation, to the linear transformation A . Because A fixes the zero vector and permutes every nonzero vector in a single cycle of length

$$2^m - 1,$$

the same orbit structure is inherited by T . Accordingly, once the control vertices are fixed, the Pulse Graph contains one fixed point together with a periodic orbit of length

$$2^m - 1 = 2^{n-3} - 1,$$

establishing the required lower bound.

It remains to estimate the graph size. Every control vertex has in-degree two, whereas each signal vertex has in-degree either two or five, depending on whether it is tapped.

Define

$$r = |\{i : 1 \leq i < m, a_i = 1\}|.$$

Since

$$a_{m-1} = 0,$$

we have

$$r \leq m - 2.$$

The construction contains six directed edges inside the control triangle, m bias edges from c_0 , another m register edges, and $3r$ additional feedback edges. Therefore,

$$\begin{aligned} 6 + m + m + 3r &\leq 6 + 2m + 3(m - 2) \\ &= 5m \\ &= 5n - 15. \end{aligned}$$

Finally,

$$2^{n-3} - 1 \geq 2^{n-4} = \frac{1}{16} 2^n, \quad n \geq 5.$$

Combining this estimate with the previously established upper bound yields

$$L(n) = \lambda(2^n),$$

which completes the proof. $\square$

4 Dynamics on Complete Directed Graphs

Throughout this section, let $K^{\rightarrow}$ denote the loopless complete directed graph on n vertices. Every ordered pair of distinct vertices is connected by a directed edge.

Definition 12. For an initial configuration $S_0 \subseteq V$, let

$$S_0, S_1, S_2, \dots$$

be the corresponding Pulse trajectory.

The transient length, denoted by $t(S_0)$, is the smallest integer for which $S_{t(S_0)}$ belongs to the eventual periodic orbit.

The largest possible transient is

$$T(n) = \max_{S_0 \subseteq V} t(S_0).$$

Theorem 13 (Uniform transient bound). For every complete directed Pulse Graph,

$$T(n) \leq 3.$$

Hence the convergence time satisfies

$$T(n) = O(1).$$

Proof. Let the initial active set be S_0 . By Theorem 6.1,

$$S_1 \in \{\emptyset, S_0, V \setminus S_0, V\}.$$

If $S_1 = \emptyset$, the trajectory has already reached the fixed empty state. If $S_1 = S_0$, then S_0 itself is a fixed point.

If $S_1 = V$, then either V is fixed or it immediately maps to $\emptyset$. Thus an attractor is reached after at most two updates.

Finally, suppose

$$S_1 = V \setminus S_0.$$

Applying the update rule once more gives

$$S_2 \in \{\emptyset, V \setminus S_0, S_0, V\}.$$

The first two possibilities produce fixed points, the third yields the complementary two-cycle

$$S_0 \leftrightarrow V \setminus S_0,$$

and the last reaches either the fixed state V or the fixed empty state after one additional step.

Therefore every orbit reaches its eventual attractor within at most three iterations, establishing

$$T(n) \leq 3.$$

□

5 Complete Directed Pulse Graphs

Let $K^{\rightarrow}$ denote the complete directed graph on n vertices without self-loops. Thus every ordered pair of distinct vertices is connected by a directed edge.

Theorem 14 (Update rule on $K^{\rightarrow}$). Let $S_t \subseteq V$ be the active set at time t , and write

$$m = |S_t|.$$

Then

$$S_{t+1} = \begin{cases} V, & \text{if } m-1 \text{ and } m \text{ are prime,} \\ S_t, & \text{if } m-1 \text{ is prime and } m \text{ is not,} \\ V \setminus S_t, & \text{if } m-1 \text{ is not prime and } m \text{ is prime,} \\ \emptyset, & \text{otherwise.} \end{cases}$$

Proof. Every active vertex has exactly $m-1$ active in-neighbors, while every inactive vertex has exactly m . Since activation depends only on whether these counts are prime, the four cases follow immediately. $\square$

The complete graph therefore reduces the dynamics to a function of the number of active vertices.

Theorem 15. Every orbit of $K^{\rightarrow}$ eventually reaches either a fixed point or a complement two-cycle

$$S \longleftrightarrow V \setminus S.$$

Proof. From the update formula, every state can only map to one of

$$\emptyset, S, V \setminus S, V.$$

Hence no periodic orbit longer than two is possible.

