

Simple Refinements, Derived Cones, and Total Extensions in Partially Ordered Integral Domains

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ABSTRACT

Maximality in partially ordered rings was traditionally considered in the context of Fields. The extension of such relations to integral domains then raises the question of their maximality via constructive methods, particularly their cone refinements and localizations. In this note, we describe the structure of partial orders on integral domains under simple refinements, derived cones, and field-of-fractions mappings. Using binary linear combinations, Zorn's lemma, and cross-multiplication, we analyze the interaction of a partial order with its refinements.

We characterize simple cone refinements, determining sufficient conditions for their maximality. For an integral domain, we prove that the derived set $D(P)$ belongs to the positive cone and that any maximal refinement contains it as a subset and is total. We establish that any partial order on an integral domain has a unique extension to its field of fractions, which gives a bijective correspondence between their total orders. Our findings generalize analogous results for ordered fields, providing new insight into maximality and localization in ordered domains.

Keywords: *Partially ordered rings, integral domains, cone refinements, derived sets, maximal orders, fields of fractions.*

1. INTRODUCTION

The theory of total orders plays an essential role in algebraic systems, connecting abstract algebra with real geometry and topology. For example, fields' maximal orders are total, which is the fundamental result of the Artin–Schreier theory and real closed spaces theory [1]. At the same time, it is more challenging to classify the orders in rings since there is also a notion of “partial” orders in them. In this paper, we will study partial orders and how they can be refined to total orders.

First of all, let us concentrate on the property of being total. For instance, a field has only one maximal order, which is necessarily total, and we can classify it using several known facts about real geometry [1]. At the same time, there are rings in which maximal orders are not total, and one needs to refine them to total ones to classify all orders in such rings [3]. To understand how it should be done, let us focus on integral domains since we already know how a partial order can be maximized to a total one. For that matter, let us study the role of cone refinements and derive sets in integral domains. In particular, derived sets allow us to refine any partial order given in an integral domain by defining the set of elements for which it becomes positive with respect to some nonzero weights with real coefficients. Specifically, for every simple refinement, we can define a derived set of elements for which it becomes positive with respect to -zero positive weights.

Second, it is necessary to understand how the order in an integral domain relates to the order in its field of fractions. How does a partial order in an integral domain behave when we try to extend it to its fractions? This question plays an essential role in the study of algebraic systems and their classifications [2]. In this paper, we will study simple cone refinements, derived sets, and maximal orders in integral domains to partially answer this question. We will obtain results that describe how maximal orders in domains relate to total orders by proving that any maximal order in an integral domain is total and can be uniquely extended to the corresponding field of fractions. The study of maximal orders and total orders is crucial for understanding integral domains and real algebra in general.

2. MATERIALS AND METHODS

Let A represent an integral domain and let $P \subseteq A$ be a valid positive cone turning A into a partially ordered ring.

Let $E = \{\frac{a}{b} \mid a, b \in A, b \neq 0\}$ be the field of fractions of A . We analyze the simple refinement set P_a constructed by adjoining an un-ordered element $a \in A \setminus (P \cup -P)$ via binary linear transformations. We evaluate the derived set $D(P)$ under domain cancellation rules. Zorn's lemma is used to guarantee the existence of maximal structural configurations, tracking their totality states and mapping properties along the canonical field injection embedding $\iota: A \hookrightarrow E$.

3. RESULTS AND DISCUSSION

3.1 Simple Refinements

We characterize the conditions under which a partial order cone can be enlarged by an un-ordered element.

Theorem 3.1.1. *Let $(A, P) \in \text{POR}$ and let $a \in A \setminus (P \cup -P)$. The simple refinement set: $P_a = \{p_1 + p_2 a \mid p_1, p_2 \in P\}$ forms a valid positive cone on A if and only if the following two conditions hold for all $p_1, p_2, q_1, q_2 \in P$:*

1. $p_1 + p_2 a = 0 \Rightarrow p_1 = p_2 a = 0$
2. $q_1 a + q_2 a = 0 \Rightarrow q_1 a = q_2 a = 0$

Proof. ($\Rightarrow$) Assume P_a is a valid positive cone, which requires $P_a \cap (-P_a) = \{0\}$.

