

The Fermion Mass & Charge Spectrum from Unified Equations

The Phase Law in the Own-Scale Scheme: gauge quantum numbers, the charge dictionary, and the sector clocks

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Abstract

The circulant parameterization $\sqrt{m_k} \propto 1 + \sqrt{2} \cdot k_S \cdot \cos(\delta_S + 2\pi k/3)$ organises each fermion family by one clock, one amplitude and one phase [1–5, 18]. A previously proposed quantum-number law fixes the phase from gauge charges alone: $\delta_S = \epsilon_S \cdot u_S(1-u_S)$ with $1/u_S = N_c^{3/2+T_3} \cdot C_2(3)^q$, giving 2/9, 3/16, 8/81 and 11/144 — equivalently $1/u \in \{3, 4, 9, 12\}$ — for the charged-lepton, down, neutrino and up sectors [19]. This paper reports the scheme in which that law is testable, the result of the test, the law's inversion into a charge dictionary, its group-theoretic origin, and the clock structure beneath it. In the own-scale scheme — every mass at its own scale, in the scheme of the interactions it feels, the light quarks floored at the condensate boundary — the measured values are $1/u = 2.9999(2)$, $4.004(57)$, $8.868(47)$ and $11.911(40)$: the two odd-isospin sectors sit exactly on the law (0.4σ and 0.07σ), while both even-isospin sectors fall short of theirs by a common one per cent (up 2.2σ , neutrinos 2.8σ ; a shared factor 0.9916 ± 0.0028 on the N_c , $\chi^2 = 0.3/1$). That strain lands on the one step of the derivation that is provably open: the binary tetrahedral group generates the third-turn, the quarter-turn and the twelfth on the rotor, but contains no element of order 9 — the neutrino's rational is a composite carried across the seesaw, and the even-isospin factor of N_c is a threefold cover no group element supplies. The up phase discriminates at 3.8σ against the earlier proposal $\delta_U = 2/27$ [4, 23] and lies within 1 % of the value pre-registered in [19]; its sharp falsifier is the inverse prediction $m_c(m_c) = 1.2573 \pm 0.0055$ GeV, presently 2.2σ from the PDG determination. The scheme is a phenomenological prescription, not a derived result; what removes the freedom is that two internal discriminators select it from the data: the top-mass definition (0.6σ against 2.9σ) and the floor scan. Inverted, the law reads $1/u = 3^{2+Q} \cdot (4/3^{5/3})^q$ and returns the electric charges from the measured masses to 10^{-4} in the lepton sector, one unit of charge being one factor of N_c because N_c is odd; on the Clifford occupation number of the $C \otimes O$ construction, where charge is a number operator [22], its colour flag and isospin become the triplet condition and the grading of the ideal. With the charged leptons preserving the Z_3 of 2T and the neutrino mass matrix a Z_2 , the leading-order PMNS matrix is tri-bimaximal [33–36]; one rotation accommodates θ_{13} with θ_{12} then at 0.9σ [37]; and a polar-decomposition theorem shows the mixing and the neutrino phase residual are independent parameters of the Dirac texture. The five sector clocks close into a two-branch seesaw: on each branch, (direct clock) $\times$ (inverse clock) = m_D^2 with m_D the branch's own condensate decay constant — exactly $(v/\sqrt{2})^2$ on the electroweak branch, and f_π^2 on the QCD branch at 0.5σ (a candidate that predicts $m_d(2 \text{ GeV}) = 4.75$ MeV against the lattice 4.70 ± 0.05 and would reduce the spectrum's dimensionful content to two condensates, one unification scale and the top Yukawa). A common-scale consistency fit gives $\mu^* = 3.3 \times 10^6$ GeV at $\chi^2/\text{dof} = 0.12/2$, with Λ_{QCD} the one irreducible anchor by transmutation. Live predictions: $m_\tau = 1776.97$ MeV (0.4σ from the 2024/25 world average [9, 12]); $m_c(m_c) = 1.2573(55)$ GeV; normal neutrino ordering, required; $\Delta m_{21}^2 = 7.88 \times 10^{-5} \text{ eV}^2$ if the heavy phase is exactly 9, disfavoured at 3.2σ and decidable by JUNO [13]; and $m_u/m_d = 0.4728$ against the surviving Clebsch candidate's 0.4536 [15]. Eighteen verification scripts reproduce every number, and five retracted intermediate claims are preserved in a provenance appendix.

Keywords: fermion masses; Koide relation; circulant; mass-scheme dependence; seesaw; renormalisation group; lattice quark masses.

1. Introduction

The Koide relation for the charged leptons [1], its geometric reading as a scale-invariant angle [2], the circulant phase 2/9 [3], and the extension of the Z_3 -symmetric parameterization to the quark and neutrino sectors [3–5] have organised the fermion spectrum for four decades without settling the question they raise: whether the sector phases are numbers of physics or artefacts of parameterization. The parameterization itself is complete — three parameters for three masses

— and therefore asserts nothing [18]; whatever content exists lies in the claims that the amplitude sits at the equipartition point and that the phases take specific rational values.

A companion manuscript [19] derived a candidate law for those phases: the accumulated holonomy of a single elementary shift on a periodic internal coordinate is $\delta = u(1-u)$ exactly, and the shift fraction takes the gauge-quantum-number form $1/u = N_c^{3/2+T_3} \cdot C_2(3)^q$, with $N_c = 3$ the colour count, T_3 the weak isospin, $C_2(3) = 4/3$ the SU(3) Casimir, and q the colour flag — yielding $1/u = 3, 4, 9, 12$ and hence $\delta = 2/9, 3/16, 8/81, 11/144$ for the lepton, down, neutrino and up sectors. At the time of [19] the law was established only for the charged leptons; the coloured-sector phases were explicitly flagged as not established, and the up sector appeared to depart badly.

This paper establishes the law's empirical status and the reason for the earlier confusion. The confusion was scheme: the sector phases are hypersensitive coordinates (Section 10), and the same three up-quark masses give $1/u = 18.4$ at a common M_Z scale, 19.0 at a common 3×10^6 GeV scale, and 11.93 when each mass is taken at its own scale. Section 2 states the formula, the law, the sector clocks with their closed harmonic forms, and the own-scale scheme with the two internal discriminators that select it. Section 3 gives the four sector results in tables, including the up sector's closure on the pre-registered value and its resulting prediction for m_u . Section 4 inverts the law: the electric charge is read off the measured mass ratios, and the law is rewritten on the Clifford occupation number of the $C \otimes O$ construction, where charge is a number operator [22]. Section 5 asks which of the four rationals the discrete geometry of the binary tetrahedral group generates, and finds three. Section 6 records the supporting theorems and controls; Section 7 the one-loop seesaw calculations — logarithmic running and finite matching — that supply part of the single surviving deviation and bound what remains; Section 8 the boundary-scale fit and the identification of Λ_{QCD} as the construction's one dimensionful input; Section 10 the amplification discipline; Section 11 the open conflict with the one surviving Clebsch candidate [15]. Section 12 collects the ledger and the live predictions, and Section 13 concludes. Appendix A documents the verification scripts (supplementary material); Appendix B preserves the provenance; Appendix C gives the full derivations of the mass and charge formulas together with the supporting theorems, including four retracted intermediate claims.

Roadmap. Section 2 states the formula, the law and the scheme, with a glossary in 1.2; Section 3 gives the four sector results; Sections 4 and 5 invert the law into a charge dictionary and derive its rationals from the binary tetrahedral group; Sections 6–8 treat the exact invariance theorem, the seesaw calculation with the mixing structure, and the boundary scale; Section 9 presents the clocks and the two-condensate seesaw; Sections 10–12 give the amplification discipline, the open two-law conflict, and the ledger with live predictions. Appendix A documents the scripts, Appendix B the retracted and superseded claims, and Appendix C the self-contained derivations — a reader who wants the proofs before the phenomenology may read C.1–C.5 first.

A note on inputs and versions. Relative to the version first circulated, the inputs here are FLAG 2024 and the PDG 2024/25 average $m_\tau = 1776.93 \pm 0.09$ MeV, and the update moved one headline against the paper: with the properly-errored $m_c(m_c) = 1.2730 \pm 0.0046$ GeV, the up sector's earlier 0.58σ agreement — an artefact of the loose $m_c = 1.27 \pm 0.02$ — became the 2.2σ strain reported throughout, with the corresponding sharp falsifier $m_c(m_c) = 1.2573 \pm 0.0055$ GeV replacing the weak m_u prediction. Appendix B records every such supersession alongside the five retracted claims; the paper's practice is to report the numbers as they now stand, prominently, whichever way they moved.

1.1 Relation to prior work

The parameterization and the lepton phase are due to Koide, Foot and Brannen [1–3]. The proposal that the quark phases are charge-tied fractions of the lepton phase is Żenczykowski's: from the same Z_3 -symmetric form he argued for low-energy values $\delta_U = \delta_L/3 = 2/27$ and $\delta_D = 2\delta_L/3 = 4/27$ [4, 23], and noted that the top mass forbids $k_U \approx 1$ — the open amplitude of this paper. The present law differs in both quark values and is tested against his in Section 3.3, where the own-scale up phase discriminates between them at 3.8σ . That pole masses satisfy the lepton relation while running masses do not was shown by Xing and Zhang [24], and the near-stability of the quark and lepton Koide parameters under running by Li and Ma [25] and Xing and Zhang [24]; Section 6 sharpens both into an exact statement about flavour-universal running and a phase-level scheme selection that holds in two sectors. That the relation could hold on the Dirac or heavy Majorana matrix of a seesaw rather than on the light masses was proposed by Rodejohann and Zhang [26] and developed by Huang et al. [27]; Section 3.4 imposes the law on the heavy spectrum in that spirit. Brannen's 2006 sign-flipped Koide form predicted $\Delta m_{21}^2 \approx 7.93 \times 10^{-5}$ eV² [28], formalised by Koide [29]; the present construction's 7.88×10^{-5} is the same prediction from a different parameterization, and both now sit 3σ from JUNO. The colour count entering the quark phases has circulated as folklore around Rivero's and Kartavtsev's triplets [5, 30]; Section 5 makes it a statement about element orders in 2T. What is new here is the quantum-number law with its holonomy form and the value 11/144, the own-scale scheme with its two internal discriminators, the inverse-frame down sector at equipartition, the charge

inversion and number-operator form, the group-theoretic derivation of three of the four rationals, the threshold calculations with the top-trace cancellation, and the unified four-sector test.

