

Some Results on Generalized Codeword Length of Variable Length Error-Correcting Codes

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Abstract: Codes of variable length that are capable of error correction are studied. A lower bound on $L_{\alpha}^{\beta}(U)$, the generalized mean length of such codes under the criterion of “promptness” is obtained. This generalizes the result due to Bernard and Sharma [5] and Baig and Zaheeruddin [4] for noisy channels. Also this generalizes the result due to Kerridge [9], which itself is a generalization of celebrated result due to Shannon [12] for noiseless channel. The bound obtained provides a measure of optimality for variable length error correcting codes.

1.Introduction

It is well known that a lower bound on the average length is obtained in terms of Shannon entropy [12] for instantaneous codes in noiseless channel (Abramson [1]). Bernard and Sharma [6] studied variable length codes for noisy channels and presented some combinatorial bounds for these variable length, error correcting codes. Bernard and Sharma [5] obtained a lower bound on average length for variable length error correcting codes satisfying the criterion of “promptness”.

In this paper a lower bound of generalized code length of order α and type β for variable length error correcting codes is obtained in terms of a measure similar to that of ‘useful’ inaccuracy of order α and type β given by Tuteja and Bhaker [14]. This generalizes the result due to Bernard and Sharma [5], Baig and Zaheeruddin [4] and well known Shannon’s inequality.

Key words and Phrases: Generalized Parametric mean length, Inaccuracy measure, Holder’s inequality, Segment decomposition.

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2. Definitions and Concepts:

In coding theory, it is assumed that Q is a finite set of alphabets and there are D code characters. A code is defined as a finite sequence of code characters and variable length code C of size n is a set of n code characters denoted by $c_1, c_2, \dots, c_n$ with lengths $l_1, l_2, \dots, l_n$ respectively. Without loss of generality it may be assumed that $l_1 \leq l_2 \leq \dots \leq l_n$.

The channel, which is considered here, is not noiseless. In other words the codes considered here are error correcting codes. The criterion for error correcting is defined in terms of a mapping α , which depends on the noise characteristics of the channel. This mapping α is called the error admissibility mapping. Given a code word c and error admissibility α , the set of code words received over the channel when c was sent, denoted by $\alpha(c)$ is the error range of c .

Various kinds of error pattern can be described in terms of mapping α . In particular α may be defined by Bernard and Sharma [6] as

$$\alpha_e(c) = \{u / W(c - u) \leq e\}$$

where e is the random substitution error and $W(c - u)$ is the Hamming weight i.e., the number of non zero co-ordinate of $c - u$. Bernard and Sharma [6] verified that the number of sequences in $\alpha_e(c)$ denoted as $|\alpha_e(c)|$ is given by

$$|\alpha_e(c)| = \sum_{i=0}^n \binom{l}{i} (D-1)^i$$

where l is the length of codeword c .

we may assume that α_0 corresponds to the noiseless cases. In other words if c is sent then c is received with respect to α_0 . Moreover it is clear that $|\alpha_e(c)|$ depends only on the length l of c when α and D are given. In noiseless coding, the class of uniquely decodable instantaneous codes are studied. It is known that these codes satisfy prefix property (Abramson [1]).

In the same way Hartnett [8] studied variable length code over noisy channel, satisfying the prefix property in the range. The codes are called α -prompt codes. Such codes have the property that they can be decoded 'promptly'.

Further, Bernard and Sharma [6] gave a combinatorial inequality that must necessarily be satisfied by codeword length of prompt codes. Two useful concepts namely, segment decomposition and the effective range $r_\alpha(c_i)$ of codeword c_i of length l_i under error mapping α as the cartesian product of ranges of the segment are also given by Bernard and Sharma [6]. The number of sequences in effective range of c_i denoted by $|r_\alpha|_{l_i}$ depends only on α and l_i .

It is given by

$$|r_\alpha|_{l_i} = |\alpha|_{l_1} |\alpha|_{l_2-l_1} \dots |\alpha|_{l_i-l_{i-1}}$$

Moreover, Bernard and Sharma [6] obtained the following inequality.

Theorem 1: An α -prompt code with n codewords of length $l_1 \leq l_2 \leq \dots \leq l_n$, satisfies the following inequality

$$(2.1) \quad \sum_{i=1}^n |r_\alpha|_{l_i} D^{-l_i} \leq 1$$

Remark 2.1: If the codes of constant length l are taken then the average inequality (2.1) reduces to Hamming sphere packing bound given by Hamming [7].

