International Journal of Scientific Research in Engineering and Management (IJSREM)
Volume: 08 Issue:12 |Dec-2024 SJIF Rating: 8.448 ISSN: 2582-3930

Properties of Homomorphisms and Centers in Multiset Rings.

Moloy Dutta

Research Scholar

Department of Mathematics

T.M. Bhagalpur University, Bhagalpur

Abstract

Multiset rings (*M*rings) form an important generalization of classical rings by allowing elements to appear with finite frequencies and frequency function satisfies some conditions. In non-commutative *M*rings, the center (called Madhya, denoted by $\mathcal{M}$(Y), where Y is an *M*ring) consists of those elements that frequency-commute with every element of the ring with respect to multiplication. This paper studies the algebraic behavior of the center of an *M*ring constructed over a ring. We prove that if Y is a non-empty *M*ring over a ring (S, +, ·) which is also form a field and satisfies the natural condition that every non-zero element has the same frequency as its multiplicative inverse, i.e., $Ƒ_Y(u) = Ƒ_Y(u^{-1})$ for all $u \neq 0$, then the center $\boldsymbol{\mathcal{M}}$(Y) itself becomes a sub-*M*ring of Y. This result shows that the center inherits the full *M*ring structure in the presence of multiplicative inverses and frequency symmetry. Additionally, we establish a homomorphism theorem for *M*ideals and centers: if ɧ: (S, ⊕, ⊙) → (T, +, ·) is a bijective *M*ring homomorphism, then the pre-image of every *M*ideal of T is an *M*ideal of S, and the centers satisfy a natural correspondence, highlighting when the center coincides with the whole *M*ring and when it is properly smaller. These results extend classical center and correspondence theorems to the multiset setting and open the door to further investigation of representation theory, idempotent structures, and applications in coding theory and cryptographic protocols based on multiplicity-aware algebraic systems.

Keywords: *Multiset ring (Mring), Mideal, Madhya and Ring homomorphism etc.*

Introduction:

The classical theory of rings has been successfully generalized in various directions to capture uncertainties, multiplicities, and fuzzy behaviors. One such significant extension is the notion of multisets (*M*sets), where an element is allowed to occur more than once with a certain multiplicity or frequency. By assigning a counting function (frequency function) Ƒ to each element of a ring, several authors introduced the concept of multiset rings (*M*rings) and studied their algebraic properties analogous to crisp rings.

In an *M*ring Y [which is an *M*set] over a ring (S, ⊕, ⊙), the frequency values satisfy the following three conditions for all u,v∈ S:

1.$Ƒ_Y(u \oplus v) \geq \min\{Ƒ_Y(u), Ƒ_Y(v)\}$,

2.$Ƒ_Y(u \odot v) \geq \min\{Ƒ_Y(u), Ƒ_Y(v)\}$.

3.$Ƒ_Y(-u) = Ƒ_Y(u)$ for all $u \in S$.

This structure preserves many ring-like properties while accommodating repeated occurrences of elements. A natural and important substructure that arises in non-commutative *M*rings is the center of an *M*ring, it can also be called the Madhya (denoted by $\boldsymbol{\mathcal{M}}$(Y)), defined as the multiset of all elements that commute with every element of the underlying ring with respect to the frequency of multiplication, i.e.,

$\mathcal{M}(Y) = \{u \in S \mid Ƒ_Y(u \odot v) = Ƒ_Y(v \odot u)\ \forall\ v \in S\}$,

with frequency $Ƒ_{\mathcal{M}(Y))}(u) = Ƒ_Y(u)$

When the underlying ring is commutative, the center coincides with the *M*ring itself, but in the general case, the center plays a role analogous to the classical center of a ring.

The study of ideals in *M*rings (called *M*ideals) and their behavior under ring homomorphisms has already shown that *M*ideals are preserved in the reverse direction under bijective homomorphisms (preimage of an *M*ideal is an *M*ideal). However, little attention has been paid to how the center behaves under homomorphisms and in special classes of rings such as fields, where every non-zero element possesses a multiplicative inverse.

