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The Impact of Toxicant on Prey Predator EcologicalSystem: A Mathematical Approach

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Abstract: In this research paper, a three species food chain mathematical model with toxicant is proposed and studied. Discussed the behavior of prey predator system in the presence of toxicant. It is considered that only prey species release toxins as its defense and also prey species is having logistic growth. All the reasonable equilibrium points of the system are discussed for local stability by using stability conditions. At the end, the numerical simulations are done for different parameters. All the graphs illustrate thestability conditions.

Keywords: Prey-Predator, Toxicant, Variational Matrix, Stability

1 Introduction

Ecology is the subject which mainly deals with the detailed study of the organisms. It is commonly considered as one of the branches of biology, which studies about living or- ganisms. Ecology is mainly connected with/as the highest proportions of organization, individual organisms, species or population, species community, the ecological community and the ecosphere as a whole. The efforts related to wildlife protection, habitat manage- ment, mitigation of ecological effects of environmental pollution, ecosystem remodelling, species reintroductions, fisheries, forestry and preservation of biodiversity are often the direct domain of applied ecology [1, 2].

Food chains tell about the relationships of prey predator systems. Food chain represents the eating relationships among different species within an ecosystem or at particular living environment. In another words, the food chain is the circulate of energy from one organism to another and so on. A food chain shows that how a living thing gets its food. In biosphere, few animals eat plants and few animals eat other animals. Many types offood chains depending on habitats or the environmental factors. Every food chain starts with a type of autotroph, let it be a plant or some other unicellular organism [1, 2, 3, 4, 5]. The lotka voltera equations are also known as the prey predator models, it is a classification of first order nonlinear differential equations, which is used to describe about the changing of biological process. In which, one as a prey and the other as a predator andthese prey predators interact with each other for food. The predator population depends on the population of prey for their survival.

In the environment there is an uncontrolled and some immoral emission toxics ac-countable for the extinction of large number of species and if there is no control overthe situation then the prey predator species which are extinction at largely and may disappear completely [6, 7]. Mainly [8] the marine populations accumulate toxins and transfer to other prey predator populations through food chain that may affect higherlevel of trophic. Interestingly, in the prey predator environment species use some tactical way or strategies to protect themselves by using toxins as their weapons. Such species release toxic substances in order to protect their habitat or populations. Therefore, it isthe significant and an important topic to discuss the effect of toxic on the prey predator populations so that it could be easier to predict the survival or extinction of any species of terrestrial or marine community in the future [9, 10, 11, 12, 13, 14, 15].

The one of the existence techniques of the prey population is to use that they release some kind of toxic substances to protect themselves from the predator populations [3, 12, 13]. In the research work of [18], the authors have studied and developed a marine model of tri level food chain (micro algae - zooplankton- fish) to take care of methyl mercury (*MeHg*) along prey predator models. Beforehand, good research has been done on tri trophic prey predator food chain models remembering the toxicant crash for the endurance or elimination of populations in the models [16, 17, 18].

Presently, a prey predator mathematical model is considered which is actually motivated by [3] to discuss the effect of toxic substance on predator populations. In the model, prey is considered as a logistically growing population and also prey Population release toxins as their defense.

2 The Mathematical Model

A prey predator mathematical model is considered to discuss the effect of toxic substance releases by prey population as defensive from predator populations [3]. In the model, it is assumed that prey grows logistically releases the toxic substance for defensive. The population of prey is denoted by $P_1(T)$ at time T. The population of first and top predatoris denoted by $P_2(T)$ and $P_3(T)$ at time T and r is known as the intrinsic growth rate, andk is the carrying capacity, α_1 and α_2 predation rates of prey by P_1 and P_2 respectively, $b_1$is a toxic substance which is released by P_1. c_1 and c_2 and c_3 are crowding effects of P_1 and P_2 and P_3 respectively. Prey is capable to produce toxic substance for its survival, which is denoted by b_1. Natural death rates of P_1, P_2 and P_3 are denoted by d_1, d_2 and d_3 respectively.