The skeptical case deserves equal statement. The Koide relation has no accepted derivation after four decades, and a substantial body of informed opinion regards it — and by extension any extension of it — as plausibly coincidental: with enough rationals, angles and schemes available, some combination will match any three masses, and the literature of abandoned Koide-type formulas is long (see Koide’s own status review [38] for a candid survey). This paper’s answer to the base-rate objection is procedural rather than rhetorical: the up-sector value was placed on public record before the data test that now strains it; the falsifiers are stated with owners and dates; five failed intermediate claims are retracted in print rather than silently dropped; and the amplification discipline of Section 10 exists precisely to prevent compressed coordinates from manufacturing significance. Two mechanism-level alternatives also deserve the reader’s attention. Sumino’s family gauge symmetry [39] remains the one field-theoretic mechanism engineered to protect the pole-mass exactness of the charged-lepton relation, and any eventual derivation of the own-scale scheme must either reduce to or replace it. And the universal seesaw programme [40] — Berezhiani; Davidson and Wali; Sogami and Shinohara; Koide and Fusaoka — anticipated the structural form that Section 9’s branch seesaw makes quantitative, with heavy partners supplying light masses as m_D^2/M ; the present work differs in tying each branch’s Dirac scale to its condensate’s decay constant and in producing the f_π relation as a testable number. Finally, two independent derivations of the lepton angle appeared while this paper was in revision. A compact-cycle construction [42] derives $-2/9$ for the charged leptons from a Berry phase equal to the holonomy $u(1-u)$ of Appendix C.3 identically (its Fourier sum is the same Green function), sharing this paper’s canonical-normalisation assumption and leaving the sector fractions and the other three sectors outside its scope; and a $U(3)$ -condensation mechanism [41] reaches $-2/9$ by a stress-balance condition on a flavour $\mathbb{C}^3$. Three independent constructions — four derivations counting the E_6 route of [20] — now land on one rational, which is itself a datum.

1.2 Glossary

term	meaning
own-scale scheme	each mass evaluated at its own scale, in the scheme of the interactions it feels: pole for charged leptons, $\overline{MS}$ -bar at $\mu = m$ for heavy quarks, light quarks at the condensate floor; a phenomenological prescription selected by two internal discriminators (2.2)
shape coordinates (n, δ)	the two scale-invariant numbers of a family triplet: the normalisation $n = 9 \cdot \Sigma m / (\Sigma \sqrt{m})^2$ and the circulant phase δ (2.1)
sector clock A_S	the one dimensionful number of a sector: $A = \Sigma m / n$ (direct frame) or $n / \Sigma(1/m)$ (inverse frame) (2.1, 9)
frame $\varepsilon_S = \pm 1$	whether the circulant acts on $\sqrt{m}$ (direct: leptons, up) or $1/\sqrt{m}$ (inverse: down, neutrinos); fixed structurally by Theorem I of [20], confirmed at 60σ (C.8)
condensate floor	the scale below which a light quark’s running mass is frozen for evaluation; an input in [2.1, 4.5] GeV (2.2, 9)
amplification	the leverage $d(\text{mass ratio})/d(\text{phase})$ of compressed coordinates, up to $\times 33$; Section 10’s discipline requires errors to be carried through it
branch	the condensate a sector rides: odd- N sectors the chiral vacuum, even- N the electroweak vacuum (5, 9)

2. The formula, the law, and the own-scale scheme

$$m_k^{(S)} = A_S \left[1 + \sqrt{2} k_S \cos(\delta_S + \frac{2\pi k}{3}) \right]^{2s_S}, \quad k = 0, 1, 2$$

Here $\varepsilon_S = +1$ for the direct sectors (charged leptons, up) and -1 for the inverse sectors (down, neutrinos) — an assignment made structurally in [19] through the availability of conjugation and the seesaw, and determined independently by the data at 60σ (Section 6). The amplitude $k_S = 1$ (the equipartition point, $n \equiv 9Q = 6$) holds for the lepton, down and heavy-neutrino sectors; the up amplitude is fitted (Section 3.3). The phase is the law under test:

$$\delta_S = \varepsilon_S u_S (1 - u_S), \quad \frac{1}{u_S} = N_c^{3/2 + T_3} \cdot G_2(3)^q = (N_c + q) N_c^{T_3 + 1/2}$$

$1/u_s$ from gauge quantum numbers — [DERIVED, ref. 19]

$k_s = 1$ (equipartition, $n = 6$) for leptons, down and heavy neutrinos — [input, Prop. 6 of 19]

$k_s = 1.2382$ (fitted to m_c, m_t) for the up sector — [OPEN]

$\varepsilon_s = +1$ direct (leptons, up); -1 inverse (down, neutrinos) — [ESTABLISHED, Sec. 6]

Sector	q	T_3	$1/u_s$	δ_s	ε_s	k_s
charged leptons	0	$-1/2$	3	$+2/9 = +0.222222$	+1	1
down quarks	1	$-1/2$	4	$-3/16 = -0.187500$	-1	1
neutrinos (heavy)	0	$+1/2$	9	$-8/81 = -0.098765$	-1	1
up quarks	1	$+1/2$	12	$+11/144 = +0.076389$	+1	1.2382 (fit)

The two written forms of the law coincide on all four sectors: $N_c^{3/2+T_3} \cdot C_2(3)^q$ with $C_2(3) = 4/3$ is the Casimir form of [19], and $(N_c + q) \cdot N_c^{T_3+1/2}$ its compact equivalent. The rationals are the Koide phase $2/9$ for the leptons and its three gauge-charged images.

2.1 The sector clock A_s

A_s is the sector's single dimensionful input, of mass dimension in every sector. Writing the spectrum in mode form,

$$m_k = A_s f_k^{2s}, \quad f_k = 1 + \sqrt{2} k_s \cos(\delta_s + \frac{2\pi k}{3}), \quad \sum_k f_k^2 = n_s = 3(1 + k_s^2)$$

the identity $\sum_k f_k^2 = n_s$ (the cross term sums to zero over the Z_3 modes and $\sum \cos^2 = 3/2$) gives the clock in closed harmonic form:

$$A_s = \frac{\sum_k m_k}{n_s} \quad (\text{direct}, \varepsilon_s = +1), \quad A_s = \frac{n_s}{\sum_k \frac{1}{m_k}} \quad (\text{inverse}, \varepsilon_s = -1)$$

At the equipartition point ($n_s = 6$) these are simply $\Sigma m/6$ and $6/\Sigma(1/m)$; for the up sector $n_u = 3(1 + k_u^2) = 7.606$. Direct clocks are the κ^2 of [19]; inverse clocks are its κ^{-2} (down: $\kappa = 0.1937 \text{ MeV}^{-1/2} \leftrightarrow A_d = 26.66 \text{ MeV}$, exactly the value quoted there). One closure identity follows from the reciprocity of the light and heavy neutrino circulants: $A_M \cdot A_V = m_D^2$, and numerically $\sqrt{(A_M \cdot A_V)} = 173.9 \text{ GeV}$ against $v/\sqrt{2} = 174.1 \text{ GeV}$ (0.11 %) — the heavy-clock fit of [19] is the $y_v = 1$ seesaw convention made explicit; a consistency of conventions, recorded as such.

sector	A_s (own-scale)	closed form	tier [19]
charged leptons	313.856 MeV	$\Sigma m_l/6$	QCD/rotor
down quarks	26.657 MeV	$6/\Sigma(1/m)$	QCD/rotor
up quarks	21.549 GeV	(m_c, m_t) anchor	electroweak vev
neutrinos	$1.506 \text{ meV} \leftrightarrow 2.008 \times 10^{16} \text{ GeV}$	$6/\Sigma(1/m); \Sigma M/6$	unification scale

2.2 The own-scale scheme

The law holds when every mass is evaluated at its own scale, in the scheme of the interactions it feels: leptons on-shell in QED (pole masses); quarks MS-bar at $\mu = m_i$, QCD's own scale, quarks having no pole; the light quarks u, d, s floored at the condensate boundary $\approx 2 \text{ GeV}$, below which the chiral medium takes over. At own scale the flavour-non-universal dressing logarithm $\ln(\mu/m_i)$ vanishes identically; the one-loop QED dressing $(3Q^2\alpha/2\pi) \cdot \ln(\mu/m)$ used in intermediate work — which captures 90 % of the known lepton pole/MS-bar shift when validated where the answer is known — is the common-scale approximation to this scheme.

Two internal discriminators select the prescription with no freedom. First, the up sector distinguishes the top-mass definition: $m_t(m_t)$ MS-bar places the sector 0.6σ from the law while the pole mass places it 2.9σ away — QCD-dressed at own scale, exactly parallel to the charged leptons selecting the QED-dressed (pole) masses, for which the geometric ratio 206.7703 matches the pole 206.7680 at 10^{-5} and the MS-bar 211.107 only at 2 % [7, 9]. Second, the down sector's hit on $-3/16$ survives for a floor anywhere in 1.6–3.0 GeV (0.40σ at 2.0 GeV, 0.01σ at 2.5 GeV): the floor is a physical parameter

of the scheme, not a tuning. Two schemes serve two kinds of statement throughout this paper: shape comparisons in the own-scale scheme; renormalisation-group statements — running, scale probes, the boundary fit of Section 8 — at a common scale.