This paper investigates the structural properties of the center (Madhya) of multiset rings constructed over fields. In particular, we explore the condition $Ƒ_Y(u) = Ƒ_Y(u^{-1})$ for all non-zero $u \in S$, which is natural in many counting or weighting scenarios on fields (e.g., finite fields, functional graphs, or incidence structures).Under this mild and interpretable hypothesis, we prove that the $\mathcal{M}(\Upsilon)$ itself forms an *M*ring over the same field, thereby becoming a sub-*M*ring with rich algebraic properties.

Furthermore, we establish homomorphism theorems for centers, showing that bijective *M*ring homomorphisms preserve the *M*ideal property in the inverse image and induce natural relationships between the centers of domain and codomain *M*rings. Concrete examples over the power set ring of a two-element set (which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$) and over $\mathbb{Z}_4$ (a commutative ring that is not a field) are provided to illustrate both commutative and non-commutative phenomena in frequency tables.

The results presented here not only generalize classical center and homomorphism theorems to the multiset environment but also lay the foundation for further study of representation theory, module theory over *M*rings, and applications in coding theory and cryptography where multiplicity and symmetry of elements are crucial.

Definition 1.1: *M*set [8]:

Let S be a classical set. A multiset or *M*set Υ is a combination of members of S and a pre-defined frequency function F_{Υ}, where F_{Υ}: S→**N** U {0}, where $F_{\Upsilon}(u)$ is the frequency of u in Υ.

Example 1.1:

If S= $\{u_1, u_2, u_3, \ldots\ldots u_n\}$ and Υ= {<2,u_1>,<5,u_2>, <4,u_3>}

Then Υ is an *M*set over S.

Here 2, 5, 4 are number of repetitions of u_1, u_2, u_3 respectively.

we can write $u_1 \in^2 \Upsilon$, $u_2 \in^5 \Upsilon$, $u_3 \in^4 \Upsilon$.

Definition 1.2: Sub *M*set [8]:

Let Υ_1 and Υ_2 are two *M*sets from S. Then Υ_1 is called sub *M*set of Υ_2, denoted by $\Upsilon_1 \subseteq \Upsilon_2$, if

$F_{\Upsilon_2}(u) \geq F_{\Upsilon_1}(u)\ \forall\ u \in S$,

And Υ_1 is called proper sub *M*set of Υ_2, denoted by $\Upsilon_1 \subset \Upsilon_2$ if,

$F_{\Upsilon_2}(u) \geq F_{\Upsilon_1}(u)\ \forall\ u \in S$ and there exist at least one u ∈ S such that $F_{\Upsilon_2}(u) > F_{\Upsilon_1}(u)$.

Example 1.2:

Let S = {a, b, c, d} and Υ_1 = {<2,a>,<3,b>,<4,d>},

Υ_2 = {<1,a>,<3,b>,<2,d>}.

Clearly, Υ_2 is a sub *M*set of Υ_1.

i.e. $\Upsilon_2 \subseteq \Upsilon_1$, more accurately $\Upsilon_2 \subset \Upsilon_1$.

Definition 1.3: Multiset Rings or *M*ring[7]:

Let (S, +,.) be a Ring and Υ be an *M*set taken from S. Then Υ is an *M*set Ring or *M*ring if the frequency function satisfies following conditions:

1. $F_{\Upsilon}(u+v) \geq min\ \{F_{\Upsilon}(u), F_{\Upsilon}(v); u, v \in S\}$.
2. $F_{\Upsilon}(u.v) \geq min\ \{F_{\Upsilon}(u), F_{\Upsilon}(v)\ ; u, v \in S\}$.
3. $F_{\Upsilon}(-u) = F_{\Upsilon}(u)\ \forall\ u \in \mho$,

Example 1.3:

Let A= {1,2}, and S= Power set of A,

i.e., S= {Φ, {1}, {2}, {1,2}}

and operation taken here symmetric difference as addition and intersection as multiplication.

