Specification and Estimation of the Sur Equation Model with First-Order Autoregressive Errors

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Abstract

This paper addresses the specification and estimation of the Seemingly Unrelated Regressions (SUR) model when the error terms exhibit first-order autocorrelation (AR(1)). The standard SUR framework, while effectively handling contemporaneous correlation across equations, assumes serially uncorrelated disturbances. This assumption is often violated in economic and financial time-series data, leading to inefficient parameter estimates if not properly addressed. We develop and implement a feasible generalized least squares (FGLS) estimation procedure tailored for the SUR-AR(1) model. The methodology involves a multi-step process: initial estimation of the individual equation autoregressive parameters (ρi), transformation of the data to correct for serial correlation, and subsequent application of the SUR technique to the transformed model to account for contemporaneous correlation. Through Monte Carlo simulations, we demonstrate that this estimator is consistent and significantly more efficient than both equation-by-equation Ordinary Least Squares (OLS) and the traditional SUR model under conditions of first-order serial correlation. The practical application of the proposed model is illustrated with an empirical example, confirming its superiority in producing reliable and efficient estimates in real-world scenarios.

Keywords: Seemingly Unrelated Regressions (SUR), First-Order Autocorrelation, AR(1) Errors, Feasible Generalized Least Squares (FGLS), Serial Correlation, Contemporaneous Correlation, Monte Carlo Simulation.

1 Introduction

The Seemingly Unrelated Regressions (SUR) model, introduced by Zellner (1962), is a cornerstone of modern econometrics, widely employed in fields such as economics and finance. Its primary strength lies in its ability to yield efficient parameter estimates for a system of equations by accounting for the **contemporaneous correlation** of the error terms across different equations. However, the classical SUR framework relies on the stringent assumption that the error terms within each equation are serially uncorrelated over time. This assumption is frequently violated in practice, particularly when analyzing **time-series** or **panel data**, where factors like economic inertia or omitted cyclical variables often lead to **serial correlation**. Ignoring this autocorrelation renders the standard SUR estimator inefficient and can result in biased standard errors, potentially leading to flawed statistical inference and incorrect policy recommendations.

To address this critical limitation, this paper specifies and develops an estimation procedure for a SUR model where the disturbances follow a **first-order autoregressive (AR(1))** process. We propose a multi-step **feasible generalized least squares (FGLS)** estimator designed to simultaneously handle both serial correlation within equations and contemporaneous correlation across them. The methodology first involves obtaining consistent estimates of the autocorrelation parameters to transform the data, thereby purging it of serial correlation.

Subsequently, the SUR estimation technique is applied to the transformed system to achieve efficiency gains. Through **Monte Carlo simulations** and an empirical application, we demonstrate that this SUR-AR(1) estimator is consistent and substantially more efficient than both the traditional SUR model and equation-by-equation OLS when errors are autocorrelated, highlighting its practical importance for empirical researchers.

2. The SUR Model with AR(1) Errors

2.1 The Standard SUR Model

Consider a system of M regression equations, where each equation i has T observations:

$Y_i = X_i\beta_i + u_i$, i=1,2,…,M

where Y_i is a (T×1) vector of observations on the dependent variable,

X_i is a (T×K_i) matrix of non-stochastic regressors,

β_i is a (K_i×1) vector of parameters,

and ui is a (T×1) vector of disturbances.

The system can be stacked as:

$$\begin{pmatrix} y_1 \\ y_2 \\ . \\ . \\ y_M \end{pmatrix} = \begin{pmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \dots & 0 \\ . & . & \ddots & . \\ 0 & 0 & \cdots & X_M \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ . \\ . \\ \beta_M \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ . \\ . \\ u_M \end{pmatrix}$$

or more compactly as y=Xβ+u.

The standard SUR model assumes the errors are contemporaneously correlated but serially uncorrelated:

$$\mathrm{E}(u_{it}u_{js}) = \begin{cases} \sigma_{ij} & if\ t = s \\ 0, & if\ t \neq s \end{cases}$$

The covariance matrix of the stacked disturbance vector u is given by Ω=E(uu′)=Σ⊗IT,

where Σ is the (M×M) contemporaneous covariance matrix with elements σ_{ij}

and I_T is a (T×T) identity matrix.

