

Solution of Nonlinear Fractional Differential Equation Using Picard's Iteration Method with Integrating Factor Approach

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Abstract

In this Article, An approximate Solution of Nonlinear Differential Equations of non-integer order using Picard's iteration method based on Integrating Factor approach is obtained and gives solution for fractional order α ($0 < \alpha < 1$) in infinite series form. Results obtained using proposed method are presented and compared with classical Picard's iteration method.

Keywords

Fractional Differential Operator, Riemann Liouville Fractional Integral, Riemann Liouville Fractional Derivative, FDE, Fractional Calculus, Numerical Solution, Lipchitz condition, Graphical Representation, MATLAB, Picard's Iterative Method, Integrating factor.

1. Introduction

Differential equations play a crucial role in modelling real-world phenomena, appearing in both linear and nonlinear forms depending on the nature of the system being studied. Since most scientific processes are inherently nonlinear, exact analytical solutions exist only for a relatively small subset of such problems [1]. As a result, many nonlinear systems require the use of numerical or approximation analytical methods [1].

As our scientific knowledge and technology progress, a growing number of phenomena cannot be effectively described using conventional differential equations. Numerous physical systems display behavior characterized by memory or hereditary effects, which traditional local operators struggle to accurately model [1] [6]. To tackle these issues, the use of fractional differential equations has gained traction—these models utilize nonlocal operators that provide a more accurate representation of such processes.

Since exact analytical solutions rarely exist for most fractional differential equations, analytical approximations and numerical techniques have become essential. In recent years, numerous effective methods for solving FDEs have been introduced [1] [6].

When nonlinear fractional differential equations lack closed-form solutions, numerical procedures are used to obtain approximate results. Examples include Picard's iteration [1] [9], Taylor series expansions [11], and Lagrange polynomial methods [2]. Generally, numerical methods provide higher accuracy compared with approximate analytical approaches.

In recent times, many techniques have been created to find numerical solutions for different kinds of fractional differential equations. Eman Ali Ahmed Ziada introduced a method for addressing nonlinear fractional differential equations through the Adomian Decomposition Method (ADM) [1] [10]. The research offers an approximate solution to the nonlinear fractional model and assesses its precision by contrasting it with the exact solution.

Using Taylor basis functions, Athassawat Kammanee devised a numerical method for solving fractional differential equations with variable coefficients [1] [11]. This approach provides a structured framework for producing precise numerical approximations and has a straightforward computing process.

With the help of controlled Picard method that incorporates the Atangana–Baleanu fractional derivative, authors obtained an approximate answer strategy for Riccati equations of fractional order [1][3]. Their approach is versatile, functioning effectively for both integer- and fractional-order differential equations. The inclusion of an auxiliary parameter enhances convergence, making the method particularly suitable for nonlinear problems.

Another method for solving differential equations was presented by the authors, which combines Picard's Fixed Point Iteration with the Integrating Factor Method [16]. The performance of this modified Picard approach has been assessed by comparing the obtained numerical results with exact solutions, as shown in tabulated data. Furthermore, its convergence characteristics were examined relative to the traditional Picard method. In this chapter, the integrating factor technique is employed together with Picard's iterative scheme to derive approximate solutions for nonlinear fractional differential equations [1] [26].

2. Preliminaries

There exist multiple definitions for fractional derivatives and integrals, making this section important for outlining the fractional calculus employed [1]. The Caputo derivative and the Riemann-Liouville integral are prominent and commonly utilized. In this work, we concentrate on the Riemann-Liouville fractional integral [1].

DEFINITION 2.1

The fractional integral operator of order α , referred to as I^α in the Riemann-Liouville sense, is defined as follows:

$$I^\alpha u(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds, & \alpha > 0 \\ u(t), & \alpha = 0 \end{cases}$$

Being a linear operator is a crucial characteristic of the Riemann-Liouville fractional integral [1].

$$I^\alpha (\lambda f(t) + g(t)) = \lambda I^\alpha f(t) + I^\alpha g(t)$$

Additionally, how the integrand's powers are affected [1].

$$I^\alpha t^n = \frac{\Gamma(n+1)}{\Gamma(n+1+\alpha)} t^{n+\alpha}, \text{ for } n > -1$$

$$I^\alpha (e^{kt}) = \frac{e^{kt}}{k^\alpha}, \text{ where } k \text{ is a constant.}$$

3 Existence of Fractional order Picard's iterative method with integrating factor approach

Let's take Caputo fractional initial value problem (IVP) of a type [1] [13]

$$\begin{cases} D^\alpha y(t) + p(t)y = g(t, y) \\ y(t_0) = y_0 \end{cases} \quad (3.1)$$