Remark 2.2: If the channel is noiseless then the inequality (2.1) reduces to well known Kraft [10] inequality.

Bernard and Sharma [5] have obtained a lower bound on average codeword length for prompt codes using a quantity similar to Shannon entropy. Also, Baig and Zaheeruddin [4] have obtained a lower bound on codeword length of order t of prompt codes using a quantity similar to Nath's [11] inaccuracy of order β . In the next section, we obtain a lower bound on generalized codeword length of order α and type β of prompt codes using a quantity similar to Tuteja and Bhaker [14] 'useful' inaccuracy measure of order α and type β .

3. A lower bound on codeword length of order α and type β of prompt codes.

Consider a generalized code length of order α and type β defined by

$$(3.1) \quad L_{\alpha}^{\beta}(U) = \frac{1}{(2^{1-\alpha} - 1) \sum_{i=1}^n p_i^{\beta}} \left[\frac{\sum_{i=1}^n (u_i p_i)^{\beta} D^{l_i \left(\frac{1-\alpha}{\alpha} \right)} \right]^{\alpha} - 1, \alpha \neq 1, \alpha > 0, \beta > 0$$

it is easy to see that for $\beta = 1, \alpha \rightarrow 1$ and $u_i = 1 \quad \forall i = 1, 2, \dots, n$. It reduces to $L = \sum_{i=1}^n l_i p_i$ given by Shannon [12].

Suppose that a person believe that the probability of i^{th} event is q_i and the code with length l_i has been constructed accordingly. But contrary to his belief the true probability is p_i .

We will now obtain a lower bound of generalized mean length $L_{\alpha}^{\beta}(U)$ under the condition

$$(3.2) \quad \sum_{i=1}^n (u_i p_i)^{\beta} q_i^{-1} D^{-l_i} \left(|r_{\alpha}|_{l_i} \right) \leq \sum_{i=1}^n (u_i p_i)^{\beta}$$

Remark 3.1: Moreover if $\beta = 1, u_i = 1 \quad \forall i = 1, 2, \dots, n$. Then (3.3) reduces to the inequality given by Autar and Soni [2]

$$(3.3) \quad \sum_{i=1}^n p_i q_i^{-1} D^{-l_i} \leq 1$$

Remark 3.2: Moreover if $p_i = q_i \quad \forall i = 1, 2, \dots, n$. (3.3) reduces to Kraft [10] inequality

$$(3.4) \quad \sum_{i=1}^n D^{-l_i} \leq 1$$

Remark 3.3: If $\beta = 1, u_i = 1 \quad \forall i = 1, 2, \dots, n$ in (3.2), It reduces to the inequality given by Baig and Zaheeruddin [4]

$$(3.5) \quad \sum_{i=1}^n p_i q_i^{-1} |r_{\alpha}|_{l_i} D^{-l_i} \leq 1$$

Remark 3.4: Moreover if $p_i = q_i \quad \forall i = 1, 2, \dots, n$ in (3.5), It reduces to the inequality given by Bernard and Sharma [6]

$$(3.6) \quad \sum_{i=1}^n |r_{\alpha}|_{l_i} D^{-l_i} \leq 1$$

Theorem 2: Let an α - prompt code encode the n messages $s_1, s_2, \dots, s_n$ into a code alphabet of D symbols and let the length of the corresponding to message s_i be l_i . Then the generalized code length of order α and type $\beta, L_{\alpha}^{\beta}(U)$ shall satisfies the inequality

$$(3.7) \quad L_{\alpha}^{\beta}(U) \geq \frac{\sum_{i=1}^n (u_i p_i)^{\beta} \left(q_i^{\alpha-1} \left(r_{\alpha} |_{l_i} \right)^{1-\alpha} - 1 \right)}{(2^{1-\alpha} - 1) \sum_{i=1}^n p_i^{\beta} \bar{U} \log D}$$

$$\text{where } \bar{U} = \sum_{i=1}^n (u_i p_i)^{\beta}$$

with equality iff

$$l_i = -\log \left(r_{\alpha} |_{l_i} \right)^{-\alpha} q_i^{\alpha} + \log \frac{\sum_{i=1}^n (u_i p_i)^{\beta} q_i^{\alpha-1} \left(r_{\alpha} |_{l_i} \right)^{1-\alpha}}{\sum_{i=1}^n (u_i p_i)^{\beta}}$$