Here, clearly S is a Ring with respect to operation symmetric difference as addition and intersection as multiplication.

Now let,

Υ= {<3,Φ>,<2,{1}>,<2,{2}>,<3,{1,2}>}.

Now frequency addition operation table is shown below:

Δ	Φ 3	{1} 2	{2} 2	{1,2} 3
Φ 3	Φ 3	{1} 2	{2} 2	{1,2} 3
{1} 2	{1} 2	Φ 3	{1,2} 3	{2} 2
{2} 2	{2} 2	{1,2} 3	Φ 3	{1} 2
{1,2} 3	{1,2} 3	{2} 2	{1} 2	Φ 3

Here addition inverse of each element is its own.

Also, Frequency multiplication operation table is shown below,

∩	Φ 3	{1} 2	{2} 2	{1,2} 3
Φ 3	Φ 3	Φ 3	Φ 3	Φ 3
{1} 2	Φ 3	{1} 2	Φ 3	{1} 2
{2} 2	Φ 3	Φ 3	{2} 2	{2} 2
{1,2} 3	Φ 3	{1} 2	{2} 2	{1,2} 3

Here frequency of each element in each cell in both cases is greater than or equal to minimum frequency of corresponding two elements from where it comes.

So, it is clear from frequency operation table that frequency function satisfies all conditions to form an *M*ring.

So Y is an *M*ring.

Definition 1.4: Commutative *M*ring [9]:

Let Y be an *M*ring taken from a ring S. Then Y is called commutative *M*ring if and only if, $Ƒ_Y(u.v) = Ƒ_Y(v.u)$ for all u,v Є S.

If any frequency multiplication table of an *M*ring is symmetric with respect to frequency then we will say that this is an example of commutative *M*ring.

Example 1.4:

Since in previous example,

frequency multiplication table is symmetric with respect to frequency, so this is an example of a commutative *M*ring.

Definition 1.5 [10]:

(a) Let U and V be two nonempty sets and ϕ: U→V be a mapping and suppose Y Є $[U]^m$ then ϕ(Y) is an *M*set where frequency function of ϕ(Y) defined as follows:

for any v Є V,

$Ƒ_{\phi(Y)}(v) = Max\ \{Ƒ_Y(u);$ if $\phi^{-1}(v) \neq \phi$ and u Є U such that, ϕ(u)=v}

= 0; otherwise.

(b) If ʙ Є $[V]^w$ then ϕ^{-1}(ʙ) is an *M*set, where frequency function of $\phi^{-1}(B)$ defined as follows,

for any u Є U, $\digamma_{\phi^{-1}(B)}(u) = \digamma_B[\phi(u)]$.

Definition 1.6: Multiset Ideal [7]:

Let (S,+, .) be a Ring and ***I*** Є *MR*[S], then ***I*** is said to be a left multiset ideal if for each k Є S, $\digamma_I(k.u) \geq \digamma_I(u)$ ∀ u Є S, and ***I*** is said to be a right multiset ideal if,

$\digamma_I(u.k) \geq \digamma_I(u)$ ∀ u Є S.

If both the conditions of left and right hold then,

It is called a multiset ideal or an *M*set ideal or an *M*ideal.

Main Results & Definition:

Theorem 2.1: Suppose (S,⊕,⊙) and (T,+,.) are two ring and a function ɧ:S→T be a one-one onto homomorphism, if *I* is an *M*ideal over T then $ɧ^{-1}(I)$ is also *M*ideal over S *i.e.* if *I* Є *MI*(T) then $ɧ^{-1}(I)$ Є *MI*(S).

Proof:

Now for all u,v Є S, $\digamma_{ɧ^{-1}(I)}(u - v)$

$= \digamma_I(ɧ(u \oplus (-v)))$

$= \digamma_I(ɧ(u) + ɧ(-v))$

$\geq min\ \{\digamma_I(ɧ(u)), \digamma_I(ɧ(-v))\}$.

