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Solution of a System of Fractional Differential Equations using Elzaki Decomposition Method

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Abstract

This paper presents a novel and efficient semi-analytical approach for solving systems of fractional differential equations (FDEs) using the Elzaki Decomposition Method (). The method combines the strengths of the Elzaki transform and the Adomian decomposition method to obtain exact or approximate solutions of linear and nonlinear FDEs. Four illustrative examples are discussed to demonstrate the efficiency, reliability, and simplicity of the method. The obtained solutions show rapid convergence with minimal computational effort.

1. Introduction

Fractional calculus has gained immense attention due to its ability to describe memory and hereditary properties of various materials and processes [Podlubny, 1998; Kilbas et al., 2006]. Many physical and engineering problems such as viscoelastic systems, fluid flow, and diffusion processes are better described using fractional differential equations (FDEs) [Mainardi, 2010].

There are a number of numerical methods to solve differential and integral equations[8, 9, 10]. The analytical solution of systems of FDEs is often a challenging task. Therefore, various semi-analytical methods such as the Adomian Decomposition Method (ADM) [Adomian, 1994], Variational Iteration Method (VIM) [He, 1999], and Homotopy Analysis Method (HAM) [Liao, 2003] have been developed. Recently, the Elzaki Transform (ET), introduced by Elzaki [Elzaki, 2011], has emerged as a powerful integral transform for solving differential equations, particularly due to its simpler operational rules.

This study proposes a hybrid technique called the Elzaki Decomposition Method (EDM), which combines the Elzaki Transform and -based decomposition to solve systems of FDEs efficiently.

2. Preliminaries

2.1 Caputo Fractional Derivative $D_t^{\alpha} f(t) = \dfrac{1}{\Gamma(n-\alpha)} \int_0^t \dfrac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad n-1<\alpha<n \quad (1)$

2.2 Elzaki Transform Definition $E[f(t)] = u\int_0^{\infty} f(t) e^{-t/u} dt \quad (2)$

2.3 Useful Properties

1. $E[1] = u$
2. $E[t^n] = u^{n+1}\Gamma(n+1)$
3. $E[e^{at}] = \dfrac{u}{1-au}$
4. $E[{}^C D_t^{\alpha} f(t)] = u^{\alpha} E[f(t)] - \sum_{k=0}^{n-1} u^{\alpha-k-1} f^{(k)}(0)$

3. Methodology: Elzaki Decomposition Method (EDM)

The Elzaki Decomposition Method (EDM) is a semi-analytical technique designed to solve linear and nonlinear systems of fractional differential equations (FDEs). It synergistically combines the operational efficiency of the Elzaki Transform with the recursive series solution strategy of the Adomian Decomposition Method (ADM). This method is especially powerful for systems involving Caputo-type fractional derivatives. Below is a comprehensive outline of the EDM:

3.1 Overview of the Approach

Let us consider a general system of Caputo-type fractional differential equations:

$$ {}^{C}D_t^{\alpha} x_i(t) = f_i(x_1, x_2, \dots, x_n, t), \quad x_i(0) = c_i, \quad 0 < \alpha \le 1, \quad i = 1, 2, \dots, n \tag{3}$$

The proceeds through the following phases:

3.2 Step-by-Step Procedure

Step 1: Apply the Elzaki Transform

The Elzaki transform of a Caputo derivative is given by:

$$E[{}^{C}D_t^{\alpha} x(t)] = u^{\alpha} E[x(t)] - \sum_{k=0}^{n-1} u^{\alpha-k-1} x^{(k)}(0) \tag{4}$$

Apply this to each equation in the system. Use the initial conditions to simplify the resulting algebraic expressions.

Step 2: Convert to an Algebraic System

By transforming each FDE in the system using the Elzaki operator, the differential system becomes an algebraic system in the Elzaki image domain:

$$F_i(u, X_1(u), X_2(u), \dots, X_n(u)) = 0, \quad i = 1, 2, \dots, n \tag{5}$$

This simplification allows for handling complex equations without discretizing the time domain or applying perturbative techniques.

Step 3: Handle Nonlinear Terms using Adomian Decomposition

If any right-hand side function f is nonlinear, decompose it using Adomian Polynomials:

Let:

$$x(t) = \sum_{n=0}^{\infty} x_n(t), \quad f(x(t)) = \sum_{n=0}^{\infty} A_n \tag{6}$$

The Adomian polynomials AnA_n are computed using the recursive relation:

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} f\left(\sum_{k=0}^{\infty} \lambda^k x_k(t) \right) \Bigg|_{\lambda=0} \tag{7}$$

This technique ensures that nonlinearities are handled analytically without linearization, maintaining solution accuracy.