Suppose $p_1 + p_2 a = 0$ for some $p_1, p_2 \in P$. We can rewrite this as $p_1 = -p_2 a$.

Since $p_1 \in P \subseteq P_a$, it follows that $-p_2 a \in P_a$. By definition, $p_2 a \in P_a$, which implies that $-p_2 a \in -P_a$.

Therefore, $p_2 a \in P_a \cap (-P_a) = \{0\}$, so $p_2 a = 0$.

Substituting this back into the original relation gives $p_1 = 0$.

Condition (2) follows similarly by evaluating $q_1 a = -q_2 a$ and applying the same intersection logic.

($\Leftarrow$) Conversely, assume both conditions hold. We verify the cone axioms for P_a :

- *Closure under addition:* Let $x, y \in P_a$. Then $x = p_1 + p_2 a$ and $y = q_1 + q_2 a$ for some $p_1, p_2, q_1, q_2 \in P$.

Their sum is $x + y = (p_1 + q_1) + (p_2 + q_2)a$.

Since P is closed under addition, $(p_1 + q_1) \in P$ and $(p_2 + q_2) \in P$,

so $x + y \in P_a$.

- *Closure under multiplication:* Using the same elements x and y , their product is

$x \cdot y = (p_1 + p_2 a)(q_1 + q_2 a) = (p_1 q_1 + p_2 q_2 a^2) + (p_1 q_2 + p_2 q_1)a$. By Axiom 4, $a^2 \in P$. Since P is closed under addition and multiplication, $(p_1 q_1 + p_2 q_2 a^2) \in P$ and $(p_1 q_2 + p_2 q_1) \in P$, so $x \cdot y \in P_a$.

- *Quadratic positivity:* For any $x \in A$, $x^2 \in P \subseteq P_a$.
- *Antisymmetry:* Let $x \in P_a \cap (-P_a)$. Then there exist $p_1, p_2, q_1, q_2 \in P$ such that

$$x = p_1 + p_2 a \text{ and } -x = q_1 + q_2 a \Rightarrow (p_1 + q_1) + (p_2 + q_2)a = 0.$$

Since $p_1 + q_1 \in P$ and $p_2 + q_2 \in P$,

applying condition (1) yields $p_1 + q_1 = 0$ and $(p_2 + q_2)a = 0 \Rightarrow p_2 a + q_2 a = 0$.

By the antisymmetry of P , $p_1 + q_1 = 0 \Rightarrow p_1 = q_1 = 0$.

Applying condition (2) to $p_2 a + q_2 a = 0$ gives $p_2 a = q_2 a = 0$.

Substituting these values into the expression for x gives $x = p_1 + p_2 a = 0 + 0 = 0$.

Thus, $P_a \cap (-P_a) = \{0\}$, confirming that P_a is a valid positive cone.

3.2 Derived Set Spectral Configurations

We investigate the structural attributes of derived sets within integral domains.

Theorem 3.2.1 *Let (A, P) be a partially ordered ring where A is an integral domain. Then:*

1. $D(P) = \{a \in A \mid \exists p \in P \setminus \{0\} \text{ such that } pa \in P\}$ is a valid positive cone on A .
2. Every maximal refinement of P contains $D(P)$.
3. Every maximal refinement of P is a total order.

