Reliability Analysis of Repairable $M^X/G/1$ Queue with Multiphase Optional Services and General Setup Time

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Abstract

Many queueing systems are characterized by the feature that all arrivals demand the first essential service, whereas only some of them demand second optional service which is provided by the same server. In this work, we study a single non reliable server $M^x/G/1$ queue with multiphase optional services. The customers arrive in batches according to a Poisson process. Two types of services are provided to the customers, the first "essential" service and second multiphase "optional" service. After the completion of the essential service, the customer either leaves the system with probability $(1-r_1)$ or join the first optional service with probability r_1; again after completing the first phase optional service, either he leaves or joins second phase of optional service with probability r_2 and similarly in continuation at the end of $(k-1)^{th}$ phase optional service, he may opt k^{th} phase of optional service with probability r_k or may leave the system with probability $(1-r_k)$. Both essential and optional services are provided by same single server. While the server is working, he is subject to breakdown according to Poisson process. When the server breaks down, he requires repair at repair facility where a repairman renders repair of failed server according to general distribution. By introducing supplementary variable technique and generating function method, some queueing and reliability characteristics of the system are derived. We facilitate numerical results to illustrate the effect of different parameters on several performance indices.

Key-words: Batch arrivals, Unreliable server, Setup time, Multiphase optional service, Supplementary variable, Generating function, Queue size, Reliability.

Introduction

During the last few decades considerable attention has been paid to studying the batch arrival queue, which has been well documented because of its interdisciplinary character in queueing systems. Takagi and Takahashi (1991) have derived the Laplace Stieltjes transform and the first two moments of the waiting time distribution for $M^x/G/1$ queues. Lee et al. (1995) have obtained the system size distribution and illustrated that the system size distribution decomposes into two random variables, one of which was the system size of ordinary $M^x/G/1$ queue. A multiple vacation model for $M^x/G/1$ queueing system with balking has been presented by Thomo (1997). Economou and Fakinos (1999) investigate, under what circumstances the stationary distribution of the number of groups of various sizes in the M/G/k group arrival group departure, loss system under a quite general acceptance policy can be obtained a closed product form. Choudhry (2002) has considered an $M^x/G/1$ queueing system with a vacation time under single vacation policy, where the server takes exactly one vacation between two successive busy periods. Madan and Rawwash (2005) have obtained steady state results in explicit and closed form in term of the probability generating functions for the number of customers in the queue. Choudhury (2007) carried out an extensive stationary analysis of the system including existence of the stationary regime, embedded markov chain, steady state distribution of the server state and number of customers in the retrial group. Choudhury (2008) dealt with the queue size distribution of the $M^x/G/1$ queue with Bernoulli vacation schedule under a restricted admissibility policy. Amar (2009) derived and explicit formula for the generating function of the number of customers in orbit for $M^x/G/1$ queue and exhibited explicit forms of stochastic decomposition property. A deterministic sample path general relationship that relates workload and batch delays for a batch arrival single-server queueing model is given by El-Taha (2014). An approximation of general multi-server queues with bulk arrivals and batch service is evaluated by Zisgen (2022).

Optional phase service systems have been discussed in the literature for their application in various areas such as computer, communication, manufacturing and other many systems. These queueing systems are characterized by the feature that all arrivals demand the first essential service, whereas only some of them demand second optional service which is provided by the same server. A single server queue with Poisson arrivals and no waiting room is considered by Madan (1994) in which two services were offered, an arrival essentially requires the first but had the option whether to have a second service or not. Medhi (2002) proposed an M/G/1 queueing system with second optional channel and developed the explicit expressions for the mean queue length and mean waiting time. Supplementary variable technique is used to develop the time dependent probability generating function in terms of their Laplace transform and the corresponding steady state results such as mean queue length and main waiting time for queueing system with second optional service was explained by Al- Jararha and Madan (2003). $M^x/G/1$ queueing system with two phases of heterogeneous service under N-policy was examined by Choudhury and Paul (2004). A discrete time Geo/G/1 retrial queue has been considered by Wang and Zhao (2007) with starting failures in which all arriving customers require a first essential service while only some of them ask for a second optional service. $M^x/G/1$ queue with two types of services is studied by Jain and Chauhan (2016).

Classical studies on queueing systems use perfect (reliable) servers. However in many real time systems, the server may meet unpredictable breakdowns. Therefore, queueing models with server breakdowns are realistic representation of the system. A queueing model subject to random breakdowns of a terminal system has been analyzed by Sztrik and Gal (1990). The steady state queue length distribution and mean queue length of Markov queueing system subject to random breakdown were computed by Hsieh and Andersland (1995). The total expected cost function per unit time was developed by Wang (1995) to obtain the optimal operating policy of a single removable and non reliable server. A single removable and non reliable server in both an infinite and a finite queueing system with Poisson arrivals has been considered by Wang et al. (1999). Ke (2003) studied the system characteristics of M/G/1 queueing system with server breakdown. The management policy of an M/G/1 queue with a single removable and no reliable server was considered by Pearn et al. (2004). Using supplementary variable method Wang (2004) obtained the transient and the steady state solutions for M/G/1 queue with second optional service. The vacation policy of an M/G/1 queueing system with an unreliable server was studied by Ke (2005). Ke (2006) analyzed the vacation policies of an unreliable server M/G/1 queue in which the length of the vacation period is controlled either by the arrivals or by timer. The operating characteristics of an $M^x/G/1$ queueing system have been studied by Ke (2007) under the assumption that the server may breakdown according to Poisson process and the repair time follows a general distribution. Wang et al. (2007) analyzed various system performance measures and investigated some performance indices and expected cost function of M/G/1 queueing system with server breakdowns. Optimal stationary policy under a suitable linear cost structure has been derived by Chaudhury et al. (2009) for an unreliable server with two phases of service.

In many realistic situations, the server takes some times (i.e. set up time) before starting the service of first customer; such systems have also been analyzed by researchers working in the area of queueing theory. Li and Yang (1995) studied M/G/1 retrial queue with setup times. An efficient iterative algorithm has been developed by Li et al. (1995) for computing the stationary queue length distribution in M/G/1/N queue with setup times and arbitrary state dependent arrival rates. The optimal N policy was determined by minimizing the total operation cost of the system by Hur and Paik (1998) for an M/G/1 queue with general setup time. Krishna et al. (1998) have derived the system size distribution and expected length of idle and busy period of a $M^x/G(a,b)/1$ queueing system with N policy and setup time. Choudhury (1998) derived analytically explicit expressions for the system state probabilities for bulk arrival Poisson queue with random set up time and vacation. Hur and Paik (1999) investigated the system operation cost and the optimal N policy by a numerical study of M/G/1 with server setup. Finite capacity single server vacation queue with setup times and batch Morkovian arrival process (BMAP) was considered by Niu et al. (2003). Lee et al. (2003) applied the factorization principle to derive the generating function of the queue length and vector Laplace Stieltjes transform of the waiting time of a BMAP/G/1 queue. $M^x/G(a,b)/1$ queueing system with multiple vacations setup time with N policy and closedown times was considered by Arumuganathan and Jeyakumar (2005). Ke (2008) obtained steady state results for system size distribution at a random epoch and at a departure epoch for an $M^x/G/1$ queueing system with startup and j additional options for services. The steady state behavior was examined of a bulk arrival queueing system with additional second phase of optional service and unreliable server by Chaudhury and Deka (2009). A single server queue with a variable service speed, batch arrivals and general setup times is studied by Yojima and Phung-Duc (2020).