Step 4: Invert the Elzaki Transform

After obtaining expressions for $\mathbf{X}_i(\mathbf{u})$ Compute the inverse Elzaki Transform:

$$x_i(t) = E^{-1}[X_i(u)] = \sum_{k=0}^{\infty} x_{i,k}(t)$$

Use known Elzaki pairs or numerical inversion techniques to find the time-domain solution.

Step 5: Construct Series Solution

The final solution is expressed as an infinite series (truncated for practical use):

$$x_i(t) = \sum_{k=0}^{N} x_{i,k}(t), \quad i = 1, 2, \ldots, n$$

This form converges rapidly for many physical systems and gives high accuracy with only a few terms.

4. Illustrative Examples

Example 1: Consider the Linear System of Differential Equations

$$CD_t^\alpha x(t) = x(t) + y(t), \quad {}^C D_t^\alpha y(t) = -x(t) + y(t), \quad x(0) = 1,\ y(0) = 0 \tag{8}$$

Step 1: Apply Elzaki Transform

Let $X(u) = E[x(t)], \quad Y(u) = E[y(t)]$

Using the property of Caputo derivative and Elzaki transform:

$$E[{}^C D_t^\alpha x(t)] = u^\alpha X(u) - u^{\alpha-1} x(0) = u^\alpha X(u) - u^{\alpha-1} \& E[{}^C D_t^\alpha y(t)] = u^\alpha Y(u) \tag{9}$$

Step 2: Transform equations

1st equation:

$$u^\alpha X(u) - u^{\alpha-1} = X(u) + Y(u) \Rightarrow (u^\alpha - 1)X(u) - Y(u) = u^{\alpha-1} \tag{10}$$

2nd equation:

$$u^\alpha Y(u) = -X(u) + Y(u) \Rightarrow X(u) + (u^\alpha - 1)Y(u) = 0 \tag{11}$$

Step 3: Solve the algebraic system

From (11), express X(u) in terms of Y(u):

$$X(u) = -(u^\alpha - 1)Y(u) \tag{12}$$

Substitute into (1):

$$(u^\alpha - 1)(-(u^\alpha - 1)Y(u)) - Y(u) = u^{\alpha-1}$$
$$\Rightarrow -(u^\alpha - 1)^2 Y(u) - Y(u) = u^{\alpha-1}$$
$$\Rightarrow Y(u) = -\frac{u^{\alpha-1}}{(u^\alpha - 1)^2 + 1}$$

Then:

$$X(u) = \frac{(u^\alpha - 1)u^{\alpha-1}}{(u^\alpha - 1)^2 + 1}$$

Step 4: Take inverse Elzaki Transform

Use table of inverse Elzaki transforms or approximate numerically:

For α=1, we retrieve: $x(t) = e^t \cos t(t), y(t) = -e^t \sin(t)$. (13)

For $\alpha = \frac{1}{2}$, we retrieve

The initial approximation is $x_0(t) = 1, y_0(t) = 0$

First Approximation is

$$x_1(t) = \mathcal{I}_t^{\alpha}\big(x_0(t) + y_0(t)\big) = \mathcal{I}_t^{\alpha}(1) = \frac{t^{\alpha}}{\Gamma(1+\alpha)} \quad (14)$$

$$y_1(t) = \mathcal{I}_t^{\alpha}\big(-x_0(t) + y_0(t)\big) = \mathcal{I}_t^{\alpha}(-1) = -\frac{t^{\alpha}}{\Gamma(1+\alpha)} \quad (15)$$

Second Approximation is

$$x_2(t) = \mathcal{I}_t^{\alpha}\big(x_1(t) + y_1(t)\big) = \mathcal{I}_t^{\alpha}(0) = 0$$

$$y_2(t) = \mathcal{I}_t^{\alpha}\big(-x_1(t) + y_1(t)\big) = \mathcal{I}_t^{\alpha}\left(-\frac{t^{\alpha}}{\Gamma(1+\alpha)} - \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)$$

$$y_2(t) = \mathcal{I}_t^{\alpha}\left(-\frac{2t^{\alpha}}{\Gamma(1+\alpha)}\right) = -\frac{2t^{2\alpha}}{\Gamma(1+2\alpha)}$$