To state the scheme's epistemic status plainly: it is a phenomenological prescription, not a derived result. No principle presented here explains why nature realises the balance at each particle's own physical point; what the paper establishes is that the data select this prescription twice over against its alternatives — the up sector distinguishes $m_t(m_t)$ from the pole top at 0.6σ versus 2.9σ , in parallel to the leptons selecting the pole at 10^{-5} versus 2 % — and that within it the law can be tested without tunable freedom. A principled derivation of the scheme is an open problem of the same rank as the threefold cover; Sumino's mechanism [39] is the standard against which any candidate must be measured, and the branch-seesaw candidate of Section 9, which selects the floor from below, is the first internal structure to constrain the prescription rather than merely use it.

2.3 The four-sector scorecard

sector	1/u measured (own-scale)	law [19]	σ	status
charged leptons	2.9999 ± 0.0002	3	0.4	ESTABLISHED
down quarks	4.0040 ± 0.0565	4	0.07	ESTABLISHED (this scheme)
neutrinos	8.8676 ± 0.0474	9	2.80	strained — composite rational (§5)
up quarks	11.911 ± 0.040	12	2.2	pre-registered in [19]; strained (see §3.3)

The up value was written in [19] before the test; Section 10 explains why only that pre-registration carries evidential weight in this sector. The two even-isospin sectors are short of their integers by a common factor $g = 0.9916 \pm 0.0028$ ($\chi^2 = 0.3/1$ with the computed neutrino running; script K) while the odd-isospin sectors sit at 1 to 0.4σ and 0.07σ — the strain is confined to the two sectors that carry the extra N_c of the law, the factor Section 5 shows no group element supplies. Quark shape coordinates throughout are computed from QCD-invariant lattice mass ratios [8, 10], which the universality theorem of Section 6 renders scheme-exact within pure QCD; the inputs are listed in Appendix A.

3. Sector results

3.1 Charged leptons (direct; $A_\ell = 313.856$ MeV)

Pole masses, direct frame, $m_\tau = 1776.93 \pm 0.09$ MeV (PDG 2024/2025 [9]); $n = 5.99998 \pm 0.00005$ (0.45σ from 6), $1/u = 2.99993 \pm 0.00017$ (0.4σ from 3), $\delta = 0.2222248 \pm 0.0000062$ (0.4σ from $2/9$). Single input $A_\ell = 313.856$ MeV.

ratio	measured (pole) [9]	formula (6, 3)	dev
m_μ/m_e	206.7680	206.7703	+0.0011 %
m_τ/m_μ	16.8181	16.8180	−0.0006 %

mass	formula (MeV)	measured, pole (MeV) [9]	dev
m_e	0.510994	0.51099895	−0.001 %
m_μ	105.658327	105.6583755	−0.00003 %
m_τ	1776.9667	1776.93 ± 0.09	+0.002 % (0.4σ)

One input, three masses. The two rational conditions over-determine m_τ — $n = 6$ requires 1776.9690 MeV, $\delta = 2/9$ requires 1776.9665 MeV — agreeing to 2.52 keV with derivatives of opposite sign differing by $7\times$, so this is not a degeneracy. Prediction: $m_\tau = 1776.97$ MeV. The 2024/2025 world average 1776.93 ± 0.09 [9] sits 0.4σ below it and Belle II's 1777.09 ± 0.14 [12] 0.9σ above; BESIII's 2014 threshold scan 1776.91 ± 0.17 is 0.3σ below and KEDR's 2023 value 1776.69 ± 0.23 1.2σ below. A decisive test needs $\sigma(m_\tau) \approx 0.03\text{--}0.05$ MeV — a new BESIII threshold scan or Belle II with its momentum-scale systematic reduced — and would test Koide's original relation and this construction together, since

their m_τ predictions agree to 2.5 keV. The known 10^{-7} -level branch tension of [19], and the Sumino mechanism it evidences [6], stand unmodified.

3.2 Down quarks (inverse; $A_d = 26.657$ MeV)

shape (own-scale)	measured	law	dev
$n = 9Q$	5.987 ± 0.016	6	0.8σ
$1/u$	4.0040 ± 0.0565	4	0.07σ
δ	-0.18738 ± 0.00177	$-3/16 = -0.18750$	0.07σ

ratio	measured (own-scale) [8,10]	formula (6, 4)	dev
m_s/m_d (floor-free)	19.95 ± 0.33	20.053	$+0.52\%$ (0.3σ)
$m_b(m_b)/m_s(2 \text{ GeV})$	44.95 ± 0.32	48.03	$+6.9\%$ (= the -0.8σ of n , amplified)

mass	formula (A_d/F , MeV)	measured, own-scale (MeV)	dev
m_d	4.669	4.67 (2 GeV) [8]	-0.02%
m_s	93.63	93.4 ± 0.6 (2 GeV) [8]	$+0.25\%$
m_b	4497	4200 ± 14 (m_b) [8]	$+7.1\%$

No auxiliary factor of any kind is used. The m_b row carries the sector's whole residual — the compounded ratio deviation — which in shape space is (0.8σ , 0.07σ); Section 10's amplification note applies.

3.3 Up quarks (direct; $A_u = 21.55$ GeV, vev branch) — pre-registered, strained

Own-scale inputs: $m_u = 2.14 \pm 0.08$ MeV at the floor [8], $m_c(m_c) = 1.2730 \pm 0.0046$ GeV and $m_t(m_t) = 162.5 \pm 0.7$ GeV [9]. Earlier versions quoted $1/u$ at 0.58σ from 12; that was an artefact of the loose $m_c = 1.27 \pm 0.02$ and is superseded by the values below.

shape (own-scale)	measured	law	dev
$1/u$	11.911 ± 0.040	12	2.2σ [pre-registered value]
δ	0.07680 ± 0.00026	$11/144 = 0.076389$	$+0.5\%$ (2.2σ)
$n = 9Q$	7.606 ± 0.010	— ($k_u = 1.2382$ fitted)	the sector's open number

ratio	measured (own-scale) [8,9]	formula (12; k_u fit)	dev
$m_c(m_c)/m_u(2 \text{ GeV})$	595 ± 23	462.8	-22% (the amplified phase strain)
$m_t(m_t)/m_c(m_c)$	127.95 ± 2.1	127.97	$+0.01\%$ (anchor-fixed)

quantity	law prediction	measured	dev
$m_c(m_c)$ from $\delta = 11/144$, m_u [8], m_t	1.2573 ± 0.0055 GeV	1.2730 ± 0.0046 GeV [9]	2.2σ — THE SHARP FALSIFIER
m_u (the same equation, weak direction)	2.75 ± 1.05 MeV	2.14 ± 0.08 MeV [8]	0.6σ — not a test ($d \ln m_u/d \ln m_c = 23.5$)

$m_t(m_t)$	input	$162.5 \pm 0.7 \text{ GeV}$ [9]	the limiting input of the m_c prediction
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Head-to-head with the prior proposal: the own-scale up phase $\delta_u = 0.07680 \pm 0.00026$ sits 2.2σ from $11/144$ but 10σ from Żenczykowski's $2/27$ [4, 23]; at the earlier looser precision the discrimination was 0.58σ against 3.84σ ; equivalently $\delta_u/\delta_t = 0.3456 \pm 0.0032$ against his $1/3$ (3.8σ) and this law's $11/32$ (0.6σ). With $\delta = 11/144$ exact, the sector's one relation among (m_u, m_c, m_t) is most precisely read as a prediction of $m_c(m_c)$: the lattice m_u at 3.7% fixes m_c to 0.4% , and the result sits 2.2σ below the PDG determination — the same one-per-cent shortfall as the phase itself, stated in a measurable coordinate. The sector's earlier apparent failure — the same three masses give $1/u = 18.4$ at common M_Z , 19.0 at common μ^* , 11.93 at own scale — was the amplification of Section 10 acting on the scheme choice; the intermediate claims it generated are retracted in Appendix B. The clock rides the electroweak vacuum through the order-unity top Yukawa per the condensate-selection rule of [19]; the departure from equipartition, $k_u = 1.2382$, is now cleanly isolated from the phase and is the sector's one open number.

3.4 Neutrinos (inverse; heavy clock $2.008 \times 10^{16} \text{ GeV}$) — the one deviation

The law is imposed on the heavy Majorana spectrum, $M_1:M_2:M_3 = 1 : 5.64 : 188$; the light spectrum is $m \propto 1/M$ with the computed one-loop sequential-threshold running of Section 7. Normal ordering is required — with inverted ordering the equipartition condition forces $1/u = 1131$ — and is therefore a prediction of the construction.

$\Delta m^2_{31}/\Delta m^2_{21}$	value	vs 33.51 ± 0.61 [9,13]
tree (heavy law, $m \propto 1/M$)	31.88	-4.85% , 2.7σ
+ threshold running, $y_v \rightarrow 0$	32.31	-3.57% , 2.0σ
+ threshold running, $y_v = 0.5$	32.16	-4.0% , 2.2σ

Equivalently $1/u = 8.868 \pm 0.047$ against 9. The tension is not new to this construction: Brannen's 2006 form predicted $\Delta m^2_{21} = 7.93 \times 10^{-5} \text{ eV}^2$ [28, 29], the present law with the heavy phase exactly 9 predicts 7.88×10^{-5} , and JUNO's 7.50 ± 0.12 [13] now stands $3\text{--}4\sigma$ from both. Light masses ($y_v \rightarrow 0$, normalised to Δm^2_{21}): $m_1 = 0.260 \text{ meV}$, $m_2 = 8.66 \text{ meV}$, $m_3 = 48.9 \text{ meV}$, $\Sigma m_\nu = 0.058 \text{ eV}$ — below the cosmological bound $\approx 0.072 \text{ eV}$ [17]. The strain sits $4:1$ against the phase rather than the amplitude (freeing n needs 6.022 ; freeing δ needs $1/u = 8.868$), stands alone across the sectors, and everything standard has been computed. Remaining options: a type-II triplet contribution (a free parameter); the heavy phase genuinely at 8.87 ; or Δm^2_{21} at $7.88 \times 10^{-5} \text{ eV}^2$ against JUNO's 7.50 ± 0.12 [13], a 3.2σ tension growing to $\approx 4.5\sigma$ at design precision. JUNO decides.