The purpose of this work is to obtain explicit expressions for various queueing and reliability indices for unreliable server queue with bulk arrival. The organization of the paper is as follows. We describe the model and introduce some notations in section 2. Section 3 is devoted for the analysis part of the problem, where we obtain probability distribution of the system state. These results are obtained by the method of supplementary variable. Since the breakdowns and repair process are independent of the servicing process, then the reliability and availability are defined in the usual way in section 4. Numerical illustration in order to validate the analytic results is given in section 5. Some concluding remarks are outlined in last section 6.

2. Model Description

$M^x/G/1$ queueing system with unreliable server, setup and k-phase optional service is considered by making the following assumptions:

- The customers arrive at the system according to a compound Poisson process with random batch size denoted by random variable 'X' with distribution a_i=Pr[X=i].
- There is a single unreliable server who provides two kinds of general heterogeneous services to the customers on a first come first served (FCFS) basis.
- The first essential service is needed to all arriving customers; the duration of essential services are general distributed. Its distribution function, density function and hazard rate function are $B_0(x)$, $b_0(x)$ and $\mu_0(x)$, respectively.
- As soon as the first essential service of the customer is completed, then with probability r_1 he may demand for first phase second optional service or may leave the system with probability $(1-r_1)$. After the completion of first phase optional service he may go for second phase optional service with probability r_2 or may leave the system with probability $(1-r_2)$. In general, the customer may opt any of k^{th} ($1 \le k \le m$) phase optional service with probability r_k or may leave the system with probability $(1-r_k)$.
- The k type second optional service time follows an arbitrary distribution and its distribution function, density function and hazard rate function are $B_k(x)$, $b_k(x)$ and $\mu_k(x)$, respectively ($1 \le k \le m$).
- We assume that the life time of a server is exponentially distributed with rate α_1 and α_2 in first essential service and second optional service, respectively.
- If the server breaks down during the service, the customer just being served before server breakdown waits for the server to complete its remaining service.
- The repair time distributions for both essential and k^{th} optional service phases are arbitrarily distributed with probability distribution functions $R_0(y)$ and $R_k(y)$, respectively. Also let $r_0(y)$, $r_k(y)$ and $\beta_0(y)$, $\beta_k(y)$ are the corresponding probability density functions and hazard rates.
- The server will be recovered after completion of the repair and starts service of the customers immediately.

Notations

λ	Mean arrival rate of the customers
X	Random variable denoting the batch size
X(z)	Generating function for batch size X
α_0, α_k	Mean failure rate of server in both phases, k=1,2,…,m
μ_0, θ_0, β_0	Service rate, setup rate and repair rate in first essential service
μ_k, θ_k, β_k	Service rate, setup rate and repair rate in k^{th} phase second optional service
$\mu_0(x), \theta_0(y), \beta_0(y)$	Hazard rates of service, setup and repair for essential service

$\mu_k(x)$, $\theta_k(y)$, $\beta_k(y)$ Hazard rates of service, setup and repair for optional service

$b_0(x)$, $s_0(y)$, $r_0(y)$ Probability density functions for service time, setup time and repair time in essential service

$b_k(x)$, $s_k(y)$, $r_k(y)$ Probability density functions for service time, setup time and repair time in k^{th} phase optional service, k=1,2,…,m

$B_0(x)$, $S_0(y)$, $R_0(y)$ Distribution functions of service time, setup time and repair time for essential service

$B_k(x)$, $S_k(y)$, $R_k(y)$ Distribution functions of service time, setup time and repair time for k^{th} phase optional service, k=1,2,…,m

$P_n^{(0)}(t,x)$ Joint probability that there are n customers in the queue at time t when the server is busy with first essential service and elapsed service time lies in (x, x+dx)

$S_n^{(0)}(t,x,y)$ Joint probability that there are n customers in the queue at time t when the server is in setup state while broken down during first essential service and the elapsed service time for the customer under service is equal to x, elapsed setup time lies in (y, y+dy)

$R_n^{(0)}(t,x,y)$ Joint probability that there are n customers in the queue at time t when the server is under repair state while broken down during first essential service and the elapsed service time for the customer under service is equal to x, elapsed repair time lies in (y, y+dy)

$P_n^{(k)}(t)$ Joint probability that there are n customers in the queue at time t when the server is busy with k^{th} phase optional service, k=1,2,..,m

$S_n^{(k)}(t,y)$ Joint probability that there are n customers in the queue at time t when the server is in setup state while broken down during k^{th} phase optional service and elapsed setup time lies in (y, y+dy), k=1,2,…,m

$R_n^{(k)}(t,y)$ Joint probability that there are n customers in the queue at time t when the server is in setup state while broken down during k^{th} phase optional service and elapsed repair time lies in (y, y+dy), k=1,2,…,m

Hazard rates are given by:

$$\mu_k(x)dx = \frac{dB_k(x)}{1-B_k(x)};\ \ \theta_k(y)dy = \frac{dS_k(y)}{1-S_k(y)};\ \ \beta_k(y)dy = \frac{dR_k(y)}{1-R_k(y)},\quad 0\le k\le m$$

In order to provide analytic solution, the following probability generating functions are defined

$$X(z)=\sum_{i=1}^{\infty}a_i z^i\ ,P^{(0)}(x,z)=\sum_{n=0}^{\infty}P_n^{(0)}(x)z^n\ ,P^{(k)}(z)=\sum_{n=0}^{\infty}P_n^{(k)}z^n\ ,S^{(0)}(x,y,z)=\sum_{n=0}^{\infty}S_n^{(0)}(x,y)z^n\ ,S^{(k)}(y,z)=\sum_{n=0}^{\infty}S_n^{(k)}(y)z^n\ ,$$

$$R^{(0)}(x,y,z)=\sum_{n=0}^{\infty}R_n^{(0)}(x,y)z^n\ ,\ R^{(k)}(y,z)=\sum_{n=0}^{\infty}R_n^{(k)}(y)z^n$$