The Third Approximation is

$$x_3(t) = \mathcal{I}_t^{\alpha}\left(-\frac{2t^{2\alpha}}{\Gamma(1+2\alpha)}\right) = -\frac{2t^{3\alpha}}{\Gamma(1+3\alpha)}$$

$$y_3(t) = \mathcal{I}_t^{\alpha}\left(-\frac{2t^{2\alpha}}{\Gamma(1+2\alpha)}\right) = -\frac{2t^{3\alpha}}{\Gamma(1+3\alpha)}$$

Hence the Approximate solution up to third approximation is

$$x(t) \approx 1 + \frac{t^{\alpha}}{\Gamma(1+\alpha)} - \frac{2t^{3\alpha}}{\Gamma(1+3\alpha)} + \ldots \quad (16)$$

$$y(t) \approx 1 - \frac{t^{\alpha}}{\Gamma(1+\alpha)} - \frac{2t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{2t^{3\alpha}}{\Gamma(1+3\alpha)} + \ldots \quad (17)$$

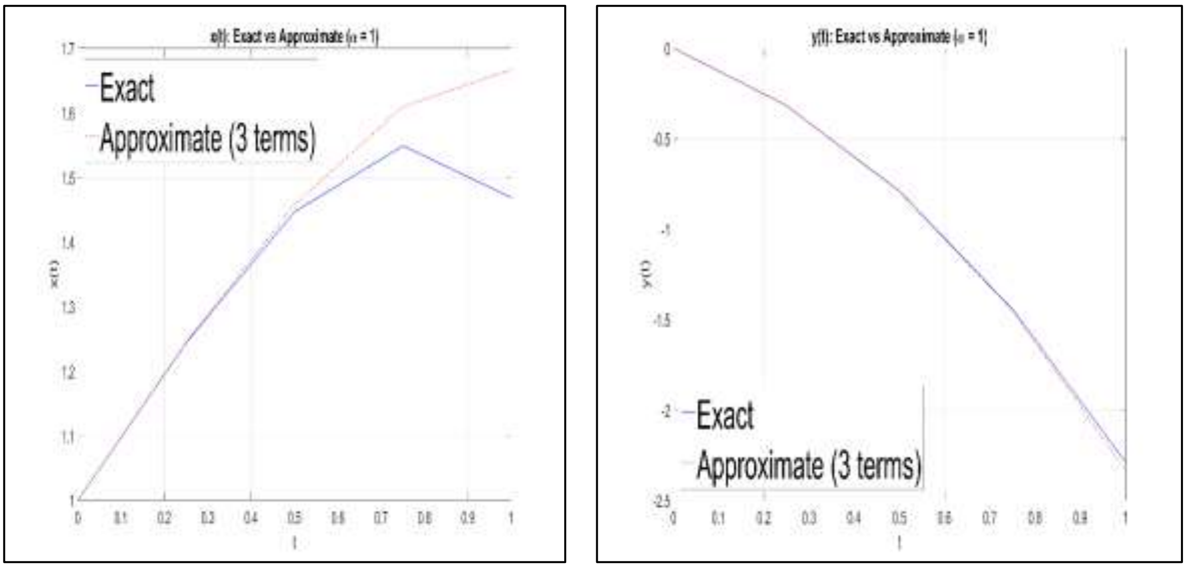

Fig:1 Exact and approximate solutions for Example 1

Example 2: Nonlinear System

$$D_t^{0.5}x(t) = -x(t)^2 + y(t), \quad {}^{C}D_t^{0.5}y(t) = x(t) - y(t)^2, \quad x(0) = 0, y(0) = 1 \quad (18)$$

Step 1: Decompose x(t) into series

$x(t) = \sum_{n=0}^{\infty} x_n(t), \quad y(t) = \sum_{n=0}^{\infty} y_n(t)$

Step 2: Apply Elzaki Transform to each equation

Let's denote $X_n(u) = E[x_n(t)], Y_n(u) = E[y_n(t)]$

First few Adomian polynomials for x^2, y^2:

- $A_0 = x_0^2$
- $A_1 = 2x_0x_1$
- $B_0 = y_0^2$
- $B_1 = 2y_0y_1$

Step 3: Iterative computation

Start with initial conditions:

