Taylor Series Expansion for Finding Time Period of Oscillation in Different Fields of Physics

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Abstract –
This communication reports the use of Taylor series for finding time period of oscillation of a particle executing simple harmonic motion in different of field of Physics. This paper helps the academician to get an in-depth understanding about Taylor series and helps to understand the point about which Taylor series is expand.

Key Words: Taylor series, Maclaurin series, time period, oscillation, large amplitude

1. INTRODUCTION:

Taylor series expansion is the mathematical root of several well-known formulas across physics ranging from one-dimensional constant acceleration kinematics equations through thermodynamics, electricity & magnetism, optics, statistical mechanics, and quantum mechanics [1]. It is used ubiquitously throughout physics to help solve problems in a tractable way. In order to gain insight into an equation, a Physicist often simplifies a function by considering only the first the first two or three terms in its Taylor series. In other words, the physicist uses a Taylor polynomial on approximation to the function [2]. In chemistry, the quasi-newton method make use of a two variable Taylor's series to approximate the equilibrium geometry of a cluster of atoms (e.g. water at 4^0C). Nowadays, Taylor series is used in machine learning, deep learning and neural network problem. It is widely applied in numerical computations when estimates of a function's values at different points are required

A Taylor series expansion is a representation of a function by an infinite series of polynomials around a point [3].

Mathematically, the Taylor series of a function, f(x) around a point x = a, is defined as

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)\,(x-a)^n}{n!}$$

Where f^n is the nth derivative of f and f^0 is the function f.

So we can write

$$f(x) = f(a) + f'(a)\,(x-a) + \frac{f''(a)}{2!}\,(x-a)^2 + \frac{f'''(a)}{3!}\,(x-a)^3 + \frac{f''''(a)}{4!}\,(x-a)^4 + \cdots \ldots..$$

A Taylor series centered at a = 0 (around a point $a = 0$) is specially named a Maclaurin series.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)\,x^n}{n!}$$

$= f(0) + x\, f'(0) + x^2\, \frac{f''(0)}{2!} + x^3\, \frac{f'''(0)}{3!} + \cdots$ ....

Taylor and Maclaurin series are fundamental tools of applied math, including many areas of math that touch on engineering applications.

In this review paper, we mainly discuss the applications of Taylor series for finding time period of oscillation of a particle executing simple harmonic motion in different of field of Physics with some example. These examples help the academician to get an in-depth understanding not only of this topic but they will be able to understand the different problems of physics using the concepts of the Taylor series. So this paper helps the researcher as well as the academician to understand the Taylor series from a physical point of view very easily.

2. *Application of Taylor series in Physics:*

Following example helps the researcher and academician to understand the point about which Taylor series applied. In these problem, the point of equilibrium is taken as the point about which Taylor series applied.

2.1. In case of simple harmonic motion for finding time period of a particle executing SHM:

For expression of a given mathematical function y = f (x) around a point x = 0 Taylor's theorem can be written as

$$f(x) = f(0) + x\, f'(0) + x^2\, \frac{f''(0)}{2!} + x^3\, \frac{f'''(0)}{3!} + .$$

If x is taken as the displacement of a particle from its mean position and the restoring force on particle depends on this x by the function $F_R = f(x)$ then it can be given as

Restoring force on particle at a distance x from mean position is

$$F_R = f(x)$$

$$= f(0) + x\, f'(0) + x^2\, \frac{f''(0)}{2!} + x^3\, \frac{f'''(0)}{3!} + \cdots$$

Here, $f(0) = 0$ as at $x = 0$ or mean position restoring force is zero and for small displacement of particle higher power of x can neglected so restoring force can be given as

$$F_R = -x\, f'(0)$$

[Negative sign shows the restoring nature]

Acceleration of particle during oscillation is

$$a = \frac{F_R}{m} = -\left(\frac{f'(0)}{m}\right) x$$

Comparing this equation with general differential equation of SHM ($a = -\omega^2 x$) we get

$$\omega^2 = \frac{f'(0)}{m} \; or \; \omega = \sqrt{\frac{f'(0)}{m}}$$

Hence time period of this SHM is

$$T = \frac{2\,\pi}{\omega} = 2\,\pi \sqrt{\frac{m}{f'(0)}} \rightarrow (1)$$

Now for clear understanding about this topic, here some examples were discussed:

Example 1.