3. The Analysis

We construct the partial differential equations governing the model for the system and assume the elapsed service time, elapsed setup time and the elapsed repair time as supplementary variables:

$$\left(\frac{d}{dt}+\lambda\right)Q(t)=\mu_k P_0^{(k)}(t)+(1-r_k)\int_0^{\infty}P_0^{(0)}(t,x)\mu_0(x) \tag{1}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\mu_0(x)+\lambda+\alpha_0\right)P_n^{(0)}(t,x)=\lambda\sum_{i=1}^{n}a_i P_{n-i}^{(0)}(x)+\int_0^{\infty}R_n^{(0)}(t,x,y)\beta_0(y)dy,\quad n\ge 1 \tag{2}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial w}+\theta_0(y)+\lambda\right)S_n^{(0)}(t,x,y)=\lambda\sum_{i=1}^{n}a_iS_{n-i}^{(0)}(t,x,y),\quad n\ge 1 \tag{3}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\beta_0(y)+\lambda\right)R_n^{(0)}(t,x,y)=\lambda\sum_{i=1}^{n}a_iR_{n-i}^{(0)}(t,x,y),\quad n\ge 1 \tag{4}$$

$$\left(\frac{d}{dt}+\mu_k+\lambda+\alpha_k\right)P_n^{(k)}(t)=\lambda\sum_{i=1}^{n}a_iP_{n-i}^{(k)}(t)+\int_0^\infty R_n^{(k)}(t,y)\beta_k(y)dy+r\int_0^\infty P_n^{(0)}(t,x)\mu_0(x)dx,\quad n\ge 1,\quad 1\le k\le m \tag{5}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial w}+\theta_k(y)+\lambda\right)S_n^{(k)}(t,y)=\lambda\sum_{i=1}^{n}a_iS_{n-i}^{(k)}(t,y),\quad n\ge 1,\quad 1\le k\le m \tag{6}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\beta_k(y)+\lambda\right)R_n^{(k)}(t,y)=\lambda\sum_{i=}^{n}a_iR_{n-i}^{(k)}(y),\quad n\ge 1,\quad 1\le k\le m \tag{7}$$

The following boundary conditions are taken into consideration:

$$P_n^{(0)}(t,0)=\mu_kP_{n+1}^{(k)}(t)+(1-r_k)\int_0^\infty P_{n+1}^{(0)}(t,x)\mu_0(x)dx,\quad n\ge 1,\quad 1\le k\le m \tag{8}$$

$$P_0^{(0)}(t,0)=\mu_kP_1^{(k)}(t)+(1-r_k)\int_0^\infty P_1^{(0)}(t,x)\mu_0(x)dx+\lambda\sum_{i=1}^{n}a_iQ_{n-i}(t),n\ge 1,\quad 1\le k\le m \tag{9}$$

$$S_n^{(0)}(t,x,0)=\alpha_0P_n^{(0)}(t,x),\quad n\ge 1,\quad 1\le k\le m \tag{10}$$

$$S_n^{(k)}(t,0)=\alpha_kP_n^{(k)}(t),\quad n\ge 1,\quad 1\le k\le m \tag{11}$$

$$R_n^{(0)}(x,0)=\int_0^\infty S_n^{(0)}(t,x,y)\theta_0(y)dy,\quad n\ge 1,\quad 1\le k\le m \tag{12}$$

$$R_n^{(k)}(0)=\int_0^\infty S_n^{(k)}(y)\theta_k(y)dy,\quad n\ge 1,\quad 1\le k\le m \tag{13}$$

$$P_n^{(k)}(t,0)=\int_0^\infty P_n^{(k-1)}(t,x)\mu_{k-1}(x)dx,\quad n\ge 1,\quad 1\le k\le m \tag{14}$$

$$S_n^{(k)}(t,0)=\int_0^\infty S_n^{(k-1)}(t,y)\theta_{k-1}(y)dy,\quad n\ge 1,\quad 1\le k\le m \tag{15}$$

$$R_n^{(k)}(t,0)=\int_0^\infty R_n^{(k-1)}(t,y)\beta_{k-1}(y)dy,\quad n\ge 1,\quad 1\le k\le m \tag{16}$$

Taking Laplace transform of eqs (1)-(7) with respect to t, we get

$$(s+\lambda)Q^*(s)-1=\mu_kP_0^{*(k)}(s)+(1-r_k)\int_0^\infty P_0^{*(0)}(s,x)\mu_0(x)dx \tag{17}$$

$$\frac{\partial}{\partial x}P_n^{*(0)}(s,x)+(s+\mu_0(x)+\lambda+\alpha_0)P_n^{*(0)}(s,x)=\lambda\sum_{i=1}^{n}a_iP_{n-i}^{*(0)}(s,x)+\int_0^\infty R_n^{*(0)}(s,x,y)\beta_0(y)dy,\quad n\ge 1,\quad 1\le k\le m \tag{18}$$

$$\frac{\partial}{\partial w}S_n^{*(0)}(s,x,y)+(s+\theta_0(y)+\lambda)S_n^{*(0)}(s,x,y)=\lambda\sum_{i=1}^{n}a_iS_{n-i}^{*(0)}(s,x,y),n\ge 1,\quad 1\le k\le m \tag{19}$$

$$\frac{\partial}{\partial y}R_n^{*(0)}(s,x,y)+(s+\beta_0(y)+\lambda)R_n^{*(0)}(s,x,y)=\lambda\sum_{i=1}^{n}a_iR_{n-i}^{*(0)}(s,x,y),n\ge 1,\quad 1\le k\le m \tag{20}$$

$$\frac{d}{dt}P_n^{(*k)}(s)+(s+\mu_k+\lambda+\alpha_k)P_n^{(*k)}(s)=\lambda\sum_{i=1}^{n}a_iP_{n-i}^{*(k)}(s)+\int_0^\infty R_n^{*(k)}(s,y)\beta_k(y)dy+r\int_0^\infty P_n^{*(0)}(s,x)\mu_0(x)dx \tag{21}$$

$$\frac{\partial}{\partial w}S_n^{*(k)}(s,y)+(s+\theta_k(y)+\lambda)S_n^{*(k)}(s,y)=\lambda\sum_{i=1}^{n}a_iS_{n-i}^{(*k)}(s,y),n\ge 1,\quad 1\le k\le m \tag{22}$$

$$\frac{\partial}{\partial y}R_n^{*(k)}(s,y)+(s+\beta_k(y)+\lambda)R_n^{*(k)}(s,y)=\lambda\sum_{i=}^{n}a_iR_{n-i}^{*(k)}(s),n\ge 1,\quad 1\le k\le m \tag{23}$$