The empty state is always fixed. The all-ones state is fixed precisely when $n-1$ is prime. A proper subset S is fixed exactly when

$$|S|-1$$

is prime but

$$|S|$$

is not. Complement two-cycles occur exactly when both

$$|S| \quad \text{and} \quad |V \setminus S|$$

are prime (excluding the exceptional case of size 3). $\square$

Definition 16. For an initial state S_0 , let $t(S_0)$ denote the number of updates required before the orbit first reaches its eventual periodic attractor. Define

$$T(n) = \max_{S_0 \subseteq V} t(S_0),$$

the maximum transient time over all initial states.

Theorem 17. For every n ,

$$T(n) \leq 3.$$

Hence the transient time of complete Pulse Graphs is bounded by a constant,

$$T(n) = O(1).$$

Proof. By the update formula,

$$S_1 \in \{\emptyset, S_0, V \setminus S_0, V\}.$$

If $S_1 = \emptyset$ or $S_1 = S_0$, the orbit has already reached a fixed point.

If $S_1 = V$, then either V is fixed or it maps once to $\emptyset$, so an attractor is reached within two steps.

Finally, if $S_1 = V \setminus S_0$, applying the update once more again yields one of

$$\{\emptyset, V \setminus S_0, S_0, V\}.$$

The first two cases give fixed points, the third gives the two-cycle

$$S_0 \leftrightarrow V \setminus S_0,$$

and the last reaches a fixed point after at most one additional update.

Thus every trajectory enters its eventual attractor after no more than three updates. □

6 Results and Discussion

The prime-count update rule produces markedly different behaviors depending on the underlying graph structure. Exhaustive enumeration shows that nontrivial periodic at-tractors first occur on four vertices, while a period-9 attractor already appears on five vertices. These examples demonstrate that a simple arithmetic activation rule can generate rich finite-state dynamics.

On general directed graphs, Pulse Graphs support information propagation, Boolean logic, and feedback mechanisms. The rotating-register construction gives attractors of linear length, while the affine feedback construction yields exponential periods of order $\lambda(2^n)$. Thus the dynamical complexity of Pulse Graphs grows exponentially with the number of vertices.

In contrast, the complete directed graph exhibits highly constrained behavior. Every trajectory reaches either a fixed point or a complement two-cycle after at most three updates. This shows that the complexity of Pulse Graphs depends not only on the prime activation rule but also on the underlying network topology.

References

Julio Aracena, Florian Bridoux, Maximilien Gadouleau, Pierre Guillon, K  vin Perrot, Adrien Richard, and Guillaume Theyssier. On the dynamics of bounded-degree automata networks. *Natural Computing*, 25(1):6, 2026. doi: 10.1007/s11047-026-10064-x.

Stephen D. Cohen. Primitive elements and polynomials with arbitrary trace. *Discrete Mathematics*, 83(1):1–7, 1990. doi: 10.1016/0012-365X(90)90215-4.

Eric Goles and Jorge Olivos. Periodic behaviour of generalized threshold functions. *Dis-crete Mathematics*, 30(2):187–189, 1980. doi: 10.1016/0012-365X(80)90121-1.

Eric Goles, Andrew Adamatzky, Pedro Montealegre, and Mart'ın R'ıos-Wilson. Gen-erating boolean functions on totalistic automata networks. *International Journal of Unconventional Computing*, 16(4):343–391, 2021.

Stuart A. Kauffman. Metabolic stability and epigenesis in randomly constructed ge-netic nets. *Journal of Theoretical Biology*, 22(3):437–467, 1969. doi: 10.1016/0022-5193(69)90015-0.

Chris J. Kuhlman, Henning S. Mortveit, David Murrugarra, and V. S. Anil Ku-mar. Bifurcations in boolean networks. *DMTCS Proceedings*, AP, 2011. doi: 10.46298/dmtcs.2975.

Carsten Marr and Marc-Thorsten Hütt. Outer-totalistic cellular automata on graphs. *Physics Letters A*, 373(5):546–549, 2009. doi: 10.1016/j.physleta.2008.12.013.

Neal Zierler. Linear recurring sequences. *Journal of the Society for Industrial and Applied Mathematics*, 7(1):31–48, 1959. doi: 10.1137/0107003.