Assuming all these as the state variables, also have formulated the prey predator mathematical model using the following nonlinear ordinary differential equations mainlyto study the toxic effect of the food-chain:

Model A

$$\frac{dP_1}{dT} = rP_1\left(1 - \frac{P_1}{k}\right) - \propto_1 P_1P_2 - b_1 P_1^2 P_2 - d_1P_1 - c_1 P_1^2 \qquad (1)$$

$$\frac{dP_2}{dT} = e_1 \propto_1 P_1P_2 - \propto_2 P_2P_3 - d_2P_2 - c_2 P_2^2 \qquad (2)$$

$$\frac{dP_3}{dT} = e_2 \propto_2 P_2P_3 - d_3P_3 - c_3 P_3^2 \qquad (3)$$

and the non-negative conditions are: $P_1(0) > 0, P_2(0) > 0, P_3(0) > 0$

There are 13 parameters in this model which is very difficult to solve. So, we have reducedparameters via use the dimensionless parameters and variable to make the system easy. In Model A, we can reduce the parameters in the system by scaling transformations via use the dimensionless parameters and variable to make the system easy. Even thou, fornumerical simulations, will continue to utilise the original system:

$$P_1 \to x, P_2 \to y, P_3 \to z, T \to \frac{rt}{k}$$

and considered rest of the values as follows:

$$u_1 = \frac{\propto_1 k}{r}, u_2 = \frac{b_1 k}{r}, u_3 = \frac{d_1 k}{r}, u_4 = \frac{c_1 k}{r}, u_5 = \frac{e_1 \propto_1 k}{r}, u_6 = \frac{\propto_2 k}{r}, u_7 = \frac{d_2 k}{r}, u_8 = \frac{c_2 k}{r},$$
$$u_9 = \frac{e_2 \propto_2 k}{r}, u_{10} = \frac{d_3 k}{r}, u_{11} = \frac{c_3 k}{r}$$

All these parameters, of course, assume only positive values, and the model could bewritten in the form of non-dimensional to reduce the number of parameters:

Model B

$$\frac{dx}{dt} = x[k - x - u_1 y - u_2 xy - u_3 - u_4 x] \quad (4)$$

$$\frac{dy}{dt} = y[u_5 x - u_6 z - u_7 - u_8 y] \quad (5)$$

$$\frac{dz}{dt} = z[u_9 y - u_{10} - u_{11} z] \quad (6)$$

And the non-negative conditions are: $x(0) > 0, y(0) > 0, z(0) > 0$

3 Equilibria of Mathematical Model

The Model B has the following four non-negative equilibria in the variables x, y and z space namely, $E_0(0,0,0), E_1(\bar{x}, 0,0), E_2(\breve{x}, \breve{y}, 0)$ *and* $E_3(\ddot{x}, \ddot{y}, \ddot{z})$. The existence of E_0 is obvious. We prove the existence of E_1, E_2 *and* E_3 as follows:

- Existence of $E_1\left(\frac{k-u_3}{1+u_4}, 0,0\right)$,
  $\frac{k-u_3}{1+u_4} > 0, \bar{x} > 0 \; if \; k > u_3$.

- Existence of $E_2(\breve{x}, \breve{y}, 0)$, is positive under conditions:
  From (5), $\breve{y} = \frac{u_5\breve{x} - u_7}{u_8}, \breve{y} > 0, u_5\breve{x} > u_7$ and
  From (4), $\breve{x} = \frac{-S_2 \pm \sqrt{S_2^2 - 4S_1S_2}}{2S_1} > 0$,
  $S_2 = u_2 u_5, S_2 = u_8 + u_1 u_5 + u_4 u_8 - u_2 u_7, S_3 = k u_8 + u_1 u_7 - u_3 u_8$
  $k > u_3$

- Existence of $E_3(\ddot{x}, \ddot{y}, \ddot{z})$ is positive under the conditions:

$\ddot{x} = \frac{-T_2 \pm \sqrt{T_2^2 - 4T_1 T_2}}{2T_1} > 0,$

Where, $T_1 = u_2 u_5 u_{11}, T_2 = u_6 u_9 + u_{11}(u_8 + u_1 u_5) + u_2(u_6 u_{10} - u_{11} u_7) + u_4(u_6 u_9 + u_8 u_{11}),$
$T_3 = (k - u_3)(u_6 u_9 + u_2 u_2) - u_1(u_6 u_{10} - u_7 u_{11}),$

$provided, \propto_2 d_3 > d_2 c_3$

From (5), $\ddot{y} = \frac{u_5 u_{11} \ddot{x} + u_6 u_{10} - u_7 u_{11}}{u_6 u_9 + u_8 u_{11}} > 0$ provided $e_1 \propto_1 \ddot{x} > d_2$ and

From (6), $\ddot{z} = \frac{u_9 \ddot{x} - u_{10}}{u_{11}} > 0$ provided $c_2 \propto_2 \ddot{y} > d_3$

Now, we will discuss the dynamical behavior of the Mathematical Model.