If two point charges with charge + Q are fixed at pts (0, r) and (0, – r) on -Y axis shown in fig.1. of a coordinate system and another small particle of mass m and charge – q is placed at origin of system, where it stays in equilibrium, then if this mass m is slightly displaced along + x direction by a small distance x and released, then we can calculate the time period of oscillations using Taylor series.

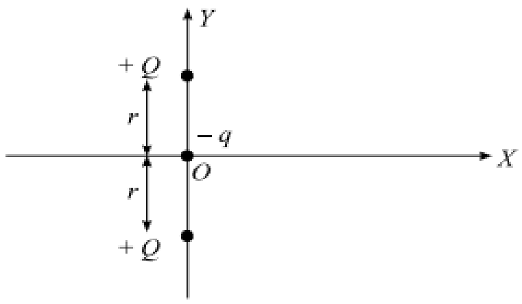

Fig. 1

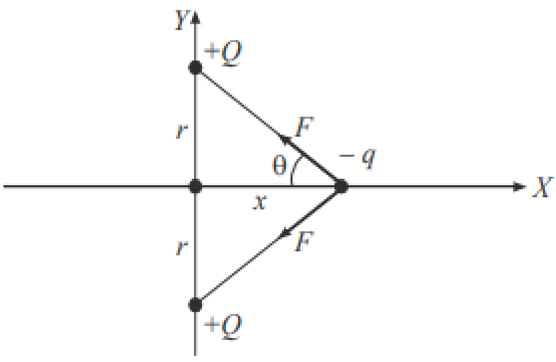

Fig. 2

Restoring force on the particle, when the particle displaced through a distance x as shown in fig. 2.

$$F_R = -\frac{2\,k\,Q\,x}{(a^2+x^2)^{3/2}}$$

Now using Taylor series, we get

$$\omega = \sqrt{\frac{f'(0)}{m}}$$

$$f'(0) = \left.\frac{dF_R}{dx}\right|_{x=0}$$

$$= 2\,k\,q\,Q\left[\frac{(r^2+x^2)^{\frac{3}{2}}.1 - x.\frac{3}{2}.(r^2+x^2)^{\frac{1}{2}}}{(r^2+x^2)^3}\right]_{x=0}$$

$$= 2\,k\,q\,Q\left[\frac{r^3}{r^6}\right] = \frac{2\,k\,q\,Q}{r^3}$$

Therefore, $\omega = \sqrt{\frac{2\,k\,q\,Q}{m\,r^3}}$

Time period

$$T = \frac{2\pi}{\omega} = 2\,\pi\sqrt{\frac{m\,r^3}{2\,k\,q\,Q}}$$

Example 2.

If the potential energy of a particle in a given force field is given as a function of its x-coordinates as

$$U\,(x) = \frac{p}{x^2} - \frac{q}{x}$$

where a and b are positive constants. Then we can calculate the period of small oscillations of the particle about its equilibrium position in the field using Taylor series.

Restoring force on particle

$$F_R = -\frac{dU}{dx} = -\left(-\frac{2p}{x^3} + \frac{q}{x^2}\right)$$

The equilibrium position of particle can have calculated by taking F=0

$$-\frac{2p}{x^3} + \frac{q}{x^2} = 0$$

$$\Rightarrow\ x = \frac{2\,p}{q}$$

So *at* $x = \frac{2p}{q}$ position particle is in equilibrium.

So now by using Taylor series expansion about the point $x = \frac{2p}{q}$ and by using equation (1) we calculate the time period of oscillation of the particle.