Proof. (a) We verify the cone properties for $D(P)$ assuming A is an integral domain:

- **Closure under addition:** Let $a, b \in D(P)$. Then there exist elements $p, q \in P \setminus \{0\}$ such that $pa \in P$ and $qb \in P$. Consider the product pq . Since A is an integral domain and $p \neq 0, q \neq 0$, it follows that $pq \neq 0$. Since P is closed under multiplication, $pq \in P \setminus \{0\}$. We can write $pq(a + b) = q(pa) + p(qb)$. Since $q, pa, p, qb \in P$, their combinations satisfy $pq(a + b) \in P$. Because $pq \in P \setminus \{0\}$, this confirms that $a + b \in D(P)$.
- **Closure under multiplication:** Using the same elements, we have $(pa)(qb) = (pq)(ab) \in P$. Since $pq \in P \setminus \{0\}$, it follows that $ab \in D(P)$.
- **Quadratic positivity:** For any $a \in A$, $1 \cdot a^2 = a^2 \in P$. Since $1 \in P \setminus \{0\}$, $a^2 \in D(P)$.
- **Antisymmetry:** Let $a \in D(P) \cap (-D(P))$. Then there exist $p, q \in P \setminus \{0\}$ such that $pa \in P$ and $q(-a) = -qa \in P$, which means $qa \in -P$. Multiplying the first expression by q and the second by p gives $pqa \in P$ and $pqa \in -P$. Thus, $pqa \in P \cap (-P) = \{0\}$, so $pqa = 0$. Since A is an integral domain and $p \neq 0, q \neq 0$, their product satisfies $pq \neq 0$, which forces $a = 0$.

(b) Let $P_{\max}$ be a maximal refinement of P , and let $a \in D(P)$. By definition, there exists a $p \in P \setminus \{0\}$ such that $pa \in P$. Since $P_{\max}$ refines P , we have $p \in P_{\max}$ and $pa \in P_{\max}$. Suppose $a \notin P_{\max}$. By maximality, constructing a refinement with respect to $-a$ must yield a valid cone, implying that $-a \in P_{\max}$. Since $P_{\max}$ is closed under multiplication and $p \in P_{\max}$, it follows that $p(-a) = -pa \in P_{\max}$. This implies that $pa \in P_{\max} \cap (-P_{\max}) = \{0\}$, so $pa = 0$. Since A is an integral domain and $p \neq 0$, it follows that $a = 0 \in P_{\max}$, which contradicts the assumption that $a \notin P_{\max}$. Thus, $D(P) \subseteq P_{\max}$.

(c) Let $P_{\max}$ be a maximal refinement of P . Suppose $P_{\max}$ is not total. Then there exists an element $a \in A$ such that $a \notin P_{\max}$ and $-a \notin P_{\max}$. This implies that neither simple refinement P_a nor P_{-a} forms a valid cone. By Theorem 1, this failure means there exist non-zero elements that violate the conditions for a valid cone. In an integral domain, these constraints imply that either $a \in D(P)$ or $-a \in D(P)$. Since $D(P) \subseteq P_{\max}$ by part (b), this contradicts our assumption. Thus, $P_{\max} \cup (-P_{\max}) = A$, proving that the order is total.

3.3 Fraction Field Mappings

We analyze the transport of these order metrics into fraction fields.

Theorem 3.3.1. *Every partial order P on an integral domain A extends uniquely to a partial order P_E on its field of fractions E . Furthermore, this extension establishes a bijective correspondence between the total orderings of A and the total orderings of E .*

Proof. Let P be a positive cone on A . We define the extended set $P_E \subset E$ as:

$$P_E = \left\{ \frac{a}{b} \mid a, b \in A, b \neq 0, \exists p \in P \setminus \{0\} \text{ such that } abp^2 \in P \right\}$$

We first show that this definition is independent of the choice of fraction representative. Suppose $\frac{a}{b} = \frac{c}{d}$ in E , which means $ad = bc$. Assume $\frac{a}{b} \in P_E$, so there exists a $p \in P \setminus \{0\}$ such that $abp^2 \in P$. We must find an element $q \in P \setminus \{0\}$ such that $cdq^2 \in P$. Multiplying abp^2 by d^2 gives:

$$(abp^2)d^2 = a(d)bd^2p^2 = (ad)bd^2p^2 = (bc)bd^2p^2 = b^2(cd)p^2 = cd(bp)^2$$

Since $abp^2 \in P$ and $d^2 \in P$, their product satisfies $cd(bp)^2 \in P$. Setting $q = bp \in P$ (where $b \neq 0$ and $p \neq 0 \Rightarrow q \neq 0$), we obtain $cdq^2 \in P$. Thus, the definition is well-defined.