Taking Laplace transforms of boundary conditions (8)-(16), we obtain

$$P_n^{*(0)}(s,0) = \mu_k P_{n+1}^{*(k)}(s) + (1-r_k)\int_0^\infty P_{n+1}^{*(0)}(s,x)\mu_0(x)dx \tag{24}$$

$$P_0^{*(0)}(s,0) = \mu_k P_1^{*(k)}(s) + (1-r_k)\int_0^\infty P_1^{*(0)}(s,x)\mu_0(x)dx + \lambda\sum_{i=1}^{n}\alpha_i Q^*{}_{n-i}(s) \tag{25}$$

$$S_n^{*(0)}(s,x,0) = \alpha_0 P_n^{*(0)}(s,x), \quad n \ge 1, \quad 1 \le k \le m \tag{26}$$

$$S_n^{*(k)}(s,0) = \alpha_k P_n^{*(k)}(s), \quad n \ge 1, \quad 1 \le k \le m \tag{27}$$

$$R_n^{*(0)}(s,x,0) = \int_0^\infty S_n^{*(0)}(s,x,y)\theta_0(y)dy, \quad n \ge 1, \quad 1 \le k \le m \tag{28}$$

$$R_n^{*(k)}(s,0) = \int_0^\infty S_n^{(*k)}(s,y)\theta_k(y)dy, \quad n \ge 1, \quad 1 \le k \le m \tag{29}$$

$$P_n^{*(k)}(s) = \int_0^\infty P_n^{*(k-1)}(s,x)\mu_{k-1}(x)dx, \quad n \ge 1, \quad 1 \le k \le m \tag{30}$$

$$S_n^{*(k)}(s,0) = \int_0^\infty S_n^{*(k-1)}(s,y)\theta_{k-1}(y)dy, \quad n \ge 1, \quad 1 \le k \le m \tag{31}$$

$$R_n^{*(k)}(t,0) = \int_0^\infty R_n^{*(k-1)}(s,y)\beta_{k-1}(y)dy, \quad n \ge 1, \quad 1 \le k \le m \tag{32}$$

Theorem 1: The Laplace Stieltjes transforms and moment generating functions when the server is in busy state, under setup state and repair state respectively, are given by

$$Q^*(s) = \frac{1}{(s+\lambda-\lambda X(z_s))} \tag{33}$$

$$P^{*(0)}(s,x,z) = P^{*(0)}(s,0,z)e^{-\phi_0(s,z)x}(1-B_0(x)) \tag{34}$$

$$P^{*(k)}(s,z) = \frac{rb*\phi_0(s,z)}{\mu_k+\psi_k(s,z)}P^{(0)}(s,0,z) \tag{35}$$

$$S^{*(0)}(s,x,y,z) = e^{-(s+\lambda-\lambda x(z))y}(1-S_0(y))S^{*(0)}(s,x,0,z) \tag{36}$$

$$S^{*(k)}(s,y,z) = e^{-(s+\lambda-\lambda x(z))y}(1-S_k(y))S^{*(k)}(s,0,z) \tag{37}$$

$$R^{*(0)}(s,x,y,z) = e^{-(s+\lambda-\lambda x(z))y}(1-R_0(y))R^{*(0)}(s,x,0,z) \tag{38}$$

$$R^{*(k)}(s,y,z) = e^{-(s+\lambda-\lambda x(z))y}(1-R_k(y))R^{*(k)}(s,0,z) \tag{39}$$

Theorem 2: The marginal generating functions are obtained as

$$P^{(0)}(s,z) = \left\{\frac{1-b^*\phi_0(s,z)}{\phi_0(s,z)}\right\}P^{(1)}(s,0,z) \tag{40}$$

$$P^{(k)}(z) = \left\{\frac{r_k b^*\phi_0(s,z)}{\mu_k+\psi_k(s,z)}\right\}P^{*(0)}(s,0,z) \tag{41}$$

$$S^{*(0)}(s,z) = \alpha_0\left\{\frac{1-b^*\phi_0(s,z)}{\phi_0(s,z)}\right\}\left\{\frac{1-s_0^*(1+\lambda-\lambda X(z))}{(1+\lambda-\lambda X(z))}\right\}P^{*(0)}(s,0,z) \tag{42}$$

$$S^{*(k)}(s,z) = \alpha_k\left\{\frac{rb^*\phi_0(s,z)}{\mu_k+\psi_k(s,z)}\right\}\left\{\frac{\prod_{n=1}^{k}1-s_n^*(s+\lambda-\lambda X(z))\}}{(s+\lambda-\lambda X(z))^n}\right\}P^{*(0)}(s,0,z) \tag{43}$$

$$R^{*(0)}(s,z) = \alpha_0\left\{\frac{1-b^*\phi_0(s,z)}{\phi_0(s,z)}\right\}\left\{\frac{1-s_0^*(s+\lambda-\lambda X(z))}{(s+\lambda-\lambda X(z))}\right\}\left\{\frac{1-r_0^*(s+\lambda-\lambda X(z))}{(s+\lambda-\lambda X(z))}\right\}P^{*(0)}(s,0,z) \tag{44}$$

$$R^{*(k)}(s,z) = \alpha_k \left\{\frac{r_k b^* \phi_0(s,z)}{\mu_k + \psi_k(s,z)}\right\}\left\{\frac{1 - s_k^*(s+\lambda-\lambda X(z))}{(s+\lambda-\lambda X(z))}\right\}\left\{\frac{\prod_{n=1}^{k}(1 - s_n^*(s+\lambda-\lambda X(z)))}{(s+\lambda-\lambda X(z))^n}\right\}P^{*(0)}(s,0,z) \quad (45)$$

where $P^{*(0)}(s,0,z) = \dfrac{(\psi_k(s,z)+\mu_k)\{(s+\lambda-\lambda X(z))Q^*(s)-1\}}{\{b^*\phi_0(s,z)-z\}\{\psi_k(s,z)+\mu_k\} - r_k b^*\phi_0(s,z)\psi_k(s,z)}$ (46)

$$\phi_0(s,z) = \{s+\alpha_0+\lambda-\lambda X(z) - \alpha_0 s_0^*(s+\lambda-\lambda X(z))\}$$

$$\psi_k(s,z) = s+\lambda-\lambda X(z)+\alpha_k - \alpha_k s_k^*(s+\lambda-\lambda X(z))\prod_{n=1}^{k-1} r_n^*(s+\lambda-\lambda X(z)), 1 \le k \le m$$

Theorem 3: Probability that server is in idle, busy with both phases of service, setup and under repair, respectively are given by

$$P_I = 1 - \rho_0\left(1+\frac{\alpha_0}{\theta_0}+\frac{\alpha_0}{\beta_0}\right) - r\sum_{k=1}^{m}\rho_k\left(1+\frac{\alpha_k}{\theta_k}+\frac{\alpha_k}{\beta_k}\right) \quad (47)$$

$$P_B = \rho_0 + r\sum_{k=1}^{m}\rho_k \quad (48)$$

$$P_S = \frac{\rho_0\alpha_0}{\theta_0} + r\sum_{k=1}^{m}\frac{\rho_k\alpha_k}{\theta_k} \quad (49)$$

$$P_R = \frac{\rho_0\alpha_0}{\beta_0} + r\sum_{k=1}^{m}\frac{\rho_k\alpha_k}{\beta_k} \quad (50)$$

where $\rho_0 = \dfrac{\lambda E[X]}{\mu_0}$ and $\rho_k = \dfrac{\lambda E[X]}{\mu_k}$, $1 \le k \le m$