Angular frequency of SHM of particle is given as

$$\omega = \sqrt{\frac{f'(\frac{2p}{q})}{m}}$$

$$Here,\qquad f'\left(\frac{2p}{q}\right) = \left.\frac{dF_R}{dx}\right|_{x=\frac{2p}{q}}$$

$$= \left.\left(\frac{6p}{x^4} - \frac{2q}{x^3}\right)\right|_{x=\frac{2p}{q}} = \frac{q^4}{8\,p^3}$$

$$\omega = \sqrt{\frac{f'(\frac{2p}{q})}{m}} = \sqrt{\frac{q^4}{8\,m\,p^3}}$$

$$\text{Time period } T = \frac{2\pi}{\omega} = \sqrt{\frac{8\,m\,p^3}{q^4}}$$

Example 3.

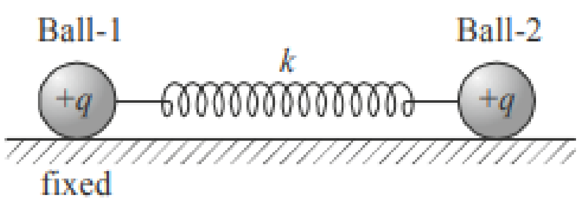

Fig. 3.

Two identical balls of mass m and charge + q. connected by an ideal spring of constant k. as shown in fig. 3. Ball-1 is fixed on a smooth surface as shown in figure. When balls are uncharged, the spring was in its natural length and as both the balls are charged, the spring length gets doubled in equilibrium position of the balls. Time period of small oscillations of the ball-2 about the new equilibrium position after charging the two balls, can be calculated using Taylor series.

If the initial length of the spring is l and after charging it becomes $2l$, from Coulomb's law we have for equilibrium of ball-2 as

$$\frac{k\,q^2}{(2l)^2} = kl$$

$$\Rightarrow\ k = \frac{k\,q^2}{4\,l^3}$$

Now if the ball is slightly displaced away from ball-1, from its equilibrium position such that the separation between two balls becomes x, then we have restoring force on ball-2 as

$$F_R = f(x) = [k\,(x-l) - \frac{k\,q^2}{x^2}$$

Now the second ball oscillates about the equilibrium position of the ball-2 at $x = 2\,l$. So here we use the Taylor series about the point $x = 2\,l$

So, angular frequency of SHM of ball-2 using Taylor's theorem as

$$\omega = \sqrt{\frac{f'(2\,l)}{m}}$$

$$Here, f'(2l) = \left|\frac{dF_R}{dx}\right|_{x=2\,l}$$

$$= \left[k + \frac{2\,k\,q^2}{x^3}\right]_{x=2\,l}$$

$$= k + \frac{2\,k\,q^2}{8\,l^3} = 2\,k$$

So, angular frequency of SHM of ball-2 is

$$\omega = \sqrt{\frac{2\,k}{m}}$$

So time period $T = \frac{2\,\pi}{\omega} = 2\,\pi\sqrt{\frac{m}{2\,k}}$

Example 4: .

One end of a spring of negligible unstretched length and spring constant k is fixed at the origin (0,0) as shown in fig 4. A point particle of mass m carrying a positive charge q is attached at its other end. The entire system is kept on a smooth horizontal surface. When a point dipole

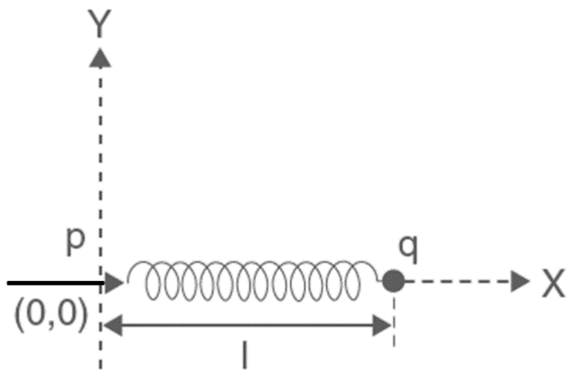

Fig.4.