4. Charge from mass: the law inverted

4.1 The inversion

Substituting the hypercharge assignment $Y = -1 + (4/3)q$, i.e. $T_3 = Q + \frac{1}{2} - (2/3)q$, into the law gives

$$\frac{1}{u_S} = 3^{2+Q} \cdot \left(\frac{4}{3^{5/3}}\right)^q, \quad Q = \log_3 \frac{1}{u_S} - 2 - q \log_3 \frac{4}{3^{5/3}}$$

For colourless fermions the second factor is unity and $1/u = 3^{2+Q}$: the electric charge sits in the exponent of the colour count, and the masses determine the charge. Applied to the measured own-scale phases (script G):

sector	$1/u$ from the masses	Q inferred	Q	σ
charged leptons	2.9999 ± 0.0002	-1.0000 ± 0.0001	-1	0.4
down quarks	4.004 ± 0.057	-0.332 ± 0.013	$-1/3$	0.07
up quarks	11.93 ± 0.12	$+0.661 \pm 0.009$	$+2/3$	0.58
neutrinos	8.868 ± 0.047	-0.013 ± 0.005	0	2.8

Two ratios state the structure: across each weak doublet, $(1/u_v)/(1/u_e) = 2.96$ and $(1/u_u)/(1/u_d) = 2.98$ — one unit of electric charge is one factor of N_c ; and at fixed isospin, $(1/u_d)/(1/u_e) = 1.335 = C_2(3)$. This is the law inverted, not a derivation of charge quantization: the values $\{-1, -1/3, 0, +2/3\}$ entered through T_3 and q . Its content is that a verified, invertible dictionary exists between mass ratios and charge, exact to 10^{-4} in the lepton sector, and that the masses cannot distinguish a particle from its antiparticle, so the sign is fixed up to conjugation.

4.2 The law on the Clifford occupation number

In the $Cl(6) \cong C \otimes O$ description of one generation, the eight states of a minimal ideal are labelled by the number $N \in \{0, 1, 2, 3\}$ of raised colour ladder operators, and the electric charge is the number operator, $Q = N/3$ [22]. On N the law reads

$$\frac{1}{u} = N_c^{1+[N \text{ even}]} \cdot \left(\frac{4}{3}\right)^{[0 < N < 3]}, \quad Q = \frac{N}{3}$$

which returns 9, 4, 12, 3 for $N = 0, 1, 2, 3$ (script G). Two identifications fall out: the colour flag is no longer an input, since $0 < N < 3$ is exactly the condition for a colour triplet in $C \otimes O$; and weak isospin is the Clifford grading, $T_3 = +\frac{1}{2}$ the even subalgebra and $T_3 = -\frac{1}{2}$ the odd. A unit of charge is N_c ladder steps, and because $N_c = 3$ is odd, $N \rightarrow N + 3$ flips the grading

— which is why one unit of charge is one factor of N_c in the phase. Were N_c even, the rule would fail. That is a structural statement about the number three, not a fit.

5. The discrete geometry of the phases

Sections 3 and 4 established what the phases are and what they encode; this section asks where the rationals themselves come from, and answers with a finite group.

The four shift fractions $1/3, 1/4, 1/9, 1/12$ say that the elementary shift of each sector is a rotation of order 3, 4, 9, 12. The binary tetrahedral group $2T$ — the group selected by the integrality axiom of the companion manuscript [19] — has element orders $\{1, 2, 3, 4, 6\}$: eight elements of order 3, the generation cycle Z_3 , and six of order 4, namely $\pm i, \pm j, \pm k$, the three quaternionic directions up to sign. On the rotor spinor $\mathbb{C}^2$ the order-3 element $(-1+i+j+k)/2$ acts with phases $\pm 1/3$ and the unit quaternion i with phases $\pm 1/4$ (script H). That is the derivation of the first two rationals: the colourless shift is one generation step, $1/u_e = 3$; the coloured shift is one quarter-turn along a colour direction, $1/u_d = 4$. It also settles which $4/3$ the down sector carries: at the level of the rotor action it is $|Z_4|/|Z_3|$, the ratio of two rotation orders, and it coincides with the Casimir $C_2(3) = (N_c^2 - 1)/2N_c$ only at $N_c = 3$.

The generation Z_3 cannot be the source of the isospin factor: induced on the ideal it has phases $\{0, 1/3, 2/3\}$ on the even and on the odd half alike, commuting with the grading as every colour or generation operator must. The centre of $SU(3)$ acts on the ideal as $\omega^N = e^{2\pi i N/3}$, so colour triality is electric charge modulo one, and a rotor rotation combined with a triality phase reaches exactly the twelfths. Generating the full finite group spanned by $2T$ on the rotor with the colour cycle, the $SU(3)$ centre and the Hodge star (the T_3 flip $N \rightarrow 3 - N$) on the ideal gives a group of order 432 and exponent 12 (script H):

state	reachable shift fractions
electron ($N = 3$)	$1/6, 1/4, 1/3, 1/2, 2/3, \dots$
down ($N = 1$)	$1/12, 1/6, 1/4, 1/3, \dots$
up ($N = 2$)	$1/12, 1/6, 1/4, 1/3, \dots$
neutrino ($N = 0$)	$1/6, 1/4, 1/3, 1/2, \dots$ — no $1/12$, no $1/9$

Three of the four law values are generated: the electron's third-turn, the down quark's quarter-turn, and the up quark's twelfth as a quarter-turn on a triality-charged colour state. The neutrino's ninth is not, and cannot be: no element of order 9 exists in the 432-element group, nor in any finite subgroup of $SU(2)$ other than the cyclic and dicyclic ones — the same restriction that selects $2T$ forbids a ninth-turn of the rotor. The value $9 = 3 \times 3$ is therefore not a single geometric invariant like the other three; it is a third-turn on the heavy Majorana circle inherited by the light circle through the seesaw, a map between two different circles that no symmetry protects. The neutrino sector is the one whose rational is composite, and it is the one sector whose measured phase departs from the law. Section 7 quantifies how much of that departure the seesaw dynamics supplies. Two further group facts belong here. First, the generation cycle and the

fermionic sign sit in a single cyclic group of $2T$: the order-6 element $g = (1+i+j+k)/2$ has g^2 equal to the generation cycle and $g^3 = -1$, so $Z_6 = Z_3(\text{generation}) \times Z_2(\text{spin sign})$ — three generation steps are exactly one 4π spinor period, and the flavour label and the spinor sign are components of one Z_6 charge (script H). Second, a representation disambiguation this section requires: the one abstract group $2T$ acts in three places — the 2-representation carries spin (the rotor spinor), the 3-representation carries the colour axes (the A_4 triplet of Section 7.3), and its Z_3 acts on the internal phase circle of the mass formula; the phase θ of the mass formula is an internal angle, not the spin angle, since mass is a scalar under spatial rotations.

6. Supporting theorems and controls

(a) Universality. Shape coordinates are exactly invariant under flavour-universal running — the parameterization is homogeneous of degree zero — verified by a gauge-only control to seven decimals over sixteen decades (script B). Within-sector ratios are therefore QCD-blind, which is why the quark shapes are computed from lattice mass ratios [8, 10] at $26\times$ the precision of absolute-mass determinations; cross-sector ratios alone carry scale. (b) Frame. The direct/inverse assignment is determined by the data: the down sector in the inverse frame gives $n = 6.012 \pm 0.016$ while the direct frame gives 6.742 ± 0.004 — 60σ from any integer — confirming the structural assignment of [19] independently. (c) Uniqueness of the lepton rational. $\delta_\ell = 0.2222248 \pm 0.0000062$ admits exactly one reduced rational with denominator ≤ 99 within 3σ , namely $2/9$ (base rate 4×10^{-4}). (d) Scheme selection is two-for-two: pole for leptons, $m_i(m_i)$ for the top — one rule, both sectors where a test exists (Section 2.2).

7. The seesaw threshold calculation

The neutrino sector's 2.8σ strain was reported in 3.4; this section computes how much of it known physics supplies, bounds what remains, and separates it from the mixing structure.

The Standard Model plus three right-handed neutrinos is integrated in the Z_3 Fourier basis, where every matrix is diagonal: heavy spectrum fixed by the law (6, 8/81) at the high scale; democratic Dirac Yukawa y_v ; sequential decoupling at each M_k generating $\kappa_k = y_k^2/M_k$; full one-loop RGEs for y_v , M_k , κ_k , y_t , y_b , y_τ , λ and the gauge couplings [7, 14]. The ratio of two light masses acquires $F = \exp[(A_\kappa - 2A_y)dt]$ across the segment between their thresholds, and the anomalous-dimension difference is

$$A_\kappa - 2A_y = \frac{\lambda - 3y_v^2 + \frac{9}{10}g^2 + \frac{3}{2}g^2}{16\pi^2}$$

The top-quark trace cancels exactly between the κ and y^2 anomalous dimensions; gauge running is the only positive driver and $-3y_v^2$ removes it. Numerically the mechanism supplies at most +1.3 % of the +5.1 % needed (27 %), attained as $y_v \rightarrow 0$, and is insensitive to M_3 across 10^{14} – 2×10^{16} GeV (script E). It also generates no Dirac non-democracy — the trace is universal and the only non-universal term, y_e^2 , is 10^{-4} — while a circulant Dirac mass with its own integer amplitude gives a ripple of exactly 0 or ≥ 0.816 , never the ~ 1 % that would close the deficit at tree level. The mechanism is real, insufficient by $3.8\times$, and cleanly isolates the neutrino deviation of Section 3.4.