$= min\ \{\digamma_I(ɧ(u)), \digamma_I(-ɧ(v))\}$.

$= min\ \{\digamma_I(ɧ(u)), \digamma_I(ɧ(v))\}$.

$= min\ \{\digamma_{ɧ^{-1}(I)}(u), \digamma_{ɧ^{-1}(I)}(v)\}$.

Now, $\digamma_{ɧ^{-1}(I)}(u \odot v) = \digamma_I(ɧ(u \odot v))$

$= \digamma_I(ɧ(u).ɧ(v))$

$\geq max\ \{\digamma_I(ɧ(u)), \digamma_I(ɧ(v))\}$

$= max\ \{\digamma_{ɧ^{-1}(I)}(u), \digamma_{ɧ^{-1}(I)}(v)\}$.

And $\digamma_{ɧ^{-1}(I)}(-u) = \digamma_I(ɧ(-u)) = \digamma_I(-ɧ(u))$

$= \digamma_I(ɧ(u))$

$= \digamma_{ɧ^{-1}(I)}(u)$, for all u ЄS.

Therefore, if *I* Є MI(T) then $ɧ^{-1}(I)$ Є *MI*(S).

Hence proved.

Definition 2.1: Madhya of an *M*ring:

Let **(S,+,.)** be a ring and ϒ be an *M*ring taken from S.

Then Madhya of ϒ is denoted by $\mathcal{M}(ϒ)$ is an *M*set such that,

$\digamma_{\mathcal{M}(ϒ)}(u) = \digamma_ϒ(u)$, where u Є S and $\digamma_ϒ(u.v) = \digamma_ϒ(v.u)$ ∀ v Є S.

$\digamma_{\mathcal{M}(ϒ)}(u) = 0$, where u Є S and $\digamma_ϒ(u.v) \neq \digamma_ϒ(v.u)$ for some v Є S.

Example 2.1:

If A= {1,2},

and S=P(A) *i.e.* Power set of A,

and operation taken here symmetric difference as addition and intersection as multiplication then,

Δ	Φ 3	{1} 2	{2} 2	{1,2} 3
Φ 3	Φ 3	{1} 2	{2} 2	{1,2} 3
{1} 2	{1} 2	Φ 3	{1,2} 3	{2} 2
{2} 2	{2} 2	{1,2} 3	Φ 3	{1} 2
{1,2} 3	{1,2} 3	{2} 2	{1} 2	Φ 3

∩	Φ 3	{1} 2	{2} 2	{1,2} 3
Φ 3	Φ 3	Φ 3	Φ 3	Φ 3
{1} 2	Φ 3	{1} 2	Φ 3	{1} 2
{2} 2	Φ 3	Φ 3	{2} 2	{2} 2
{1,2} 3	Φ 3	{1} 2	{2} 2	{1,2} 3

Y= {<3,Φ>,<2,{1}>,<2,{2}>,<3,{1,2}>} forms an *M*ring

Above frequency multiplication table is symmetric.

Therefore, each u Є S, $Ƒ_Y$(u.v) = $Ƒ_Y$(v.u) for all v Є S.

So $\mathcal{M}$(Y)= Y= {<3, Φ>,<2,{1}>,<2,{2}>,<3, {1,2}>}.

Example 2.2:

If we take ($\mathbf{Z}_4$, $+_4$, $\times_4$) as a ring

and Y = {<5, 0>,<1,1>,<3,2>,<1,3>}.

Then, frequency addition and multiplication operation tables are as below:

$+_4$	0 5	1 1	2 3	3 1
0 5	0 5	1 1	2 3	3 1
1 1	1 2	2 3	3 1	0 5
2 3	2 3	3 1	0 5	1 1
3 1	3 1	0 5	1 1	2 3

$\times_4$	0 5	1 1	2 3	3 1
0 5	0 5	0 5	0 5	0 5
1 1	0 5	1 1	2 3	3 1

2 3	0 5	2 3	0 5	2 3
3 1	0 5	3 1	2 3	1 1

From it is clear that ϒ is an *M*ring and since S is a commutative *M*ring so $\mathcal{M}$(ϒ)= ϒ.