$\vec{p}$ pointing towards the charge q is fixed at the origin, the spring gets stretched to a length l and attains a new equilibrium position (see figure below). If the point mass is now displaced slightly by $\Delta l \ll l$ from its equilibrium position and released, it is found to oscillate simple harmonically. Time period of oscillation of the point mass cab be calculated by Taylor series as:

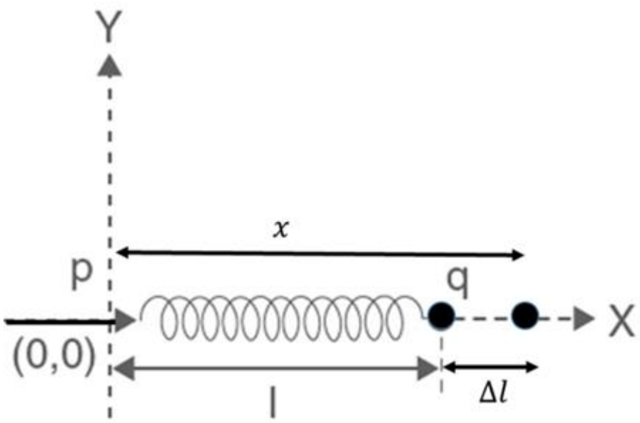

Fig. 5.

Here, since the spring is of negligible unstretched length so when the spring gets stretched to a length l,as shown in fig 5. we can write,

$$k\,l = \frac{2\,k\,p\,q}{l^3}$$

Restoring force

$$F_R = f(x) = k\,x - \frac{2\,k\,p\,q}{x^3}$$

Here the charge q oscillates about the point $x = l$

So, here we use the Taylor series about the equilibrium point $x = l$

Angular frequency of SHM of charge 'q', using Taylor's theorem as

$$\omega = \sqrt{\frac{f'(l)}{m}} = \sqrt{\frac{k + \frac{6\,k\,p\,q}{;^{4}}}{m}}$$

$$So, \omega = \sqrt{\frac{k + 3\,k}{m}} = \sqrt{\frac{4\,k}{m}}$$

So time period

$$T = \frac{2\,\pi}{\omega} = 2\,\pi\sqrt{\frac{m}{4k}}$$

Academician, may solve this equation using the concept of potential energy as,

Potential energy of a system after a small displacement x

$$U = \frac{1}{2}\,k\,x^2 + q\ \times potential\ \ at\ equitorial\ position$$

$$due\ to\ dipole\ of\ dipole\ moment\ \vec{p}$$

$$= \frac{1}{2}\,k\,x^2 + q \times \frac{k\,p}{(x)^2}$$

$$Force\ \ F = \left|\frac{dU}{dx}\right|$$

$$= k\ \ x - \frac{2\,k\,p\,q}{x^3}$$

3. Time period of a simple pendulum in case of large amplitude of oscillation using Taylor series:

We know that, when a pendulum of mass m and length l oscillating in the (constant) gravitational field of the Earth obeys the nonlinear differential equation

$$\ddot{\theta} + \frac{g}{l}\,sin\theta = 0 \ldots\ldots (i)$$

here θ is the angle that the pendulum forms with the vertical direction, and g ≈ 9.81 m/s^2 is the acceleration due to gravity.

For small oscillation, we can write $sin\theta \sim \theta$

and equation (i) can be written as

$$\ddot{\theta} + \frac{g}{l}\,\theta = 0$$

In this case, time period becomes

$$T_0 = 2\,\pi\sqrt{\frac{l}{g}}$$

A. Belendez et al. [3], find the time period of a simple pendulum in case of large amplitude of oscillation using **Taylor series**. Using Kidd-Fogg approximation with the Taylor expansion, they find the exact time period of a simple pendulum when amplitude of oscillation is large.

$$T = T_0\left(1 + \frac{\theta_0^2}{16}\right) = 2\,\pi\sqrt{\frac{l}{g}}\left(1 + \frac{\theta_0^2}{16}\right)$$

4. Conclusion:

Taylor series can be used to approximate functions that are difficult to evaluate directly. By truncating the series at a certain term, you can get an approximation of the function within a specific range. In this report we discussed the applications of Taylor series for finding time period of oscillation of a particle executing simple harmonic motion in different of field of Physics with some example about different equilibrium position which helps the researcher as well as the academician to understand the Taylor series from a physical point of view very easily.

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