Using integrating factor (I.F.), equation (3.1) can be,

$$ye^{\int p(t)dt} = e^{\int p(t)dt} g(t, y).$$

Applying Picard's approach we get

$$y_{n+1} = y_0 K e^{-\int p(t)dt} + e^{-\int p(t)dt} \int_{t_0}^t e^{\int p(x)dx} g(x, y_n) dx$$

where $K = [e^{\int p(t)dt}]_{t=t_0}$

$$y_{n+1} = y_0 K e^{-\int p(t)dt} + \int_{t_0}^t e^{\int p(x)-p(t)dx} g(x, y_n) dx.$$

Picard's iterative method

Consider the IVP.

$$\begin{cases} y' = f(t, y), \\ y(t_0) = y_0. \end{cases} \quad (3.2)$$

Where non-linear term satisfy Lipschitz condition ([1],[7]).

Solution of (3.2) using Picard's iteration method is given by

$$y(t) = y_0 + \int_{t_0}^t f(t, y(t)) dt$$

The limit of a series of functions $y_n(t)$ produced by the recurrence formula [1] yields the solution $y(t)$

$$y_n(t) = y_0 + \int_{t_0}^t f(t, y_{n-1}(t)) dt, \quad n = 1, 2, 3 \dots$$

Picard's iterative method (Based on Integrating Factor Approach)

Consider a class of equation of fractional order.

$$\begin{cases} D^\alpha y(t) + p(t)y = g(t, y), \\ y(t_0) = y_0 \end{cases} \quad (3.3)$$

Where non-linear term satisfy Lipschitz condition ([1],[7]).

Using integrating factor, equation (3.3) can be written as [16]

$$y e^{\int p(t)dt} = e^{\int p(t)dt} g(t, y)$$

Applying Picard's approach we have

$$y_{n+1} = y_0 K e^{-\int p(t)dt} + e^{-\int p(t)dt} \int_{t_0}^t e^{\int p(x)dx} g(x, y_n) dx$$

where $K = [e^{\int p(t)dt}]_{t=t_0}$

4 Numerical examples

Example 1: Consider nonlinear FDE.

$$D^\alpha y(x) = xy + x + y^2 \quad (4.1)$$

with initial condition $y(0) = 0$, fractional order α where $(0 < \alpha < 1)$

Picard's method

Solution: Approximate solution of (4.1) by Picard's iteration method in infinite series form,

$$\text{Formula : } y_n = y_0 + \int_{x_0}^x f(x, y_{n-1}) d^\alpha x$$

using $n=1$ and the initial condition $x_0 = 0, y_0 = 0$, we get

$$y_1(x) = y_0 + \int_{x_0}^x f(x, y_0) d^\alpha x$$

Using Reimann Liouville fractional order integration [12], we get

$$y_1(x) = \frac{\Gamma 1}{\Gamma(2+\alpha)} x^{1+\alpha}$$

$$y_2(x) = y_0 + \int_0^x f(x, y_1) d^\alpha x$$

$$= \frac{\Gamma 1}{\Gamma(2+\alpha)\Gamma(3+2\alpha)} x^{2+2\alpha} + \left(\frac{\Gamma 1}{\Gamma(2+\alpha)} x^{1+\alpha} \right) + x^{2+3\alpha} \frac{1}{\Gamma(2+\alpha)^2\Gamma(3\alpha+3)}$$

$$y_3(x) = y_0 + \int_0^x f(x, y_2) d^\alpha x$$

$$= \frac{\Gamma 1}{\Gamma(2+\alpha)} x^{1+\alpha} + \frac{x^{3+4\alpha}}{\Gamma(2+\alpha)^2\Gamma(3+3\alpha)\Gamma(4+4\alpha)} + \frac{x^{4+5\alpha}}{\Gamma(2+\alpha)^2\Gamma(3+3\alpha)^2\Gamma(5+5\alpha)} + \frac{x^{2+3\alpha}}{\Gamma(3+3\alpha)\Gamma(2+\alpha)^2} + \frac{x^{4+7\alpha}}{\Gamma(2+\alpha)^4\Gamma(3+3\alpha)^2\Gamma(5+7\alpha)}$$

and so on.

We can find approximate solution by putting different value of x and $\alpha[0,1]$ in above solution.