4. Reliability Analysis

In order to analyze reliability indices, we consider set up and breakdown states as absorbing states. Then using notations and assumptions as defined in sections 2 and 3, we get the following set of equations:

$$\left(\frac{d}{dt}+\lambda\right)Q(t) = \mu_k P_0^{(k)}(t) + (1-r)\int_0^\infty P_0^{(0)}(t,x)\mu_0(x), \quad 1 \le k \le m \quad (51)$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\mu_0(x)+\lambda+\alpha_0\right)P_n^{(0)}(t,x) = \lambda\sum_{i=1}^{n} a_i P_{n-i}^{(0)}(x) \quad (52)$$

$$\left(\frac{d}{dt}+\mu_k+\lambda+\alpha_k\right)P_n^{(k)}(t) = \lambda\sum_{i=1}^{n} a_i P_{n-i}^{(k)}(t) + r\int_0^\infty P_n^{(0)}(t,x)\mu_0(x)dx, \quad n \ge 1, \ 1 \le k \le m \quad (53)$$

Boundary conditions:

$$P_n^{(0)}(t,0) = \mu_k P_{n+1}^{(k)}(t) + (1-r)\int_0^\infty P_{n+1}^{(0)}(t,x)\mu_0(x)dx, \quad n \ge 1, \ 1 \le k \le m \quad (54)$$

$$P_0^{(0)}(t,0) = \mu_k P_1^{(k)}(t) + (1-r)\int_0^\infty P_1^{(0)}(t,x)\mu_0(x)dx + \lambda\sum_{i=1}^{n} a_i Q_{n-i}(t), \quad n \ge 1, \ 1 \le k \le m \quad (55)$$

$$P_n^{(k)}(t,0) = \int_0^\infty P_n^{(k-1)}(t,x)\mu_{k-1}(x)dx, \quad n \ge 1, \ 1 \le k \le m \quad (56)$$

Theorem 5: The Laplace Stieltjes transforms and moment generating functions of the state probabilities are given by:

$$Q^*(s) = \frac{1}{(s+\lambda-\lambda X(z_w))} \quad (57)$$

$$P^{*(0)}(s,x,z) = e^{-(s+\alpha_0+\lambda-\lambda X(z))x}(1-B_0(x))P^{*(0)}(s,0,z) \quad (58)$$

$$P^{(k)}(s,z)=\left\{\frac{r\prod_{n=0}^{k}\mu_n(x)b^*(s+\alpha_0+\lambda-\lambda X(z))}{(s+\lambda-\lambda X(z)+\mu_k)}\right\}P^{*(0)}(s,0,z),1\le k\le m \qquad (59)$$

where $P^{*(0)}(s,0,z)=\dfrac{(\xi_k(s,z)+\mu_k)\{(s+\lambda-\lambda X(z))Q^*(s)-1\}}{\{(1-r)b^*(\xi_0(s,z))-z\}\{\xi_k(s,z)+\mu_k\}+rb^*(\xi_0(s,z))\prod_{n=1}^{k}\mu_{n-1}(x)}$ and

$(s+\alpha_k+\lambda-\lambda X(z))=\xi_k(s,z),\ \ 0\le k\le m$

and z_w is the root of equation $x=b^*\xi_0(s,z)\left\{(1-r)+\dfrac{rb^*\xi_0(s,z)\prod_{n=1}^{k}\mu_{n-1}(x)}{\xi_k(s,z)+\mu_k}\right\}$

Proof: Taking Laplace Transform of equations (51)-(56), we get:

$$(s+\lambda)Q^*(s)-1=\mu_k P_0^{*(k)}(s)+(1-r)\int_0^\infty P_0^{*(0)}(s,x)\mu_0(x)dx \qquad (60)$$

$$\frac{\partial}{\partial x}P_n^{*(0)}(s,x)+(s+\mu_0(x)+\lambda+\alpha_0)P_n^{*(0)}(s,x)=\lambda\sum_{i=1}^{n}a_iP_{n-i}^{*(0)}(s,x),\ \ n\ge 1 \qquad (61)$$

$$\frac{d}{dt}P_n^{(*k)}(s)+(s+\mu_k+\lambda+\alpha_k)P_n^{(*k)}(s)=\lambda\sum_{i=1}^{n}a_iP_{n-i}^{*(k)}(s)+r\int_0^\infty P_n^{*(0)}(s,x)\mu_0(x)dx,\ n\ge 1,1\le k\le m \qquad (62)$$

And boundary conditions become

$$P_n^{*(0)}(s,0)=\mu_kP_{n+1}^{*(k)}(s)+(1-r)\int_0^\infty P_{n+1}^{*(0)}(s,x)\mu_0(x)dx,\quad n\ge 1,\ \ 1\le k\le m \qquad (63)$$

$$P_0^{*(0)}(s,0)=\mu_kP_1^{*(k)}(s)+(1-r)\int_0^\infty P_1^{*(0)}(s,x)\mu_0(x)dx+\lambda\sum_{i=1}^{n}a_iQ^*{}_{n-i}(s),\ \ n\ge 1,\ \ 1\le k\le m \qquad (64)$$

$$P_n^{*(k)}(s)=\int_0^\infty P_n^{*(k-1)}(s)\mu_{k-1}(x)dx,\quad n\ge 1,\quad 1\le k\le m \qquad (65)$$

Multiplying (61) with suitable power of z and some over n and using defined generating functions, we have

$$P^{*(0)}(s,x,z)=e^{-(s+\alpha_0+\lambda-\lambda X(z))x}(1-B_0(x))P^{*(0)}(s,0,z) \qquad (66)$$

From (63) and (64) we get

$$\begin{aligned}zP^{*(0)}(s,0,z)&=(1-r_k)\int_0^\infty P^{*(0)}(s,x,z)\mu_0(x)dx+\mu_kP^{*(k)}(s,z)+\lambda X(z)Q^*(s)\\&\quad-(1-r_k)\int_0^\infty P^{*(0)}\mu_0(x)dx-\mu_kP_0^{*(k)}(s,z)\end{aligned} \qquad (67)$$

Eq. (60) gives

$$[\mu_kP^{*(k)}(s)+(1-r_k)\int_0^\infty P_0^{*(0)}(s,x)\mu_1(x)dx]=1-(s+\lambda)Q^*(s) \qquad (68)$$

Eqs (67) and (68) give

$$zP^{*(0)}(s,0,z)=(1-r_k)\int_0^\infty P^{*(0)}(s,x,z)\mu_0(x)dx+\mu_kP^{*(k)}(s,z)+(\lambda X(z)-\lambda-s)Q^*(s) \qquad (69)$$