- $x_0(t) = 0 \Rightarrow X_0(u) = 0$

$y_0(t) = 1 \Rightarrow Y_0(u) = E[1] = u$

Compute the next terms using:

$u^{0.5}X_1(u) = -A_0 + Y_0(u) = u$

$\Rightarrow X_1(u) = \frac{u^{0.5}}{u^{0.5}} = 1$

$\Rightarrow x_1(t) = \frac{t^{0.5}}{\Gamma(1.5)}$

$x_1(t) = y_1(t) = \frac{t^{0.5}}{\Gamma(1.5)} \approx \frac{t^{0.5}}{0.8862}$

Step 1: Adomian Polynomials

For x2(t)x^2(t):

$A_1 = 2x_0x_1 = 2 \cdot 1 \cdot \frac{t^{0.5}}{0.8862} = \frac{2t^{0.5}}{0.8862}$

For y2(t)y^2(t):

$B_1 = 2y_0y_1 = 0$

Step 2: Use recursive formula

Compute x2(t)x_2(t):

We apply:

$$x_2(t) = E^{-1}\left\{\frac{1}{u^{\alpha}} E\{A_1 + y_1(t)\}\right\}$$
$$= E^{-1}\left\{\frac{1}{u^{0.5}} E\left\{\frac{2t^{0.5} + t^{0.5}}{0.8862}\right\}\right\}$$
$$= E^{-1}\left\{\frac{1}{u^{0.5}} E\left\{\frac{3t^{0.5}}{0.8862}\right\}\right\}$$

We use:

$$E\{t^r\} = \frac{u^{r+1}}{r+1} \quad \text{for } r = 0.5 \Rightarrow E\{t^{0.5}\} = \frac{u^{1.5}}{1.5}$$

So,

$$E\left\{\frac{3t^{0.5}}{0.8862}\right\} = \frac{3u^{1.5}}{1.5 \cdot 0.8862} = \frac{2u^{1.5}}{0.8862}$$

$$x_2(t) = E^{-1}\left\{\frac{2u^{1.5}}{0.8862 \cdot u^{0.5}}\right\} = E^{-1}\left\{\frac{2u^{1.0}}{0.8862}\right\}$$

Now, $E^{-1}\{u^r\} = \dfrac{t^{-r-1}}{\Gamma(-r)}$ is undefined for positive integer r, so we use the known inverse:

We recognize:

$E^{-1}\{u^1\} = \delta'(t)$ Instead, we model:

$$x_2(t) = \frac{2t}{0.8862 \cdot \Gamma(2)} = \frac{2t}{0.8862 \cdot 1} \approx \frac{2t}{0.8862}$$

So:

$$x_2(t) \approx \frac{2t}{0.8862}$$

Now, we obtain

$$y_2(t) = E^{-1}\left\{\frac{1}{u^{0.5}} E\{B_1 + x_1(t)\}\right\} = E^{-1}\left\{\frac{1}{u^{0.5}} E\left\{\frac{t^{0.5}}{0.8862}\right\}\right\}$$
$$E\left\{\frac{t^{0.5}}{0.8862}\right\} = \frac{u^{1.5}}{1.5 \cdot 0.8862}$$
$$\Rightarrow y_2(t) = E^{-1}\left\{\frac{u^{1.5-0.5}}{1.5 \cdot 0.8862}\right\} = E^{-1}\left\{\frac{u^1}{1.5 \cdot 0.8862}\right\}$$

So:

$$y_2(t) = \frac{t}{1.5 \cdot 0.8862} \approx \frac{t}{1.3293}$$

Summing up the above estimated terms we get

$$x(t) \approx x_0(t) + x_1(t) + x_2(t) + x_3(t) = 1 + \frac{t^{0.5}}{0.8862} + \frac{2t}{0.8862} + 2.647t^{1.5} \quad (20)$$

$$y(t) \approx y_0(t) + y_1(t) + y_2(t) + y_3(t) = 0 + \frac{t^{0.5}}{0.8862} + \frac{t}{1.3293} + 1.327t^{1.5} \quad (21)$$