2.2 Incorporating AR(1) Errors

We relax the assumption of serial independence and specify that the disturbance term in each equation follows a stationary first-order autoregressive process:

$u_{it} = \rho_i u_{i,t-1} + \epsilon_{it}$

where $|\rho_i|<1$ for stationarity. The innovation term ϵ_{it} is assumed to be white noise with the following properties:

$\mathrm{E}(\epsilon_{it})=0$

$$\mathrm{E}(\epsilon_{it}\epsilon_{js}) = \begin{cases} \sigma_{ij} & if\ t = s \\ 0, & if\ t \neq s \end{cases}$$

This structure implies that the innovations are serially uncorrelated but may be contemporaneously correlated across equations. The covariance matrix for the disturbances of a single equation,

$E(u_i u_i^1)$, is:

$$E(u_i u_i^1) = \frac{\sigma_{ii}}{1-\rho_i^2}\begin{pmatrix} 1 & \rho_i & \cdots & \rho_i^{T-1} \\ \rho_i & 1 & \ldots & \rho_i^{T-2} \\ . & . & \ddots & . \\ . & . & . & . \\ \rho_i^{T-1} & \rho_i^{T-2} & \cdots & 1 \end{pmatrix}$$

The full covariance matrix of the stacked disturbance vector u, denoted Ω, is a block matrix where the $(i,j)^{th}$ block is $E(u_i u_j^1)$.

This matrix is no longer $\Sigma \otimes I_T$, making standard SUR estimation inefficient.

3. Estimation Procedure

The efficient estimator for the model is the Generalized Least Squares (GLS) estimator:

$$\widehat{\beta}_{GLS} = (X^1\Omega^{-1}X)^{-1}X^1\Omega^{-1}y$$

This estimator is infeasible as Ω depends on the unknown parameters ρ_i and σ_{ij}. Therefore, we propose a multi-step Feasible GLS (FGLS) procedure.

Step 1**:** Estimate each of the M equations separately using OLS. This provides consistent, though inefficient, estimates of β_i and the residuals $\widehat{u_i}$.

Step 2: For each equation i, obtain a consistent estimate of the autocorrelation coefficient, $\widehat{\rho_i}$, by regressing the OLS residuals on their lags:

$$\widehat{u_{it}} = \rho_i\, \hat{u}_{i,t-1} + Error$$

Step 3: Transform the data for each equation using a Prais-Winsten transformation to correct for serial correlation.

For t=2,…,T:

$$y_{it}^* = y_{it} - \widehat{\rho_i} y_{i,t-1}$$

$$X_{ikt}^* = X_{ikt} - \widehat{\rho_i} X_{ik,t-1}$$

For t=1:

$$y_{i1}^* = \sqrt{1-\hat{\rho}_i^2}\, y_{i1}$$

$$X_{ik1}^* = \sqrt{1-\hat{\rho}_i^2}\, X_{ik1}$$

This transformation results in a system $y^* = c\beta + \epsilon$, where the new error vector ϵ is approximately serially uncorrelated but retains its contemporaneous correlation structure. Its covariance matrix is $\Sigma \otimes I_T$.

Step 4: Apply the standard SUR procedure to the transformed model.

a. Estimate the transformed system equation-by-equation using OLS to obtain a new set of residuals $\widehat{\epsilon_i}$.

b. Use these residuals to estimate the contemporaneous covariance matrix $\hat{\Sigma}$, where the typical element is $\widehat{\sigma_{ij}} = \frac{\hat{\epsilon}_i^1 \epsilon_j^1}{T}$

c. Construct the FGLS estimator for the SUR-AR(1) model:

$$\widehat{\beta_{FGLS}} = (X^{*1}(\hat{\Sigma}^{-1} \otimes I_T)\ X^{*})^{-1} X^{*1}(\hat{\Sigma}^{-1} \otimes I_T)\ y^{*}$$

This estimator is asymptotically efficient under standard assumptions.

4. Monte Carlo Simulation

To evaluate the finite-sample performance of the proposed estimator, we conduct a Monte Carlo study.

4.1 Experimental Design

We specify a two-equation system (M=2):

$$y_{1t} = \beta_{10} + \beta_{11} X_{11t} + u_{1t}$$

$$y_{2t} = \beta_{20} + \beta_{21} X_{21t} + u_{2t}$$

The true parameters are set to $\beta_{10} = 5$, $\beta_{11} = 1.5$, $\beta_{20} = 10$, $\beta_{21} = 2$. The regressors X_{11t} and X_{21t} are drawn from a uniform distribution U(0,20). The AR(1) errors are generated as $u_{it} = \rho_i u_{i,t-1} + \epsilon_{it}$,

where the innovations ($\epsilon_{1t}, \epsilon_{2t}$) are drawn from a bivariate normal distribution with mean zero and covariance matrix:

$$\Sigma = \begin{pmatrix} 10 & 0.7\sqrt{10.10} \\ 0.7\sqrt{10.10} & 10 \end{pmatrix}$$

We consider various scenarios by varying the sample size T∈{30,100} and the autocorrelation coefficients $\rho = \rho_1 = \rho_2 \in \{0.3, 0.6, 0.9\}$. We perform 1,000 replications for each scenario.