Picard's Iterative Method[1] [16] (Applying integrating factor approach)

Consider IVP,

$$D^\alpha y(x) = xy + x + y^2, \quad y(0) = 0.$$

On comparing with standard form, we get

$$D^\alpha y(x) + P(x)y = f(x, y)$$

$$P(x) = -x, \quad f(x, y) = x + y^2$$

$$\text{Integrating Factor (I.F.)} = e^{\int -x d^\alpha x} = e^{\frac{-x^{1+\alpha}}{\Gamma 2+\alpha}}$$

Solution of IVP (4.4.1) using Picard's iteration method with integrating factor is given by

$$y_n I.F. = y_0 + \int_{x_0}^x I.F. f(x, y_{n-1}) d^\alpha x$$

Substituting $n=1$ and $x_0 = 0, y_0 = 0$, we get

$$y_1(x) I.F. = y_0 + \int_{x_0}^x I.F. f(x, y_0) d^\alpha x$$

$$y_1(x) e^{\frac{-x^{1+\alpha}}{\Gamma(2+\alpha)}} = \int_{x_0}^x e^{\frac{-x^{1+\alpha}}{\Gamma(2+\alpha)}} f(x, y_0) d^\alpha x$$

$$y_1(x) = \frac{x^{1+\alpha}}{\Gamma(2+\alpha)}$$

$$y_2(x) I.F. = y_0 + \int_0^x I.F. f(x, y_1) d^\alpha x$$

$$y_2(x) = \frac{x^{1+\alpha}}{\Gamma(2+\alpha)} + \frac{x^{2+3\alpha}}{\Gamma(2+\alpha)^2\Gamma 3 + 3\alpha}$$

$$y_3(x) = \frac{x^{1+\alpha}}{\Gamma(2+\alpha)} + \frac{x^{2+3\alpha}}{\Gamma(2+\alpha)^2\Gamma 3 + 3\alpha} + \frac{x^{4+7\alpha}}{\Gamma(2+\alpha)^4\Gamma(3+3\alpha)^2\Gamma(5+7\alpha)} + \frac{2x^{3+5\alpha}}{\Gamma(2+\alpha)^3\Gamma(3+3\alpha)\Gamma(4+5\alpha)}$$

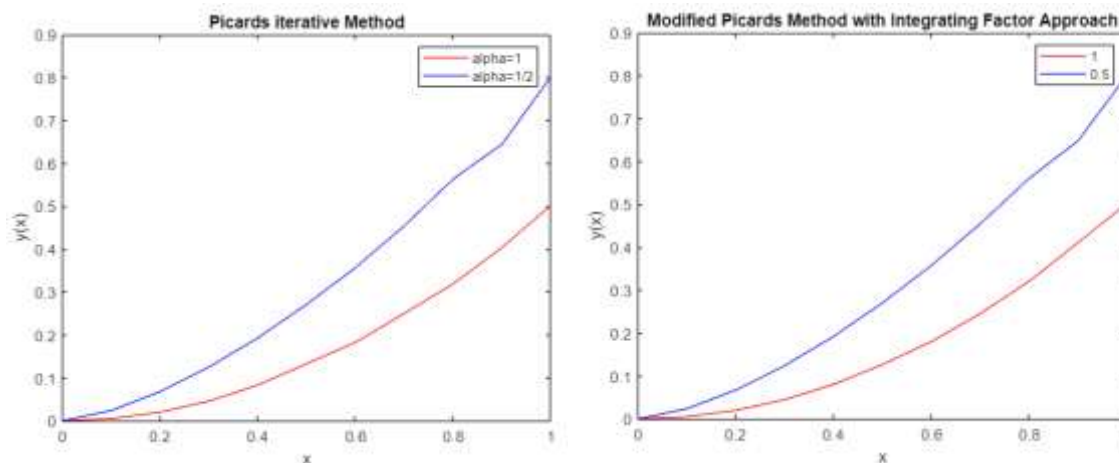
and so on.

Table 2: Comparison between results obtained with presented method and Picard's iteration method.

x	$\alpha = 1$		$\alpha = 1/2$	
	Picard's Iterative method	Picard's Method with I.F.	Picard's Iterative method	Picard's Method with I.F.
0	0	0	0	0
0.1	5.013×10^{-3}	5×10^{-3}	0.023920	0.0238037
0.2	2.02×10^{-2}	2×10^{-2}	0.068288	0.06746
0.3	4.603×10^{-2}	4.512×10^{-2}	0.125210	0.12433
0.4	8.32×10^{-2}	8.051×10^{-2}	0.193270	0.192277
0.5	0.1332	0.1266	0.271234	0.270267
0.6	0.18270	0.180162	0.35701	0.357755
0.7	0.25000	0.245350	0.45402	0.454527
0.8	0.318769	0.32068	0.563338	0.560547
0.9	0.403567	0.41076	0.644279	0.6476748
1.0	0.501696	0.50208	0.800562	0.800908

5 Graphical Representation

Figure 1: Uses Picard's approach and Picard's method with Integrating Factor to represent the IVP (4.1) solution.