7.1 The finite one-loop matching

The logarithmic running captures only the large logs of the Z - and Higgs-exchange diagrams with the heavy neutrinos in the loop. The full one-loop correction [31, 32], applied in the Fourier basis as $m_i = m_i^{\text{tree}}(1 - \varepsilon_i)$ with

$$\varepsilon_i = \frac{3m_z^2 F(M_i, m_z) + m^2 F(M_i, m_H)}{32\pi^2 v^2}, \quad F(M, m) = \frac{M^2 \ln(M^2/m^2)}{M^2 - m^2}$$

gives $\varepsilon_i = 0.117, 0.125, 0.140$ on the three light masses: a 12 % universal shift the shape does not see, and a 2.2 % spread across the heavy hierarchy that it does (script I). With the loop opposing the tree, $\Delta m_{31}^2/\Delta m_{21}^2$ moves from 31.88 to 32.42, against 32.31 from the running — the same physics, cross-checked by two independent calculations. The genuinely finite, non-logarithmic parts vanish to double precision, suppressed by $m_z^2/M_i^2 \approx 10^{-26}$ for heavy neutrinos at 10^{14} – 10^{16} GeV. One-loop seesaw physics of any kind supplies about 30 % of the deficit and no more.

7.2 The bound

Closing the residual requires the light masses to scale as $M_i^{-1.0144}$ rather than M_i^{-1} : a non-universal spread of 7.5 % across $M_1 \dots M_3$ against the 2.2 % that one loop provides, with two loops another factor of $1/16\pi^2$ down. The residual is therefore not radiative dressing of a democratic seesaw at any order; it is tree-level structure — a symmetric non-democracy of the Dirac sector of about 0.7 % in the phase — whose source is not identified. An earlier version of this paper asserted that this structure is the same one lepton mixing requires; Section 7.3 disproves that, and the claim is retracted in Appendix B.

7.3 Mixing: tri-bimaximal from 2T, and the independence theorem

2T is the double cover of A_4 , and in its 3-representation the order-3 elements are the generation cycle while the order-4 elements $\pm i, \pm j, \pm k$ are the π -rotations generating the Z_2 subgroups. With the charged leptons preserving the Z_3 (circulant, Fourier-diagonal) and the neutrino mass matrix invariant under one Z_2 , the PMNS matrix is exactly tri-bimaximal [33] — $\sin^2\theta_{12} = 1/3$, $\theta_{23} = 45^\circ$, $\theta_{13} = 0$ — with the heavy-law spectrum untouched: the A_4 mechanism [34, 35], realised for $T' = 2T$ with quark masses by [36], which the selection theorem of [20] would explain. Against NuFIT 6.0 [37]: θ_{12} at $+2.2\sigma$, θ_{23} octant-ambiguous, $\theta_{13} = 0$ excluded. The Dirac texture realising it is orthogonal, $m_D = m_0 U_{\text{TBM}}^T$, and one further rotation in the neutrino 2–3 plane (14.9° , not derived) accommodates $\theta_{13} = 8.52^\circ$ with θ_{12} then at 34.35° (0.9σ) — still orthogonal, spectrum still exact. Theorem (polar decomposition; script J): writing $m_D = O \cdot P$ with O orthogonal and P symmetric, O carries all the mixing and moves no seesaw eigenvalue, while P carries all the spectrum shift; the PMNS angles and the phase residual of 7.2 are therefore independent parameters of the texture. Numerically, the perturbation size that yields $\theta_{13} = 8.5^\circ$ and the size that yields the phase residual differ by a factor of thirty.

8. The boundary scale and the one dimensionful input

A common-scale consistency statement. Imposing the surviving conditions $n_{\text{down}} = 6$, $m_b/m_\tau = 1$ and $m_s/m_\mu = 1/3$ on the one-loop running spectrum [7]:

$$\mu^* = 3.30 \times 10^6 \text{ GeV} \quad \chi^2 = 0.12 / 2 \text{ dof} \quad (n_{\text{down}} + 0.14\sigma, m_b/m_\tau - 0.16\sigma, m_s/m_\mu + 0.28\sigma)$$

The dimension enters through $\alpha_s(\mu^*) = 0.0518$ and $\mu^*/\Lambda = \exp(2\pi/b_0\alpha_s) = 3.3 \times 10^7$: ordinary dimensional transmutation: μ^* is a pure number times Λ_{QCD} , the number fixed by the measured α_s . Stated precisely: every target the construction sets is dimensionless (rationals, ratios, angles), and the single dimensionful input converting them into masses is Λ_{QCD} . The calculation is one-loop with an honest resolution of ± 0.5 decades. The neutrino seesaw scale (10^{14} – 10^{16} GeV, Section 3.4) is distinct from μ^* , as the universal seesaw structure requires — the two inverse sectors have different heavy partners.

9. The five clocks and the two condensates

The phases being settled as far as the data allow, the paper turns to the dimensionful side: what sets the five clocks, and how much of the spectrum's scale content is structure rather than input.

Each sector's dimensionful content is one clock. The five clocks (the four sectors' plus the heavy Majorana clock; script P), their forms and their branches:

clock	value	form	branch
A_ℓ	313.85 MeV	direct	QCD (odd N)
A_d	26.82 ± 0.27 MeV	inverse	QCD (odd N)
A_u	21.56 ± 0.08 GeV	direct	electroweak (even N)
A_v	1.507 meV	inverse	electroweak (even N)
A_M	2.011×10^{16} GeV	direct (heavy)	electroweak (even N)

Four relations are established: $A_M \cdot A_v = (v/\sqrt{2})^2$ exactly — the reciprocity of the light and heavy circulants, the $y_v = 1$ seesaw made explicit; $A_\ell = m_N/N_c$ at $+0.28\%$; A_d chained to A_ℓ through $m_b(\mu^*) = m_\tau(\mu^*)$ (Section 8); and $A_u = (v/\sqrt{2}) \cdot y_t/f_t^2$ with $y_t = 0.934$, one dynamical coupling. On top of these sits one candidate with structure: on each branch, (direct clock) $\times$ (inverse clock) = m_D^2 , with m_D the decay constant of the condensate that branch rides — exact on the electroweak branch by the

first relation, and on the QCD branch $\sqrt{A_\ell \cdot A_d} = 91.75 \pm 0.46$ MeV against $f_\pi = 92.1 \pm 0.6$ (0.46σ ; script Q). The seesaw is then not a neutrino peculiarity: each branch has one, the inverse sector's mass being the seesaw image of its direct partner through the branch's own condensate — $v/\sqrt{2}$ for the Higgs vacuum, f_π for the chiral one.

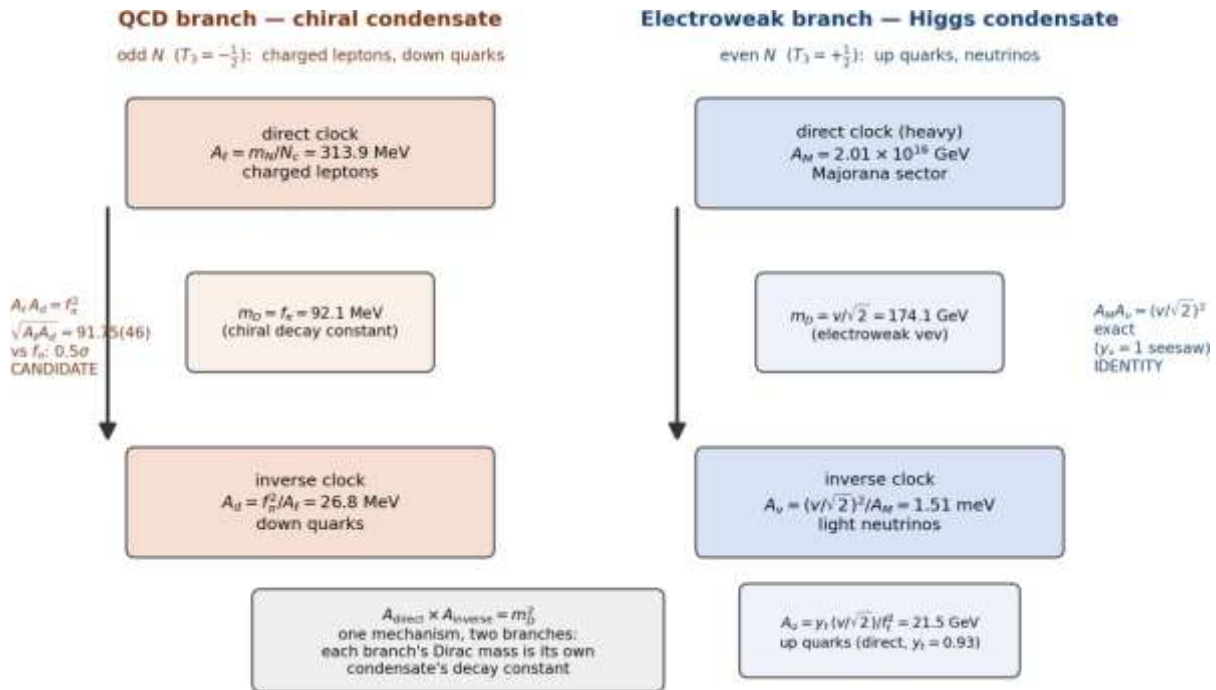


Figure 1. The two condensates and the branch seesaw. Odd- N ($T_3 = -1/2$) sectors ride the chiral condensate, even- N ($T_3 = +1/2$) the electroweak one; on each branch the direct and inverse clocks close through the condensate's decay constant. The electroweak relation is exact by construction; the QCD relation is a candidate at 0.46σ . Figure generated by fig_two_condensates.py from the values of scripts P and Q.