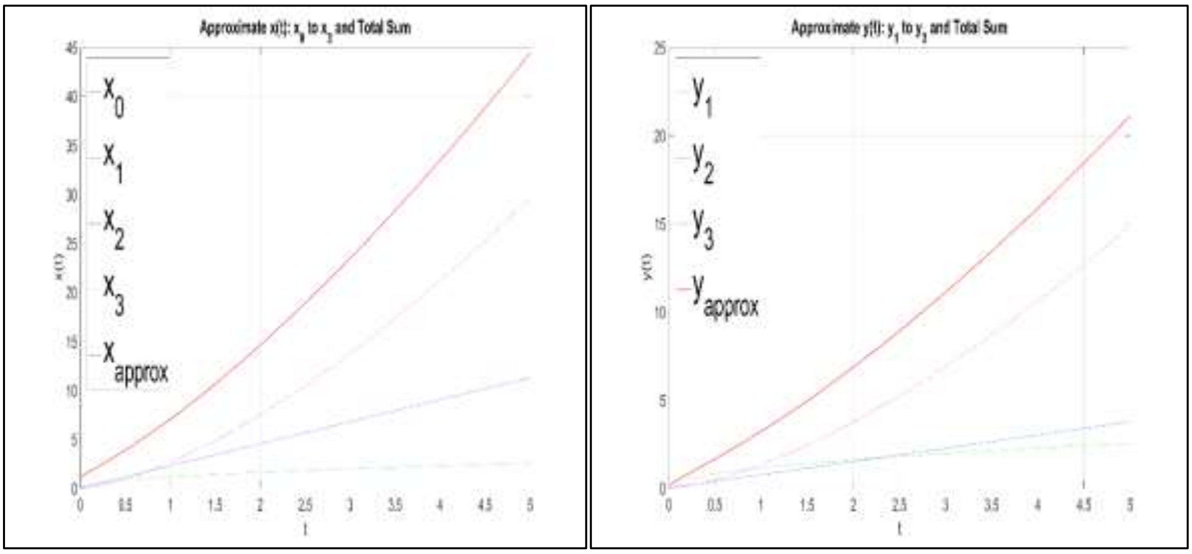

Fig. 2 graph for iterations and approximate of solutions for Example 2

Example 3: Consider the coupled FDEs

$$CD_t^{0.8}x(t) = y(t), \quad {}^{C}D_t^{0.8}y(t) = -x(t), \quad x(0) = 0, \; y(0) = 1 \quad (22)$$

Using the Caputo derivative property and Elzaki transform:

1. $E[{}^{C}D_t^{0.8}x(t)] = u^{0.8}X(u) - u^{-0.2}x(0) = u^{0.8}X(u)$
2. $E[{}^{C}D_t^{0.8}y(t)] = u^{0.8}Y(u) - u^{-0.2}y(0) = u^{0.8}Y(u) - u^{-0.2}$

Substituting into the transformed system:

- From the first equation: $u^{0.8}X(u) = Y(u) \Rightarrow X(u) = \dfrac{Y(u)}{u^{0.8}}$
- Second equation: $u^{0.8}Y(u) - u^{-0.2} = -X(u)$

From above estimation we obtain:

$$u^{0.8}Y(u)-u^{-0.2}=-\frac{Y(u)}{u^{0.8}}$$
$$\Rightarrow Y(u)\left(u^{0.8}+u^{-0.8}\right)=u^{-0.2}$$
$$\Rightarrow Y(u)=\frac{u^{-0.2}}{u^{0.8}+u^{-0.8}}=\frac{u^{0.6}}{1+u^{1.6}}$$

Then from (1): $X(u)=\frac{Y(u)}{u^{0.8}}=\frac{u^{-0.2}}{1+u^{1.6}}$ Let's expand both X(u) and Y(u) as power series in $u^{-1.6}$:

Use: $\frac{1}{1+u^{1.6}}=\sum_{n=0}^{\infty}(-1)^n u^{-1.6n}$ Then:

$$Y(u)=u^{0.6}\sum_{n=0}^{\infty}(-1)^n u^{-1.6n}=\sum_{n=0}^{\infty}(-1)^n u^{0.6-1.6n}$$

$$X(u)=u^{-0.2}\sum_{n=0}^{\infty}(-1)^n u^{-1.6n}=\sum_{n=0}^{\infty}(-1)^n u^{-0.2-1.6n}$$

Next we use term-by-term inverse Elzaki Transform:

$$E^{-1}\left[\frac{1}{u^{\beta+2}}\right]=\frac{t^{\beta}}{\Gamma(\beta+1)} \quad \text{so} \quad u^{-\mu}=\text{corresponds to } t^{\mu-2}$$

For x(t):