From eq. (62), we have

$$(s+\lambda-\lambda X(z)+\alpha_k+\mu_k)P^{*(k)}(s,z)=r_k\int_0^\infty P^{*(0)}(s,x,z)\mu_0(x)dx \qquad (70)$$

By solving eqs (65) and (66) and using (70), we get

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$$P^{(k)}(s,z)=\left\{\frac{r_k \prod_{n=0}^{k}\mu_n(x)\,b^*(s+\alpha_0+\lambda-\lambda X(z))}{(s+\lambda-\lambda X(z)+\mu_k)}\right\}P^{*(0)}(s,0,z),\ 1\le k\le m \tag{71}$$

Let $(s+\alpha_k+\lambda-\lambda X(z))=\xi_k(s,z), 0\le k\le m$

Now put the value of $P^{(k)}(s,z)$ in (67)

$$P^{*(0)}(s,0,z)=\frac{(\xi_k(s,z)+\mu_k)\{(s+\lambda-\lambda X(z))Q^*(s)-1\}}{\{(1-r)b^*(\xi_0(s,z))-z\}\{\xi_k(s,z)++\mu_k\}+rb^*(\xi_0(s,z))\prod_{n=1}^{k}\mu_{n-1}(x)} \tag{72}$$

Let $P^*(s,z)=P^{*(0)}(s,z)+\sum_{k=1}^{m}P^{*(k)}(s,z)$ denote the probability generating function of the number of customers in the queue when setup and breakdown states are assumed to be absorbing states. Therefore,

$$P^*(s,z)=\frac{\left\{(1-b^*(\xi_0(s,z)))(\xi_k(s,z)+\mu_k)+rb^*(\xi_0(s,z))\prod_{n=1}^{k}\mu_{n-1}(x)\right\}\{(s+\lambda-\lambda X(z))Q^*(s)-1\}}{\xi_0(s,z)\left[\{(1-r)b^*(\xi_0(s,z))-z\}\{\xi_k(s,z)++\mu_k\}+rb^*(\xi_0(s,z))\prod_{n=1}^{k}\mu_{n-1}(x)\right]} \tag{73}$$

We can find $Q^*(s)$ by solving above equation for z=1 with the help of Rouche's theorem. Thus we get

$$Q^*(s)=\frac{1}{(s+\lambda-\lambda X(z_w))} \tag{74}$$

where z_w is the root of equation $x=b^*\xi_0(s,z)\left\{(1-r)+\frac{rb^*\xi_0(s,z)\prod_{n=1}^{k}\mu_{n-1}(x)}{\xi_k(s,z)+\mu_k}\right\}$

(i) The Laplace Stieltjes transform of system availability (A(t)) at time t is given by

$$A^*(s)=Q^*(s)+P^{*(0)}(s,1)+\sum_{k=1}^{m}P^{*(k)}(s,1)$$

$$=\frac{1}{(s+\lambda-X(z_s))}+\frac{\left(\frac{s}{(s+\lambda-X(z_s))}-1\right)\{rB^*\phi_0(s,1)\phi_0(s,1)+(\psi_k(s,1)+\mu_k)(1-B^*\phi_0(s,1))\}}{\phi_0(s,1)[\{B^*\phi_0(s,1)-1\}\{\psi_k(s,1)+\mu_k\}-rB^*\phi_0(s,1)\psi_k(s,1)]} \tag{75}$$

(ii) The steady state availability of the server is given by

$$A=Q+P=1-\rho_0\left(\frac{\alpha_0}{\theta_0}+\frac{\alpha_0}{\beta_0}\right)-r_k\sum_{k=1}^{m}\rho_k\left(\frac{\alpha_k}{\theta_k}+\frac{\alpha_k}{\beta_k}\right) \tag{76}$$

(iii) The Laplace Stieltjes transforms of the expected number of failures of the server in the first essential service and the second optional service up to time t ($M_0(t)$, $M_k(t)$) are given by:

$$M_0^*(s)=\int_0^\infty \alpha_0 P_n^{*(0)}(s,x)dx=\alpha_0 P^{*(0)}(s,1)$$

$$=\frac{\alpha_0\{1-B^*\phi_0(s,1)\}\{\psi_k(s,1)+\mu_k\}(sQ^*(s)-1)}{\phi_0(s,1)\{B^*\phi_0(s,1)-1\}\{\psi_k(s,1)+\mu_k\}-rB^*\phi_0(s,1)\psi_k(s,1)} \tag{77}$$

$$M_k^*(s)=\int_0^\infty \alpha_k P_n^{*(k)}(s,x)dx=\alpha_k P^{*(k)}(s,1)$$

$$=\frac{r_k\alpha_k B^*\phi_0(s,1)\}sQ^*(s)-1)}{\{B^*\phi_0(s,1)-1\}\{\psi_k(s,1)+\mu_k\}-r_kB^*\phi_0(s,1)\psi_k(s,1)} \tag{78}$$

(iv) The steady state failure frequency of the server is given by

$$M_f = \lim_{z\to 1} \int_0^\infty \left(\alpha_0 P^{*(0)}(s,z) + \sum_{k=1}^{m} P^{*(k)}(s,z) \right) dx = \alpha_0 \rho_0 + r \sum_{k=1}^{m} \alpha_k \rho_k \qquad (79)$$

(v) The Laplace transform of the reliability function R(t) of the server is given by

$$R^*(s) = Q^*(s) + \lim_{z\to 1} \int_0^\infty P^{*(0)}(s,x,z)dx + \sum_{k=1}^{m} P^{*(k)}(s,z)$$

$$= \frac{1}{(s+\lambda-\lambda X(z_w))}$$

$$+ \frac{\left\{ (1-b^*(\xi_0(s,z)))(\xi_k(s,z)+\mu_k) + rb^*(\xi_0(s,z)) \prod_{n=1}^{k} \mu_{n-1}(x) \right\}}{\xi_0(s,z)\left[\{(1-r)b^*(\xi_0(s,z)) - z\}\{\xi_k(s,z)++\mu_k\} + rb^*(\xi_0(s,z)) \prod_{n=1}^{k} \mu_{n-1}(x) \right]} \times \left\{ (s+\lambda-\lambda X(z))Q^*(s) - 1 \right\} \qquad (80)$$

z_w is the root of equation $x = b^* \xi_0(s,z) \left\{ (1-r) + \frac{rb^* \xi_0(s,z) \prod_{n=1}^{k} \mu_{n-1}(x)}{\xi_k(s,z)+\mu_k} \right\}$

(vi)The mean time to the first failure (MTTFF) of the server is given by

$$MTTFF = \int_0^\infty R(t)dt = R^*(s), \lim_{s\to 0} sQ^*(s) = P_I$$

Therefore

$$MTTFF = Q^*(0) + \left(\rho_0 + \sum_{k=1}^{m} r_k \rho_k \right) \frac{\left\{ (1-b^*\alpha_0)(\alpha_k + \mu_k) + rb^*(\alpha_0) \prod_{n=1}^{k} \mu_{n-1}(x) \right\}}{\alpha_0 \left[\{1-(1-r)b^*(\alpha_0)\{\alpha_k ++\mu_k\} + rb^*(\alpha_0) \prod_{n=1}^{k} \mu_{n-1}(x) \right]} \qquad (81)$$