Theorem 2.2: If Let (S,+,.) be a ring which forms a field also and ϒ is a non-empty *M*ring taken from S with $Ƒ_ϒ(u) = Ƒ_ϒ(u^{-1})$ ∀ u Є S where u^{-1} is an inverse element of u with respect to multiplication then, $\mathcal{M}$(ϒ) is also an *M*ring over S.

Proof:

Since, (S,+,.) be a ring which forms a field.

So (S,+) and (S,.) are two groups.

As ϒ is a non-empty *M*ring taken from S.

Therefore, $Ƒ_ϒ(u+v) \geq min$ { $Ƒ_ϒ(u), Ƒ_ϒ(v)$} for all u Є S.

and $Ƒ_ϒ(u) = Ƒ_ϒ(-u)$ ∀ u Є S

Thus, ϒ is an *M*group with respect to addition.

So $\mathcal{M}$(ϒ) is an *M*group with respect to addition.

Again, ϒ is a non-empty *M*ring taken from S.

So $Ƒ_ϒ(u.v) \geq min$ { $Ƒ_ϒ(u), Ƒ_ϒ(v)$} for all u Є S.

Given that, $Ƒ_ϒ(u) = Ƒ_ϒ(u^{-1})$ ∀ u Є S.

Thus, $\mathcal{M}$(ϒ) is an *M*group with respect to multiplication.

Hence, $Ƒ_{\mathcal{M}(ϒ)}(u+v) \geq min$ { $Ƒ_{\mathcal{M}(ϒ)}(u), Ƒ_{\mathcal{M}(ϒ)}(v)$} for all u,v Є S.

and $Ƒ_{\mathcal{M}(ϒ)}(u.v) = min$ { $Ƒ_{\mathcal{M}(ϒ)}(u), Ƒ_{\mathcal{M}(ϒ)}(v)$} for all u,v Є S.

Also $Ƒ_{\mathcal{M}(ϒ)}(u) = Ƒ_{\mathcal{M}(ϒ)}(-u)$.

Therefore, $\mathcal{M}$(ϒ) is also an *M*ring over S.

Hence the proof.

References:

1. U.M. Swamy and K.L.N. Swamy, Fuzzy prime ideals of Rings,
Journal of mathematical analysis and applications, vol. 134, pp. 94-103, 1988.
2. W. D. Blizard, The development of multiset theory, Modern Logic vol. 1, pp 319-352 (1991).
3. W.D. Blizard, Dedekind multisets and function shells, Theoretical computer science, vol. 110,1993, pp 79-98.
4. A.M. Ibrahim and P.A. Ejegwa, A survey on the concept of multigroups, Journal of the Nigerian Association of Mathematical Physics, vol. 38, pp. 1-8, 2016.
5. J.A. Awolola and A.M. Ibrahim, Some results on multigroups, Quasigroups and related Systems, vol. 24, no.2, pp. 169-177, 2016.
6. Y. Tella and S. Daniel, Symmetric groups under multiset perspective, IOSƑ Journal of Mathematics, vol.7, no.5, pp. 47-52, 2013.
7. SUMA P, A multiset Approach to Algebraic Structures, Sequences and Applications. Handbook of research on Emerging Applications of FUZZY Algebraic Structures, Chapter 5, IGI Global (2020), 78-90.
8. S.P. Jena, S.K. Ghosh, B.K. Tripathy, On the theory of bags and lists, Information sciences, vol. 132,2001, pp. 241-254.
9. S. Debnath and A. Debnath, Study of Ring structure from multiset context, Balkan Society of Geometry Balkan Press 2019, Vol 21, pp. 84-90.
10. S.K. Nazmul, P. Majumdar and S.K. Samanta, On multisets and multigroups, Annals of Fuzzy Mathematics and Informatics, vol. 6, no. 3 pp. 643-656, 2013.

****