Proof: By Holders inequality [13]

$$(3.8) \quad \sum_{i=1}^n x_i y_i \geq \left[\sum_{i=1}^n x_i^p \right]^{\frac{1}{p}} \left[\sum_{i=1}^n y_i^q \right]^{\frac{1}{q}}$$

for all $x_i > 0, y_i > 0, i = 1, 2, \dots, n$ when $p < 1 (\neq 0), q < 0$ or $q < 1 (\neq 0), p < 0$ and $\frac{1}{p} + \frac{1}{q} = 1$, with equality iff

there exists a positive number c such that $x_i^p = c y_i^q$.

$$\text{Put } x_i = \left[\frac{(u_i p_i)^{\beta}}{\sum_{i=1}^n (u_i p_i)^{\beta}} \right]^{\frac{\alpha}{\alpha-1}} D^{-l_i}, y_i = \left[\frac{(u_i p_i)^{\beta}}{\sum_{i=1}^n (u_i p_i)^{\beta}} \right]^{\frac{1}{1-\alpha}} q_i^{-1} \left(r_{\alpha} |_{l_i} \right)$$

$$p = \frac{\alpha-1}{\alpha}, q = 1-\alpha$$

in (3.8), we get

$$\frac{\sum_{i=1}^n (u_i p_i)^{\beta} D^{-l_i} q_i^{-1} \left(r_{\alpha} |_{l_i} \right)}{\sum_{i=1}^n (u_i p_i)^{\beta}} \geq \left[\frac{\sum_{i=1}^n (u_i p_i)^{\beta} D^{l_i \left(\frac{1-\alpha}{\alpha} \right)}}{\sum_{i=1}^n (u_i p_i)^{\beta}} \right]^{\frac{\alpha}{\alpha-1}} \left[\frac{\sum_{i=1}^n (u_i p_i)^{\beta} q_i^{\alpha-1} \left(r_{\alpha} |_{l_i} \right)^{1-\alpha}}{\sum_{i=1}^n (u_i p_i)^{\beta}} \right]^{\frac{1}{1-\alpha}}$$

using inequality (3.2), we get

$$\left[\frac{\sum_{i=1}^n (u_i p_i)^{\beta} D^{l_i \left(\frac{1-\alpha}{\alpha} \right)}}{\sum_{i=1}^n (u_i p_i)^{\beta}} \right]^{\frac{\alpha}{1-\alpha}} \geq \left[\frac{\sum_{i=1}^n (u_i p_i)^{\beta} q_i^{\alpha-1} \left(r_{\alpha} |_{l_i} \right)^{1-\alpha}}{\sum_{i=1}^n (u_i p_i)^{\beta}} \right]^{\frac{1}{1-\alpha}}$$

Let $0 < \alpha < 1$. Raising both sides to the power $(1-\alpha)$, and after suitable operations, we get

$$L_{\alpha}^{\beta}(U) \geq \frac{\sum_{i=1}^n (u_i p_i)^{\beta} (q_i^{\alpha-1} (|r_{\alpha}|_{l_i})^{1-\alpha} - 1)}{(2^{1-\alpha} - 1) \sum_{i=1}^n p_i^{\beta} \bar{U} \log D}, \alpha \neq 1, \alpha > 0, \beta > 0$$

4. Particular case:

(a) For $\beta = 1, \alpha \rightarrow 1$, the inequality (3.7) reduces to

$$(4.1) \quad L(U) \geq \frac{\sum_{i=1}^n (u_i p_i) \log \left(\frac{|r_{\alpha}|_{l_i}}{q_i} \right)}{\sum_{i=1}^n u_i p_i},$$

(b) For $u_i = 1 \quad \forall i = 1, 2, \dots, n$, Inequality (4.1) reduces to the inequality given by Baig and Zaheeruddin [4]

$$(4.2) \quad L \geq \sum_{i=1}^n p_i \log \left(\frac{|r_{\alpha}|_{l_i}}{q_i} \right),$$

(c) Moreover if $p_i = q_i \quad \forall i = 1, 2, \dots, n$, Inequality (4.2) reduces to inequality given by Bernard and Sharma [5]

$$(4.3) \quad L \geq \sum_{i=1}^n p_i \log \left(\frac{|r_{\alpha}|_{l_i}}{p_i} \right),$$

(d) For noiseless channel $|r_{\alpha}|_{l_i} = 1$, Inequality (4.2) reduces to inequality given by Kerridge [9]

$$(4.4) \quad L \geq - \sum_{i=1}^n p_i \log q_i,$$

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