5. Sensitivity Analysis

In this section we validate our analytical results by taking numerical examples. Reliability indices are summarized in tables 1-4 with respect to different descriptors. For numerical results illustrated in tables 1-4, we set default parameters as

Table 1: λ=0.5, k=1, μ_0=6, μ_1=5, θ_0=0.8, θ_1=0.6, β_0=2.5, β_1=3.5, r=0.4

Table 2: λ=0.5, k=1, α_0=0.2, α_1=0.5, θ_0=0.8, θ_1=0.6, β_0=2.5, β_1=3.5, r=0.4

Table 3: λ=0.5, k=1, α_0=0.2, α_1=0.5, θ_0=0.8, θ_1=0.6, μ_0=6, μ_1=5, r=0.4

Table 4: λ=0.5, k=1, α_0=0.2, α_1=0.5, β_0=2.5, β_1=3.5, μ_0=6, μ_1=5, r=0.4

In tables 1-4 we display the availability and failure frequency of the server by varying failure rates, service rates, repair rates and setup rates. It is noted from table 1 that the availability decreases but failure frequency increases as we increase failure rates α_0, α_1; therefore higher failure rates deteriorates system reliability. The system reliability can be maintained by keeping lower failure rate of the server.

A remarkable increment in availability *A* can be seen in table 2 on increasing the service rates μ_0 and μ_1, which is quite obvious. The failure frequency *f* shows a decreasing pattern on increasing μ_0 and μ_1. The effect of server's repair rates is illustrated in table 3. As we increase the repair rates, the availability increases but at the same time the failure frequency follows a decreasing trend. Table 4 depicts that server's availability *(A)* decreases and failure frequency *f* increases for higher values of setup rates.

6. Concluding Remarks

We have discussed an unreliable $M^x/G/1$ queueing system with second multiphase optional service with setup. All customers demand the first "essential" service, whereas only some of them demand the k-phase "optional" service. By using the supplementary variable method, we have modeled the system as a Markov chain to obtain the stationary queue indices and reliability measures of interest. For our model, we have been able to derive the state probabilities that we can use to calculate the commonly used relevant performance measures. Many existing queueing systems dealing with customer service problems are special cases of our model. Efforts have also been made to illustrate the system indices numerically to validate the analytical results.

The considered queueing model represents many practical problems in many manufacturing, production, and computer and communication systems etc. wherein the server is not continuously available for providing service for the customers, such as service interruptions due to server breakdowns. Our model dealt stochastically those with situations arising in daily life when a batch of customers appear in the system to get service and the service time consists of preliminary service phase followed by a second optional service phase. The model investigated in this paper is more realistic than those existing ones, since it takes the behavior of arriving customers as well as optional service rendered by the server.

References

1. Al-Jọ̈rarha, J. and Madan, K.C. (2003): An M/G/1 queue with second optional service with general service time distribution. Information and Management Sciences, Vol. 14, pp. 17-56.
2. Amar, A. (2009): An Mx/G/1 energetic retrial queue with vacation and its control. Electronics notes in theoretical computer science, Vol. 253 (3), pp. 33-44.
3. Arumuganathan, R. and Jeyakumar, S. (2005): Steady state analysis of bulk queue with multiple vacations, setup times with N-policy and closedown times, Appl. Math. Model., Vol. 29 (10), pp. 972-986.
4. Chaudhry, M.L. and Hasham, A. (1987): Software package for $M^x/G/1$ queueing system, Oper. Res. Lett., Vol. 16 (4), pp. 195-196.
5. Choudhury, G. (1998): On a batch arrival Poisson queue with a random setup time and vacation period, Comp. Oper. Res., Vol. 25 (2), pp. 1013-1026.
6. Choudhury, G. (2002): A batch arrival queue with a vacation time under single vacation policy, Comp. Oper. Res., Vol. 29 (14), pp. 1941-1955.
7. Choudhury, G. (2007): A two phase batch arrival retrial queueing system with Bernoulli vacation schedule, Appl. Math. Comp., Vol. 188 (2), pp. 1455-1466.
8. Choudhury, G. (2008): A note on $M^x/G/1$ queue with a random setup time under a restricted admissibility policy with a Bernoulli vacation schedule, Stat. Method., Vol. 5 (1), pp. 21-29.
9. Choudhury, G. and Deka, K. (2009): An $M^x/G/1$ unreliable retrial queue with two phases of service and Bernoulli admission mechanism. Appl. Math. Comp., Vol. 215 (3), pp. 936-949.
10. Choudhury, G., Ke, J.C. and Tadj, L. (2009): The N-policy for an unreliable server with delaying repair and two phases of service. J. Comp. Appl. Math., Vol. 231 (1), pp. 349-364.
11. Choudhury, G. and Paul, M. (2004): A batch arrival queue with an additional service channel under N-policy. Appl. Math. Comp., Vol. 156, pp. 115-130.
12. Dimitriou, I. and Langaris, C. (2010): A reparable queueing model with two phase service start-up times and retrial customers. Comp. Oper. Res., Vol. 37 (7), pp. 1181-1190.
13. Economou, A. and Fakinos, D. (1999): Product form stationary distributions for the M/G/k group arrival group departure loss system under a general acceptance policy, Euro. J. Oper. Res., Vol. 112 (1), pp. 196-206.
14. El-Taha, M. (2014): Sample-path analysis of queues with batch arrivals, Computers and Industrial Engineering, Vol. 70, pp. 98-106.

15. Hsieh, Y.C. and Andersland, M.S. (1995): Repairable single server systems with multiple breakdown modes, Microelectron. Reliab., Vol. 5 (2), pp. 309-318.

16. Hur, S. and Paik, S.J. (1998): A steady-state analysis of optimal threshold policy with heterogeneous arrivals to the M/G/1 queue, Copmt.Indus. Eng., Vol. 35 (1-2), pp. 335-338.

17. Hur, S. and Paik, S.J. (1999): The effect of different arrival rates on the N-policy of M/G/1 with server setup, Appl. Math. Model., Vol. 23 (4), pp. 289-299.

18. Niu, Z., Shu, J. and Takahashi, J. (2003): A vacatin queue with setup and closedown times and batch markovian arrival process, Performnace Evaluation, Vol. 54 (3), pp. 225-248.

19. Jain, M. and Chauhan, D. (2016): Unreliable M^X/ (G_1, G_2)/Vs/1(BS)/N-Policy Queue", International Journal of Applied Science and Mathematics, Vol. 3 (5), pp. 165-169.