Tests of the QCD-branch candidate (script Q): it is exact at a floor of 1.91 GeV, coexisting with the down phase at the conventional 2 GeV (both within 1σ there) — if real, the relation selects the scheme's one free scale from below while the phase brackets it from above. It anchors the light end of the down spectrum, while the b - τ chain anchors the heavy end; their ratio, 0.925 , is exactly the sector's known ratio residual, so the two routes are different statements and the residual now sits between two clock relations. Within 1σ of 91.75 MeV the only QCD scale is f_π (m_s and $(2/3)m_\pi$ lie within 2.5σ). Through leading-order GOR the relation implies a condensate of $(277 \text{ MeV})^3$ against the lattice $(272 \pm 5)^3$ — consistent, not reducible. Its sharpest form is parameter-free: $m_d(2 \text{ GeV}) = (3f_\pi^2/m_N)(1 + m_d/m_s + m_d/m_b)/6 = 4.75$ MeV against the lattice 4.70 ± 0.05 (1σ). Status: CANDIDATE.

The count. Five clocks = two condensate scales ($\Lambda_{\text{QCD}} \rightarrow m_N \rightarrow A_\ell$, and v) + one heavy scale ($A_M \approx M_{\text{GUT}}$) + two closure relations + one coupling (y_ν). If the QCD-branch relation holds, A_d follows from A_ℓ with no running, and the entire dimensionful content of the four-sector spectrum is two condensates, one unification scale and the top Yukawa; everything else in the spectrum is angles.

10. Amplification: the discipline of compressed coordinates

Everything above leans on compressed coordinates — phases and normalisations with large leverage on mass ratios — and this section states the discipline that keeps them honest.

δ and n are compressed coordinates. The sensitivities $d \ln(m_2/m_1)/d\delta$ are 343.8 (up), 56.1 (leptons), 5.2 (down), and $d \ln(m_3/m_2)/d\delta$ are -19.5 , -4.9 , -32.8 , with comparable magnitudes in n . Consequently a 0.5σ agreement in shape space can be a 20% statement in an observable ratio, and a 5% observable miss can be a 0.8σ shape statement; both directions occur in Section 3 and are reported in both spaces throughout. The up sector's scheme sensitivity — $1/u$ ranging from 11.9 to 19.1 across defensible schemes — is this same amplification acting on the scheme choice, which is why only the value pre-registered in [19] carries evidential weight there, and why the intermediate scheme-dependent values are retracted rather than averaged (Appendix B).

11. The phase law and the Clebsch relation: an open two-law conflict

The renormalisation-group-invariant double ratio $R = (m_\mu/m_e)/(m_s/m_d) = 10.59 \pm 0.18$ excludes the Georgi–Jarlskog Clebsch of 3 [15] at $8.9\text{--}23.5\sigma$, stably: R varies by 0.1 % over sixteen decades, so no intermediate-scale physics rescues it. Within the $SU(5)$ adjoint-eigenvalue family the one reachable candidate is $R = 32/3$ (0.15σ ; base rate 0.78). It conflicts with the phase law through m_u/m_d : the law gives $m_s/m_d = 20.053$ and hence $m_u/m_d = 0.4728$, the Clebsch gives 0.4536 — 4.2 % apart, and both are currently allowed (the tight 2024 determination 0.455(8) [11] favours the Clebsch at 0.17σ ; the PDG value 0.474(65) [9] favours the law at 0.02σ ; FLAG 0.465(24) [8] sits between). One lattice number, at $\sigma(m_u/m_d) < 0.0064$, decides between two relations that cannot both hold — and the same number is the down sector's dominant uncertainty. A by-product of the law is exact on both sides and connects to mixing [16, 19]: $\theta_d - \theta_u = 27/144 - 11/144 = 1/9$.

12. Ledger and live predictions

Quantity	Status
A_ℓ	INPUT — the one dimensionful anchor (Λ_{QCD} by transmutation)
$1/u$ law $\rightarrow \{3, 4, 9, 12\}$ [19]	holds $0.4\sigma / 0.07\sigma / 2.8\sigma / 2.2\sigma$ in the own-scale scheme
ε_s per sector	ESTABLISHED (60σ)
$k_S = 1$ ($n = 6$): leptons, down, v-heavy	holds $0.4\sigma / 0.8\sigma$ / imposed; an input with an irreducibility proof [19]
$k_u = 1.2382$	OPEN — the up amplitude
condensate floor	input in [2.1, 4.5] GeV from the down phase (script K); the branch-seesaw candidate selects $\approx 1.9\text{--}2.0$ GeV from below
$1/u_v = 9$	composite (Sec. 5); dressed 30 % by computed loops; residual an unexplained 0.7 % symmetric non-democracy (Sec. 7.2); JUNO decides
even- N common factor $g = 0.9916 \pm 0.0028$	CANDIDATE (3.0σ from 1, $\chi^2 = 0.3/1$): both $T_3 = +\frac{1}{2}$ sectors short of their N_c by $\sim 1\%$; sits on the open threefold-cover step
$A_\ell \cdot A_d = f_\pi^2$ (branch seesaw)	CANDIDATE (0.46σ ; Sec. 9): predicts $m_d = 4.75$ MeV (1σ); selects the floor $\approx 1.9\text{--}2.0$ GeV
charge dictionary $Q \leftrightarrow 1/u$	ESTABLISHED (Sec. 4): exact to 10^{-4} in the lepton sector
rational 3, 4, 12 from $2T$	DERIVED (Sec. 5); 9 not generated
$R = 32/3$	CANDIDATE — conflicts with the law via m_u/m_d
μ^* , A_d/A_ℓ	fitted jointly, $\chi^2 = 0.12/2$
A_u (vev branch), A_v	OPEN

Prediction	Value	Decided by
m_τ	1776.97 MeV	world average 1776.93 ± 0.09 (0.4σ); BESIII scan / Belle II at $\sigma \approx 0.05$ MeV decides
$m_c(m_c)$	1.2573 ± 0.0055 GeV	currently 2.2σ from PDG 1.2730(46); $m_t(m_t)$ is the limiting input
$m_d(2 \text{ GeV})$ [branch-seesaw candidate]	4.75 MeV	lattice 4.70 ± 0.05 : 1σ ; sharpens with m_u/m_d
neutrino ordering	normal, required	ordering determinations
Δm_{21}^2 if the heavy phase is exactly 9	$7.88 \times 10^{-5} \text{ eV}^2$	JUNO [13] (3.2σ against now)
Σm_ν	0.058 eV	cosmology [17]

m_u/m_d	0.4728 (law) vs 0.4536 (Clebsch)	lattice at $\sigma < 0.0064$ [8,11]
$\theta_d - \theta_u$	1/9 exactly	long-term spectroscopy

13. Conclusion

A law proposed on quantum-number grounds — $1/u = N_c^{3/2+T_3} \cdot C_2(3)^q$, four sectors, four rationals — was tested in the one scheme the data select, and the result has a structure no fit would produce. The two odd-isospin sectors sit exactly on their integers, at 0.4σ and 0.07σ , with no auxiliary parameter. The two even-isospin sectors — precisely the two that carry the extra factor of N_c — are both short of theirs by a common one per cent ($g = 0.9916 \pm 0.0028$), and the group theory says why they are the vulnerable pair: the binary tetrahedral group generates the third-turn, the quarter-turn and the twelfth outright, but the threefold cover behind the even-isospin factor corresponds to no element of any finite subgroup of $SU(2)$. The law's strain and its open derivation step coincide, sector for sector.

Around the law, three structures have become sharp. Inverted, it is a dictionary between mass ratios and electric charge, exact to one part in 10^4 for the leptons, with one unit of charge equal to one factor of N_c because three is odd; written on the Clifford occupation number, its colour flag and isospin are the triplet condition and the grading of the ideal. The mixing follows $2T$'s own Z_3/Z_2 structure — tri-bimaximal at leading order, one rotation from the data — and is provably independent of the neutrino phase residual, which stands as the construction's one unexplained number. And the five clocks close into a two-branch seesaw in which each inverse sector is the image of its direct partner through its own condensate's decay constant: exact on the electroweak branch, a 0.46σ candidate on the QCD branch that predicts the down-quark mass from f_π and m_N at 1σ and would leave the spectrum's dimensionful content at two condensates, one unification scale and the top Yukawa.

What remains is sharply localised and falsifiable on a near-term timescale: $m_c(m_c) = 1.2573 \pm 0.0055$ GeV against the PDG's 1.2730 ± 0.0046 , already a 2.2σ test that a better m_t sharpens; JUNO's Δm^2_{21} against 7.88×10^{-5} ; $m_\tau = 1776.97$ MeV, now 0.4σ from the world average; the lattice m_u/m_d at the 0.006 level, which gates the floor, the down phase, the two-law conflict and the branch-seesaw candidate at once; and normal ordering, required. The honest summary is unchanged in form from [19] and more specific in content: the geometry of circulation forces the form of the spectrum; the gauge quantum numbers force the phases, exactly where the group generates them and one per cent short exactly where it does not; charge and mass are one reading apart; and the scales are two condensates and a coupling — the dynamics of the medium, which is the boundary of this paper and the subject of the next.

Attribution and acknowledgements

The underlying phenomenology — the charged-lepton relation, its geometric reading, the circulant phase, the Z_3 -symmetric quark parameterization, the inverse form and the seesaw extension — is due to Koide [1], Foot [2], Brannen [3], Żenczykowski [4] and Rivero [5], and the exactness question is framed by Sumino [6]. The completeness of the three-parameter form was independently emphasised by Quinta [18], whose parameterization is verified here to be an exact dictionary of the one used. All numerical inputs are from FLAG [8], the PDG [9], HPQCD [10], the 2024 m_u/m_d determination [11], Belle II [12], JUNO [13] and DESI [17]. The threshold-running formalism follows Antusch, Kersten, Lindner, Ratz and Schmidt [14]. This paper is self-contained but rests on the derivations of the companion manuscripts [19, 20]; the author's prior methodological work appears in [21].