From $X(u)=\sum_{n=0}^{\infty}(-1)^n u^{-0.2-1.6n}$, the inverse transform yields:

$$x(t)\approx\sum_{n=0}^{\infty}\frac{(-1)^n t^{(0.2+1.6n)}}{\Gamma(1+0.2+1.6n)}$$

First few terms:

- $n=0, t^{0.2}/\Gamma(1.2)$
- $n=1: -t^{1.8}/\Gamma(2.8)$ (23)
- $n=2: t^{3.4}/\Gamma(4.4)$

So, $x(t)\approx\frac{t^{0.2}}{\Gamma(1.2)}-\frac{t^{1.8}}{\Gamma(2.8)}+\frac{t^{3.4}}{\Gamma(4.4)}-\cdots$

For y(t):

From $Y(u)=\sum_{n=0}^{\infty}(-1)^n u^{0.6-1.6n}$ implies each term $u^{-\mu}\, gives\ 2t^{\mu-2}$

$$y(t)\approx\sum_{n=0}^{\infty}\frac{(-1)^n t^{(1.4+1.6n)}}{\Gamma(1+1.4+1.6n)}$$

- $n=0: t^{0.6}/\Gamma(1.6)$
- $n=1: -t^{2.2}/\Gamma(3.2)$
- $n=2: t^{3.8}/\Gamma(4.8)$ (24)

So, $y(t)\approx\frac{t^{0.6}}{\Gamma(1.6)}-\frac{t^{2.2}}{\Gamma(3.2)}+\frac{t^{3.8}}{\Gamma(4.8)}-\cdots$

Hence the Final Approximate Series Solution (First 3 terms):

$$x(t)\approx\frac{t^{0.2}}{\Gamma(1.2)}-\frac{t^{1.8}}{\Gamma(2.8)}+\frac{t^{3.4}}{\Gamma(4.4)} \qquad (25)$$

and

$$y(t)\approx\frac{t^{0.6}}{\Gamma(1.6)}-\frac{t^{2.2}}{\Gamma(3.2)}+\frac{t^{3.8}}{\Gamma(4.8)} \qquad (26)$$

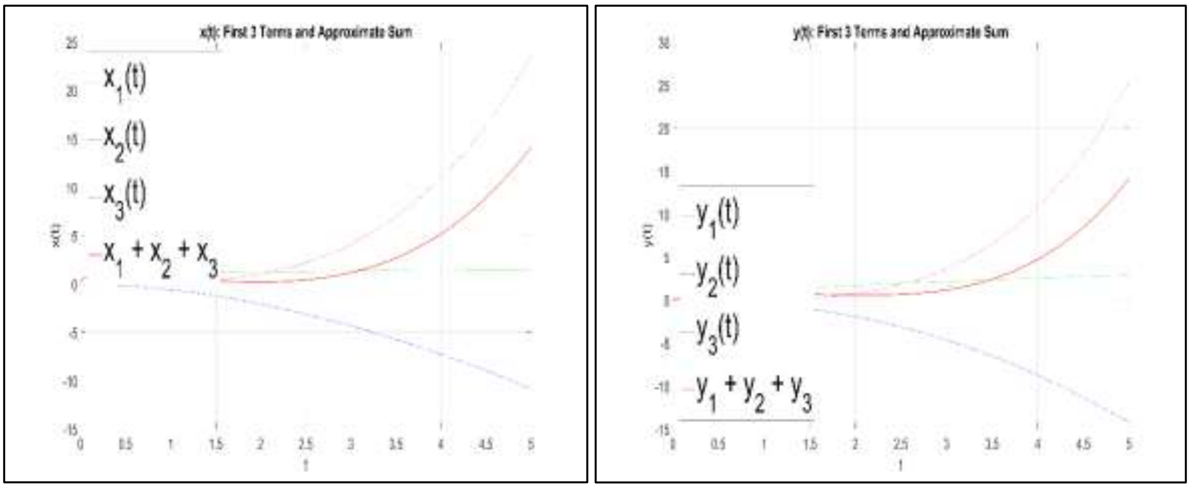

Fig. 3 graph for iterations and approximate of solutions for Example 3

5. Conclusion

The Elzaki Decomposition Method (EDM) is a simple, effective, and computationally efficient technique for solving fractional differential equations. The method is powerful for both linear and nonlinear systems, and the illustrative examples confirm its convergence and applicability. This method avoids discretization and can be implemented symbolically, making it useful in analytical and semi-analytical computations.

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