4.2 Results

We compare three estimators: (1) Equation-by-equation OLS, (2) Standard SUR, and (3) our proposed SUR-AR(1) FGLS. Performance is measured by the Root Mean Squared Error (RMSE).

Table 1: RMSE of Estimators for β_{11} (True Value = 1.5)

T	ρ	OLS	Std. SUR	SUR-AR(1) FGLS
30	0.3	0.251	0.239	0.203
	0.6	0.362	0.341	0.211
	0.9	0.689	0.654	0.224
100	0.3	0.133	0.125	0.110
	0.6	0.198	0.185	0.115
	0.9	0.380	0.353	0.121

The simulation results in Table 1 clearly demonstrate the superiority of the SUR-AR(1) FGLS estimator. While standard SUR provides a modest improvement over OLS by accounting for contemporaneous correlation, its performance degrades significantly as serial correlation (ρ) increases. In contrast, the SUR-AR(1) FGLS estimator maintains a low RMSE across all levels of autocorrelation. The efficiency gains are substantial, particularly in cases of high serial correlation (ρ=0.9), where the RMSE is more than halved compared to the other estimators. As expected, the performance of all estimators improves with a larger sample size (T=100).

5. Empirical Application: Investment Demand

We apply our model to a classic dataset on the investment behavior of two major U.S. firms: General Electric (GE) and Westinghouse (WH). We use annual data from 1935-1954 (T=20). It is plausible that the investment decisions of these competing firms are contemporaneously related and that firm-level investment exhibits inertia, suggesting serial correlation.

5.1 The Model

We estimate a simple investment model for each firm:

$$I_{i1} = \beta_{i0} + \beta_{i1} F_{it} + \beta_{i2} C_{it} + u_{it}$$

where I ∈{GE, WH}, I is gross investment, F is the market value of the firm, and C is the value of the capital stock.

Preliminary diagnostics on the OLS residuals showed Durbin-Watson statistics of 0.85 (GE) and 0.91 (WH), strongly indicating positive serial correlation. A Breusch-Pagan LM test on the system yielded a chi-squared statistic of 8.45 (p-value < 0.01), confirming significant contemporaneous correlation.

5.2 Estimation Results

We estimate the system using OLS, standard SUR, and our SUR-AR(1) FGLS method. Results are presented in Table 2 (standard errors in parentheses).

Table 2: Estimation Results for Investment Model

Variable	Method	General Electric (GE)	Westinghouse (WH)
Intercept	OLS	-15.2 (19.1)	-2.8 (5.9)
	Std. SUR	-10.1 (16.5)	-0.5 (4.8)
	SUR-AR(1)	-28.5 (18.8)	-4.2 (5.3)
F (Value)	OLS	0.041 (0.015)**	0.062 (0.019)**
	Std. SUR	0.035 (0.013)**	0.058 (0.016)**
	SUR-AR(1)	0.029 (0.011)*	0.051 (0.015)*
C (Capital)	OLS	0.138 (0.024)	0.081 (0.045)
	Std. SUR	0.145 (0.021)	0.089 (0.039)*
	SUR-AR(1)	0.151 (0.019)*	0.092 (0.035)*

| ρ^ | SUR-AR(1) | 0.78 | 0.69 |

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| *p <0.05, **p<0.01, ***p<0.001

The results show a marked difference between the methods. The SUR-AR(1) model estimates high degrees of serial correlation ($\hat{\rho}_{GE}$=0.78, $\hat{\rho}_{WH}$=0.69). Correcting for this has a notable impact on the coefficients and their standard errors. For Westinghouse, the effect of capital stock (C) on investment is insignificant under OLS but becomes statistically significant at the 5% level under the SUR-AR(1) model. Furthermore, the standard errors for the significant coefficients are generally smallest for the SUR-AR(1) estimator, reflecting its superior efficiency. This application demonstrates that failing to account for serial correlation can lead to different and potentially erroneous conclusions about the economic drivers of investment.

6. Conclusion

This paper addressed a common but often overlooked problem in the application of Seemingly Unrelated Regressions: the presence of serially correlated errors. We specified a SUR model with first-order autoregressive (AR(1)) disturbances and detailed a multi-step FGLS procedure for its estimation.

Our findings, supported by both Monte Carlo simulations and an empirical application, are clear. When both contemporaneous and serial correlation are present, the proposed SUR-AR(1) FGLS estimator is substantially more efficient than both equation-by-equation OLS and the standard SUR model. The simulations show that the efficiency gains increase with the degree of serial correlation. The empirical example highlights the practical importance of this approach, as correcting for autocorrelation altered the statistical significance of key economic variables.

For researchers working with time-series or panel data in a systems-of-equations context, the methodology presented here provides a robust tool for obtaining reliable and efficient estimates. Future research could extend this framework to handle higher-order autoregressive processes, moving average components, or dynamic panel data specifications with endogenous regressors.

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