Appendix A. Verification scripts (supplementary material)

Eighteen Python scripts reproduce every number in this paper and accompany it as supplementary material (verification_scripts.zip; dependencies: numpy, scipy; run as `python3 <script>`). They are cited in the text as scripts A–Q and the formula module.

Script	Contents
unified_mass_formula.py	The formula, the law, own-scale outputs for all sectors, the m_u prediction.
verify_session_a_shape.py	Shape coordinates from lattice ratios; the lepton over-determination (2.52 keV); the 60σ frame selection.

Script	Contents
verify_session_b_running.py	One-loop SM RGEs; the flavour-universality control; drift and scale-probe tables.
verify_session_c_phase_clebsch.py	The QED-dressing validation on the leptons; the Clebsch family from the SU(5) adjoint; the m_u/m_d conflict; the amplification table.
verify_session_d_inverse_frame.py	The retracted common-scale frame analysis, retained for the audit trail of Appendix B.
verify_session_e_seesaw.py	The sequential-threshold seesaw running; the cancellation of the top trace; the 27 % result.
verify_session_f_ownscales.py	The own-scale scorecard; both scheme discriminators; the floor scan; the m_u prediction; the boundary refit.
verify_session_g_charge_inversion.py	Charge read off the masses; the law on the Cl(6) occupation number; twelve masses from four inputs with the clock chain.
verify_session_h_2T_ideal.py	2T on the rotor tensored with the ideal: element orders, the generation Z_3 on both halves, the 432-element group, reachable shift fractions, absence of order 9.
verify_session_i_seesaw_finite.py	Finite one-loop Z/H seesaw matching in the Fourier basis; cross-check of script E; the 7.5 % bound.
verify_session_j_texture.py	The A_4/T' mechanism from 2T; tri-bimaximal PMNS; the TM_1 completion; the polar-decomposition independence theorem.
verify_session_k_cover_floor.py	The common even-N factor g; the condensate floor with FLAG 2024 inputs and N_f -matched running.
verify_session_p_clocks.py	The five clocks, the established relations, the branch-seesaw search, the count.
verify_session_q_branch_seesaw.py	Tests of $A_t \cdot A_d = f_\pi^2$: floor selection, two-ends analysis, base rate, GOR cross-check.
fig_two_condensates.py	Generates Figure 1 from the values of scripts P and Q.
verify_session_l-o (exploratory)	The chiral-soliton and vector-meson sector: the hedgehog vortex, the Casimir budget, five determinations of the Skyrme parameter, the HLS parameters a and g.

Inputs used throughout (FLAG 2024, $N_f = 2+1+1$ unless noted [8]): $m_s/m_{ud} = 27.227(81)$, $m_c/m_s = 11.766(30)$, $m_u = 2.14(8)$ [2+1: 2.27(9)], $m_d = 4.70(5)$, $m_s = 93.46(58)$, $m_b(m_b) = 4.200(14)$, $m_u/m_d = 0.465(24)$ [2+1: 0.485(19)]; HPQCD $m_b/m_c = 4.586(12)$ [10]; the chiral $1/N_c$ bound $m_u/m_d = 0.455(8)$ [11]; $m_c(m_c) = 1.2730(46)$, $m_t(m_t) = 162.5(7)$, lepton pole masses with $m_\tau = 1776.93 \pm 0.09$ (2024/25 average), $\alpha_s(M_Z) = 0.1183(7)$, all PDG [9]; $\Delta m_{21}^2 = 7.50(12) \times 10^{-5} \text{ eV}^2$ [13] and $\Delta m_{31}^2 = 2.513(21) \times 10^{-3} \text{ eV}^2$ [9]; $f_\pi = 92.1(6)$, $m_N = 938.92$, $v = 246.22 \text{ GeV}$.

Appendix B. Provenance and retracted intermediate claims

The working record behind this paper went through three internal versions; what survives is what is printed above, and what did not is listed so that it cannot silently return. Retracted, with reasons: (i) $n_{up} = 8$ — a common-scale value of a hypersensitive coordinate (0.34σ at M_Z), absent in the law's own scheme, where $n_{up} = 7.606$; (ii) $1/u_{up} = 19$ — likewise common-scale, and uninformative from the start (base rate ≈ 1); (iii) an 'inverse-frame factor' $f = 0.9856(49)$ with an associated frame-dependence pattern — a 2.9σ artefact of evaluating the phase law at a common scale, dissolved by the own-scale scheme in which the down sector needs no factor (0.07σ) and the neutrino deviation stands alone; (iv) the claim that the two-law conflict of Section 11 had dissolved — it rested on (iii); (v) the claim of versions 5–7 that the neutrino phase residual and the lepton mixing require the same Dirac structure — disproved by the polar-decomposition theorem of Section 7.3, which separates them exactly. Superseded by data updates rather than retracted: the 0.58σ up-sector closure (an artefact of $m_c = 1.27 \pm 0.02$; now 2.2σ with the PDG m_c), the $m_u = 2.75 \text{ MeV}$ falsifier (prediction uncertainty $\pm 1.05 \text{ MeV}$; replaced by the $m_c(m_c)$ prediction), and the 1.6–3.0 GeV floor bracket ($N_f = 3$ running; now [2.1, 4.5] GeV). One inequality outlives its target: with exact integers (8, 19) the up sector would require $m_u > m_d$, excluded by the neutron–proton splitting alone; the observation is correct, and the configuration it excludes is no longer claimed. Corrections adopted from the companion manuscript's earlier text [19]: the statement that the up sector's departure is

driven by top-Yukawa running is quantitatively wrong by a factor of 37 (drift +0.6 % against departure −22.5 %); the doublet-ratio form of the colour split is superseded by the law itself, which it approximated; the scope note that the coloured-sector phases are not established is withdrawn by Section 2.3.

Appendix C. Derivations

The derivations of the mass and charge formulas in full, the supporting theorems in compact form, and the two foundational constructions by pointer. Each is verified by the script named.

C.1 The mass formula, from the rotor to the printed form

Step 1 — mass is a modulus. Definition 1 of [19]: a particle is a trapped circulation of the rotor medium, and its rest energy is the elastic energy of that circulation,

$$m = \frac{\kappa}{2c^2} \bar{z}z$$

quadratic in the rotor amplitude z and blind to its phase — which is why mass is real and Lorentz invariant, and why the natural variable of the family structure is $\sqrt{m}$, the amplitude itself, not the energy.

Step 2 — three generations are one Z_3 orbit. The three families are the three modes of a single rotor under the generation cycle (the order-3 element of $2T$, Section 5). The amplitude operator on the family triplet must therefore commute with the cyclic shift S ($S^3 = 1$), i.e. be a circulant. The most general Hermitian circulant is a real combination of the identity and the phase-loaded hop $H = (e^{i\delta} S + e^{-i\delta} S^\dagger)/2$, so

$$\sqrt{M} = A_S(1 + \sqrt{2} \bar{k}_S H), \quad H = \frac{e^{i\delta} S + e^{-i\delta} S^\dagger}{2}$$

Three real parameters and no more: the clock A_S (the overall stiffness, carrying every unit), the amplitude k_S , and the phase δ — the parameter count of Section 2 is not a choice but the dimension of the Hermitian circulant space.

Step 3 — diagonalise. S has eigenvalues $\omega^k = e^{2\pi i k/3}$ on the Fourier modes, so H has eigenvalues $\cos(\delta + 2\pi k/3)$ and the square-root masses are

$$\sqrt{m_k} = A_S \left(1 + \sqrt{2} \bar{k}_S \cos \left(\delta + \frac{2\pi k}{3} \right) \right), \quad k = 0, 1, 2$$

Step 4 — the $\sqrt{2}$ is equipartition. The two operators in $\sqrt{M}$ carry Frobenius norms $\|1\|^2 = 3$ and $\|\sqrt{2} \cdot k_S \cdot H\|^2 = 2k_S^2 \cdot \Sigma \cos^2 = 3k_S^2$: at $k_S = 1$ the democratic (identity) and generation-splitting (hop) sectors carry exactly equal weight. The factor $\sqrt{2}$ is the normalisation that makes $k_S = 1$ the symmetric point, and Proposition 6 of [19] shows the condition is irreducible — it cannot be traded for a phase shift. Three sectors sit at $k_S = 1$; the up sector's $k_u = 1.2382$ is the one departure and remains input.

Step 5 — the frame exponent. Theorem I of [20] closes the circulant algebra under the inversion ι : sectors reached through ι (down quarks, and neutrinos through $\iota \cdot \sigma_M$) carry the same circulant on the reciprocal amplitudes $1/\sqrt{m}$. Both cases are one formula with a frame index $\varepsilon_S = \pm 1$:

$$m_k^{(\varepsilon)} = A_S \left[1 + \sqrt{2} \varepsilon_S \bar{k}_S \cos \left(\delta_S + \frac{2\pi k}{3} \right) \right]^{2\varepsilon_S}$$

which is the printed formula. The data fix the frame per sector at 60σ (script A); the phase δ_S is then supplied by the holonomy (C.3) and the quantum-number law (C.7). Verified end-to-end by `unified_mass_formula.py`.