20. Ke, J.C. (2003): The optimal control of an M/G/1 queueing system with server vacations, startup and breakdowns, Comput. Indust. Eng., Vol. 44 (4), pp. 567-579.

21. Ke, J.C. (2005): On M/G/1 systems under NT policies with breakdowns, startup and closedown. Appl. Math. Model., Vol. 30, pp. 49-66.

22. Ke, J.C. (2006): Optimal NT policies for M/G/1 system with a startup and unreliable server, Comput. Indust. Eng., Vol. 50 (3), pp.248-262.

23. Ke, J. (2007): Operating characteristics analysis of the M^x/G/1 system with variant vacation policy and balking. Appl. Math. Model., Vol. 31, pp. 1321-1337.

24. Ke, J.C. (2008): An M^x/G/1 system with startup server and j additional options for server. Appl. Math. Model., Vol. 32 (4), pp. 443-458.

25. Krishna Reddy, G.V., Nadarajan, R and Arumuganathan, R. (1998): Analysis of a bulk queue with N-policy multiple vacations and setup times, Comp. Oper. Res., Vol. 25 (11), pp. 957-967.

26. Lee, S.S., Lee, H.W., Yoon, S.H. and Chae, K.C. (1995): Batch arrival queue with N-policy single vacation, Comp. Oper. Res., Vol. 22 (2), pp. 173-189.

27. Lee, H.W., Park, N.I. and Jeon, J. (2003): A new approach to the queue length and waiting time of BMAP/G/1 queues, Comp. Oper. Res., Vol. 30 (13), pp. 2021-2045.

28. Li, H. and Yang, T. (1995): A single-server retrial queue with server vacations and a finite number of input sources, Euro. J. Oper. Res., Vol. 85 (17), pp. 149-160.

29. Li, H., Zhu, Y. and Yang, P. (1995): Computational analysis of M(n)/G/1/N queues with setup time, Comp. Opr. Res., Vol. 22 (8), pp. 829-840.

30. Madan, K.C. (1994): An M/G/1 queueing system with additional optional service and no waiting capacity. Microelectron. Reliab., Vol. 34, pp. 521-527.

31. Madan, K.C. and Rawwash, M.A. (2005): On the M^x/G/1 queue with feedback and optional server vacations based on a single vacation policy, Appl. Math. Comp., Vol. 160 (3), pp. 909-919.

32. Medhi, J. (2002): A single server Poisson input queue with second optional channel. Queueing System Theory Appl., Vol. 42, pp. 239-242.

33. Niu, Z., Shu, J. and Takahashi, J. (2003): A vacation queue with setup and closedown times and batch markovian arrival process, Performance Evaluation, Vol. 54 (3), pp. 225-248.

34. Pearn, W.L., Ke, J.C. and Chang, Y.C. (2004): Sensitivity analysis of the optimal management policy for a queuing system with a removable and non-reliable server, Comput. Indust. Eng., Vol. 46 (1), pp. 87-99.

35. Sztrik, J. and Gal, T. (1990): A recursive solution of a queueing model for a multi-terminal system subject to breakdowns, Performance Evaluation, Vol. 11 (1), pp. 1-7.

36. Thomo, L. (1997): A multiple vacation model $M^x/G/1$ with balking, Nonlinear analysis, Vol. 30 (4), pp. 2025-2030. Takagi, H. and Takahashi, Y. (1990): Priority queue with batch Poisson arrivals, Oper. Res. Lett., Vol. 10 (4), pp. 225-232.Wang, K.H. (1995): Optimal operation of a Markovian queueing system with a removable and non-reliable server, Microelectron. Reliab., Vol. 35 (8), pp. 1131-1136.

37. Wang J. (2004): An M/G/1 queue with second optional service and server breakdowns. Computer & Mathematics with Applications, Vol. 47, pp. 1713-1723.

38. Wang, K.H., Chang, K.W. and Sivazlian, B.D. (1999): Optimal control of a removable and non-reliable server in an infinite and a finite $M/H_2/1$ queueing system, Appl. Math. Model., Vol. 23 (8), pp. 651-666.

39. Wang, J., Liu, B. and Li, J. (2007): Transient analysis of an M/G/1 retrial queue subject to disasters and server failures. Euro. J. Oper. Res..

40. Wang, J. and Zhao, Q. (2007): A discrete time Geo/G/1 retrial queue with starting failures and second optional service, Comp. Math. Appl., Vol. 53 (1), pp. 115-127.

41. Yojima, M., Phung-Duc, T. (2020): Analysis of a variable service speed single server queue with batch arrivals and general setup time**,** Performance Evaluation, Vol. 138.

42. Zisgen, H. (2022): An approximation of general multi-server queues with bulk arrivals and batch service Operation research letters, Vol 50 (1), pp. 57-63,

α_0	α_1	*A*	*F*
0.1	0.4	0.2541	0.5179
0.2	0.4	0.2315	0.5379
0.3	0.4	0.2092	0.5582
0.4	0.4	0.1873	0.5786
0.5	0.4	0.1659	0.5995
0.6	0.1	0.1443	0.6418
0.6	0.2	0.1233	0.6635
0.6	0.3	0.1024	0.6854
0.6	0.4	0.0820	0.7077
0.6	0.5	0.0617	0.7303

Table 1: Availability and failure frequency of the server by varying α_0 and α_1

μ_0	μ_1	*A*	*f*
2	4	0.2543	0.0979
4	4	0.3311	0.0879
6	4	0.4090	0.0582
8	4	0.5876	0.0486
10	4	0.6659	0.0395
6	2	0.3449	0.0818
6	4	0.4237	0.0635
6	6	0.4090	0.0582
6	8	0.5828	0.0177
6	10	0.6619	0.0003

Table 2: Availability and failure frequency of the server by varying μ_0 and μ_1

β_0	β_1	*A*	*f*
0.5	2.5	0.1942	0.0979
1.0	2.5	0.2910	0.0879
1.5	2.5	0.3799	0.0582
2.0	2.5	0.4896	0.0486
2.5	2.5	0.5699	0.0395
4.5	0.5	0.2449	0.0418
4.5	1	0.2237	0.0335
4.5	1.5	0.4020	0.0254
4.5	2	0.4826	0.0211
4.5	2.5	0.5605	0.0101

Table 3: Availability and failure frequency of the server by varying β_0 and β_1

θ_0	θ_1	*A*	*f*
0.1	0.3	0.1581	0.3178
0.3	0.3	0.1365	0.4309
0.5	0.3	0.1092	0.5082
0.7	0.3	0.0893	0.5981
0.9	0.3	0.0699	0.6005
0.5	0.1	0.0943	0.4418
0.5	0.3	0.0793	0.5455
0.5	0.5	0.0087	0.6844
0.5	0.7	0.0054	0.7197
0.5	0.9	0.0099	0.7913

Table 4: Availability and failure frequency of the server by varying θ_0 and θ_1