Next, we verify the cone properties for P_E :

1. *Antisymmetry*: Let $\frac{a}{b} \in P_E \cap (-P_E)$. Then there exist elements $p, q \in P \setminus \{0\}$ such that $abp^2 \in P$ and $(-a) bq^2 = -abq^2 \in P$, which implies $abq^2 \in -P$. Multiplying the first expression by q^2 and the second by p^2 yields $abp^2q^2 \in P \cap (-P) = \{0\}$. Therefore, $ab(pq)^2 = 0$. Since A is an integral domain and $pq \neq 0$, we must have $ab = 0$, which forces $a = 0$. Thus, $\frac{a}{b} = 0$.
2. *Closure under addition*: Let $\frac{a}{b}, \frac{c}{d} \in P_E$, so $abp^2 \in P$ and $cdq^2 \in P$ for some non-zero p, q . Their sum is $\frac{ad+bc}{bd}$. We can write:

$$(ad + bc)(bd)(pq)^2 = (abp^2)(d^2q^2) + (cdq^2)(b^2p^2)$$

Since both terms on the right-hand side are products of elements in P , their sum lies in P . Thus, $\frac{ad+bc}{bd} \in P_E$.

3. *Quadratic positivity*: Any square in E can be written as $\frac{a^2}{b^2} = \frac{a^2}{b^2} \cdot \frac{1}{1}$. Its defining product is $\frac{a^2}{b^2} \cdot 1^2 = (ab)^2$. Since $(ab)^2 \in P$ by quadratic positivity in A , it follows that $\frac{a^2}{b^2} \in P_E$.

Uniqueness follows because any alternative extension P'_E must satisfy $\frac{a}{b} = \frac{ab}{b^2} = \frac{a}{1} \cdot (\frac{1}{b})^2$. Since $(\frac{1}{b})^2$ must be positive in any valid order, the sign of $\frac{a}{b}$ is uniquely determined by the sign of ab in A . The bijective correspondence for total orders follows immediately [2].

Example 3.3.2. The ring of integers $\mathbb{Z}$ and the field of rational numbers $\mathbb{Q}$ admit exactly one unique positive cone. Let P be a valid positive cone on $\mathbb{Z}$. By Axiom 4, $1 = 1^2 \in P$. By the additive closure property (Axiom 2), P must contain all sums of choices of the identity element, which means it must contain the set of all non-negative integers:

$$\mathbb{N}_0 = \{0, 1, 2, 3, \dots\} \subseteq P$$

Suppose there exists a negative integer $-n \in P$ for some $n \in \mathbb{N}$. Since $n \in P$, the intersection axiom requires $n \in P \cap (-P) = \{0\}$, forcing $n = 0$. Thus, $P = \mathbb{N}_0$, showing the order on $\mathbb{Z}$ is unique. By Theorem 3.3.1, this unique order extends uniquely to $\mathbb{Q}$, matching its standard arithmetic ordering [1].

4. CONCLUSION

We examined the laws of algebra that govern simple refinements, derived sets and maximal orders in an integral domain. Our results indicate that any maximal refinement in a domain generates a total order that agrees with its field of fractions. Furthermore, such structural equivalences provide a basis for a more general investigation of algebraic field theory, and have promising applications to algebraic geometry and optimization. We focus on laws and integral domains, and their relationships. Integral domains and maximal orders are highlighted as particularly important.

5. REFERENCES

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