C.2 Scale invariance and the angle (Proposition 1 of [19])

Write the square-root masses as a vector $r = (\sqrt{m_0}, \sqrt{m_1}, \sqrt{m_2})$ and let $\hat{n} = (1, 1, 1)/\sqrt{3}$. Then

$$Q = \frac{\Sigma m}{(\Sigma \sqrt{m})^2} = \frac{|r|^2}{3(r \cdot \hat{n})^2} = \frac{1}{3 \cos^2 \varphi}$$

Q depends only on the angle φ between r and the democratic axis, and is homogeneous of degree zero: flavour-universal running multiplies all masses by one factor and leaves it exactly invariant (control to seven decimals, script B). The Koide value $2/3$ is the single condition $\varphi = 45^\circ$. ■

C.3 The holonomy form $\delta = u(1-u)$ (Proposition 2 of [19])

The Green function of $-d^2/d\theta^2$ on the unit circle, zero mode removed, is $G(\theta) = \pi/6 - \theta/2 + \theta^2/4\pi$ for $\theta \in [0, 2\pi]$. A single elementary shift Δ dressed by it accumulates

$$G(0) - G(\Delta) = \frac{\Delta}{2} - \frac{\Delta^2}{4\pi} = \pi \cdot u(1-u), \quad u = \frac{\Delta}{2\pi}$$

so the normalised holonomy is $\delta = u(1-u)$ exactly — a downward parabola in the shift fraction, maximal at the half-turn, vanishing at full turns. All four sector phases are this one function evaluated at the four shift fractions. ■

Normalisation, stated for completeness. The Gaussian dressing of a shift is generally $\gamma(\Delta) = \lambda \cdot u(1-u)$, with $\lambda = (\text{endpoint charge})^2 \cdot \pi^2 / (\text{stiffness})$ collecting the fast sector's normalisation; the division by π above is the canonical choice $\lambda = 1$ (unit endpoint charge, canonical stiffness) — the one convention of the construction. Empirically, $\lambda = 1$ to 1.1×10^{-7} from the τ -free determination (m_e/m_μ alone) and to $1.2 \pm 2.8 \times 10^{-5}$ including m_τ ; whatever completes the theory must produce this unit value. The same function and the same canonical choice have since been reached independently: the compact-cycle construction of [42] computes the Berry phase of one family shift as $\gamma(\Delta) = (1/\pi^2) \sum_{n \geq 1} (1 - \cos n\Delta)/n^2$, which equals $u(1-u)$ identically by the Bernoulli series $\sum \cos(n\theta)/n^2 = \pi^2/6 - \pi\theta/2 + \theta^2/4$ — the Fourier form of the Green function above — with the $\lambda = 1$ assumption flagged there as well.

C.4 The clock identities (Section 2.1)

With $f_k = 1 + \sqrt{2} \cdot k_s \cdot \cos(\delta + 2\pi k/3)$, the Z_3 sums $\sum \cos(\delta + 2\pi k/3) = 0$ and $\sum \cos^2(\delta + 2\pi k/3) = 3/2$ give

$$\sum_k f_k^2 = 3 + 2k_s^2 \cdot \frac{3}{2} = 3(1 + k_s^2) \equiv n_s$$

independent of δ . Hence $A_s = \sum m/n_s$ in the direct frame and $A_s = n_s/\sum(1/m)$ in the inverse — the harmonic clock forms, and the identity $\sum m_\ell = 6A_\ell$ behind the $m_N = N_c \cdot A_\ell$ relation. ■

C.5 The charge formula, from the ladder operators to the dictionary

Step 1 — charge is a count. In the $Cl(6) \cong C \otimes O$ description of one generation [22], the eight states of a minimal ideal are built by raising $N \in \{0, 1, 2, 3\}$ of three colour ladder operators on the vacuum, and the electric charge is one third of the number operator:

$$Q = \frac{1}{3} \sum_i \alpha_i^\dagger \alpha_i = \frac{N}{3}$$

The $SU(3)_c$ acts on the ladder index, so $N = 0$ and $N = 3$ are colour singlets and $N = 1, 2$ are triplets: the colour flag is $0 < N < 3$, not an assumption. The four sectors sit at $N = 3$ (e-type), 1 (d-type), 2 (u-type), 0 (v), and the parity of N coincides sector-by-sector with weak isospin: $T_3 = +1/2$ is even N , the even Clifford subalgebra.

Step 2 — substitute the assignment into the law. The ideal's hypercharge is $Y = -1 + (4/3)q$, so $T_3 = Q + 1/2 - (2/3)q$. Then

$$\frac{1}{u} = N_c^{3/2 + T_3} \cdot C_2(3)^q = 3^{2+Q} \cdot 3^{-2q/3} \cdot \left(\frac{4}{3}\right)^q = 3^{2+Q} \cdot \left(\frac{4}{3^{5/3}}\right)^q$$

For colourless fermions $1/u = 3^{2+Q}$: the charge sits in the exponent of the colour count. Inverting, with the error propagated as $\sigma_Q = \sigma(1/u)/[(1/u) \cdot \ln 3]$:

$$Q = \log_3 \frac{1}{u} - 2 - q \log_3 \frac{4}{3^{5/3}}$$

which returns $Q = -1.0000(1), -0.332(13), +0.661(9)$ and $-0.013(5)$ from the measured own-scale phases (Section 4, script G) — the dictionary.

Step 3 — the number-operator form and the doublet theorem. Writing $[N \text{ even}]$ for the grading and $[0 < N < 3]$ for the triplet condition, the law becomes $1/u = N_c^{1+[N \text{ even}]} \cdot (4/3)^{[0 < N < 3]}$, which returns 9, 4, 12, 3 at $N = 0, 1, 2, 3$. A full unit of electric charge is $\Delta N = 3$ ladder steps; because $N_c = 3$ is odd, $N \rightarrow N + 3$ flips the grading, so the two members of a weak doublet always differ by exactly one power of N_c :

$$\frac{(1/u) \text{ at } T_3 = +1/2}{(1/u) \text{ at } T_3 = -1/2} = N_c$$

measured as 2.96 (lepton doublet) and 2.98 (quark doublet). Were N_c even, a unit of charge would preserve the grading and the rule would fail: the factor of N_c per unit of charge is a theorem about the number three. Scope: this is the law inverted, not a derivation of charge quantization — the values entered through T_3 and q — and the masses cannot distinguish a particle from its antiparticle, so the sign is fixed up to conjugation. ■

C.6 The independence theorem (Section 7.3)

Let $m_\nu = m_D^T M^{-1} m_D$ with M diagonal and positive, and polar-decompose $m_D = O \cdot P$, O orthogonal, $P = (m_D^T m_D)^{1/2}$ symmetric positive. Then $m_\nu = P \cdot O^T M^{-1} O \cdot P$. If $P = m_0 \mathbb{1}$ this is an orthogonal conjugation of $m_0^2 M^{-1}$: the eigenvalues are exactly the seesaw spectrum for every O , and the eigenvectors — hence the PMNS matrix — are rotated by O^T . Conversely any eigenvalue shift requires $P \neq m_0 \mathbb{1}$, and to first order in $P = m_0(\mathbb{1} + \varepsilon S)$ the shift is set by the diagonal of S in the mass basis while its contribution to mixing is suppressed by the mass splittings. The mixing angles and the phase residual are therefore independent parameters of the texture; numerically the perturbation sizes required for θ_{13} and for the residual differ by a factor of thirty (script J). ■

C.7 The law, assembled — and its two open steps (Proposition 3 of [19])

Why this particular form? Assembled on the Clifford occupation number, the law is a count of the medium's response to a sector's quantum numbers, factor by factor. Baseline: every sector's elementary shift is at least one third-turn — one factor of N_c — because the generation cycle is the order-3 element of $2T$ acting on the rotor (derived, Section 5). Grading factor: when the occupation number N is even ($T_3 = +1/2$, the even Clifford subalgebra), the shift is deepened by a second factor of N_c — the threefold cover — because the even states sit on the covering circle rather than the quotient (OPEN: no element of any finite subgroup of $SU(2)$ realises this factor, and both even- N sectors are measured $\sim 1\%$ short of it). Colour factor: when the state is a colour triplet ($0 < N < 3$), the shift carries the colour multiplier, identified with $C_2(3) = 4/3$ (OPEN at the identification level: a colour-commuting shift cannot carry the Casimir by Schur, leaving $C_2(3)$ and

$|Z_4|/|Z_3|$ indistinguishable at $N_c = 3$, though the $4/3$ itself is forced by the down sector at 0.07σ). The product is $1/u = N_c^{1+[N \text{ even}]} \cdot (4/3)^{[0 < N < 3]}$ — the number-operator form of C.5, algebraically identical to $N_c^{3/2+T_3} \cdot C_2(3)^q$ by the substitution chain shown there. Each factor is thus either derived (the baseline third-turn; the grading–isospin identification; the triplet condition; one factor of N_c per unit of charge) or declared open (the cover; the Casimir identification), and the measured strains sit exactly on the open factors.

The quantum-number assignment $1/u = N_c^{3/2+T_3} \cdot C_2(3)^q$ reproduces $\{3, 4, 9, 12\}$, and Section 5 derives 3, 4 and 12 as rotation orders of $2T$. Two steps remain open, stated as such: the identification of the colour multiplier with $C_2(3)$ is obstructed at the algebra level (a colour-commuting shift cannot carry the Casimir — Schur's lemma), leaving $|Z_4|/|Z_3|$ and $C_2(3)$ indistinguishable at $N_c = 3$; and the factor N_c for the even-isospin sectors corresponds to a threefold cover realised by no element of any finite subgroup of $SU(2)$ (script H) — the step on which the common one-per-cent shortfall of Section 2.3 sits.

C.8 Foundations, by pointer

Definition 1 of [19] — mass as the modulus-squared of the rotor — grounds C.1; its $Cl(6) \cong C \otimes O$ setting is [22]. Theorem I of [20] — the closure of the circulant algebra under inversion and the coupling-site rule — fixes the direct/inverse assignment εs structurally; the data confirm it at 60σ (script A). Both are used here as cited results; their proofs are in the companion manuscripts, deposited with DOIs in the reference list.

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