Example 2: Consider nonlinear Riccati differential equation of fractional order.

$$\left. \begin{aligned} D^\alpha y(t) &= 2y(t) - y^2(t) + 1, \\ y(0) &= 0 \end{aligned} \right\} \quad (5.1)$$

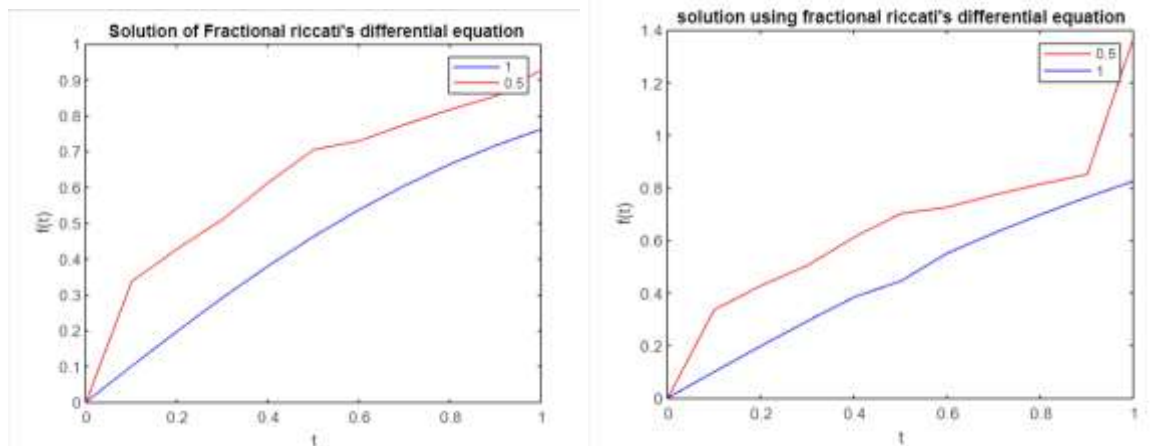
Solution of (5.1) using Picard's method with I.F. is given by

$$y(t) = \frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)} \frac{t^{3\alpha}}{\Gamma(1+\alpha)^2} - \left(\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} \right)^2 t^{5\alpha} \frac{\Gamma(5\alpha+1)}{\Gamma(6\alpha+1)} + \frac{2}{\Gamma(1+2\alpha)} \cdot t^{4\alpha} \frac{\Gamma(4\alpha+1)}{\Gamma(5\alpha+1)} \dots$$

Table 3: Comparison between Results obtained with Modified Picard's Method and classical Picard's Iterative Method. ($\alpha = 0.5$)

t	Picard's Iterative Method		Picard's Method using Integrating Factor	
	($\alpha = 1$)	($\alpha = 0.5$)	($\alpha = 1$)	($\alpha = 0.5$)
0	0	0	0	0
0.1	0.0996680	0.3360562	0.0996842	0.33604663
0.2	0.1973753	0.4272109	0.1976399	0.42686749
0.3	0.2913126	0.5098723	0.2925187	0.50464705
0.4	0.3799490	0.6123543	0.3833633	0.61109472
0.5	0.4621172	0.7045361	0.4445312	0.70123721
0.6	0.5370496	0.7287650	0.5506800	0.72531386
0.7	0.6043678	0.7745632	0.6266871	0.77164970
0.8	0.6640368	0.8167854	0.697600	0.81322282
0.9	0.7162979	0.8542109	0.7636162	0.85040792
1.0	0.7616080	0.92773134	0.8250334	1.37418563

Figure 2: Shows solution of IVP (5.1) using both Picard's technique and Picard's method with Integrating Factor.



6 Conclusion

This study investigates the resolution of non-linear differential equations of non-integer order by utilizing Picard's Iteration Method in conjunction with the Integrating Factor technique. Numerical examples are solved to showcase the precision of the proposed method. The approach is simple and particularly well-suited for tackling nonlinear fractional order issues, providing an easier option compared to other semi-analytic techniques. Importantly, it eliminates the requirement to compute generic Lagrange multipliers or Adomian polynomials. This method can also be applied to nonlinear systems of fractional differential equations for additional problem-solving scenarios.

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