4 Local Stability of Mathematical Model

The stability behavior of E_1, E_2 and E_3 can be studied by computing variational matrices.The general variational matrix is:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix} \quad (7)$$

To check the stability behavior of Model B, first let us consider the f_1, f_2 and f_3 as follows:

$$\frac{dx}{dt} = x[k - x - u_1 y - u_2 xy - u_3 - u_4 x] = f_1$$
$$\frac{dy}{dt} = y[u_5 x - u_6 z - u_7 - u_8 y] = f_2$$
$$\frac{dz}{dt} = z[u_9 y - u_{10} - u_{11} z] = f_3$$

Let us use f_1, f_2 and f_3 in (7), we get the general variational matrix corresponding to theModel B is:

$$J = \begin{bmatrix} j_{11} & -u_1 x - u_2 x^2 & 0 \\ u_5 y & u_5 x - u_6 z - u_7 - 2u_8 y & -u_6 y \\ 0 & u_9 y & u_9 y - u_{10} - 2u_{11} z \end{bmatrix} \quad (8)$$

Where, $j_{11} = k - 2x(1+u_4) - u_1 y - u_3 - 2u_2 xy$

The local stability of equilibrium of Model B is determined by computing the eigenvalues of the variational matrix (8) about the equilibrium points E_1, E_2 and E_3. The followingresults are derived:

- About the equilibrium point$E_1(\bar{x},0{,}0)$, the eigen values of the characteristic equation are $k-u_3-2\bar{x}(1+u_4)$, $u_5\bar{x}-u_7$ and $-u_{10}$, which shows that E_1 is locally asymptotically stable if $\frac{k-u_3}{2(1+u_4)} < \bar{x}, r > d_1$ and $\bar{x} < \frac{d_2}{e_1\propto_1}$ hold good.

Remark 1: From the stability conditions of E_1 it may be noted that if (i) the intrinsic growth rate is greater than the natural death rate of prey population, and

(ii) The prey population is less than the division of natural death rate of intermediate predator population, and the product of predation rate and the conversion rateof prey and predator population, then only prey population will survive.

- About the equilibrium point$E_2(\breve{x},\breve{y},0)$, the eigen values of the characteristic equation are $k-2\breve{x}-u_1\breve{y}-2u_2\breve{x}\breve{y}-u_3-2u_4\breve{x}$, $u_5\breve{x}-u_7-2u_8\breve{y}$ and $u_9\breve{y}-u_{10}$, which shows that E_2 is locally asymptotically stable if $k<(2\breve{x}+u_1\breve{y}+2u_2\breve{x}\breve{y}+u_3+2u_4\breve{x})$, $u_5\breve{x}<(u_7+2u_8\breve{y})$ and $\breve{y}<\frac{d_3}{e_2\propto_2}$ hold good.

Remark 2: From the stability conditions of E_2 it may be noted that if (i) the predator population is less than the division of natural death rate of intermediate predator population, and the product of predation rate and the conversion rate of intermediate and top predator populations, then only prey population will survive.

- About the equilibrium point $E_3(\ddot{x},\ddot{y},\ddot{z})$, the characteristic equation is given by

$$\lambda^3 + T_1\lambda^2 + T_2\lambda + T_3 = 0$$

Where

$$T_1 = 2\ddot{x}(1+u_4) + (u_1-u_9)\ddot{y} + u_3 + 2u_2\ddot{x}\ddot{y} + u_7 + 2u_8\ddot{y} + u_{10} + (2u_{11}+u_6)\ddot{z} - k - u_5\ddot{x}\,,$$

$$T_2 = (k-2\ddot{x}(1+u_4) - u_1\ddot{y} - u_3 - 2u_2\ddot{x}\ddot{y}) + (u_5\ddot{x} - u_6\ddot{z} - u_7 - 2u_8\ddot{y} + u_9\ddot{y} - u_{10} - 2u_{11}\ddot{z}) + (u_5\ddot{x} - u_6\ddot{z} - u_7 - 2u_8\ddot{y})(u_9\ddot{y} - u_{10} - 2u_{11}\ddot{z}) + u_6u_9\ddot{y}^2 + (u_1\ddot{x} + u_2\ddot{x}^2)\,u_5\ddot{y}\,,$$

$$T_3 = (k-2\ddot{x}(1+u_4) - u_1\ddot{y} - u_3 - 2u_2\ddot{x}\ddot{y})(-u_6u_9\ddot{y}^2 - (u_5\ddot{x} - u_6\ddot{z} - u_7 - 2u_8\ddot{y})(u_9\ddot{y} - u_{10} - 2u_{11}\ddot{z})) - (u_1\ddot{x} + u_2\ddot{x}^2)\,(u_5\ddot{y})(u_9\ddot{y} - u_{10} - 2u_{11}\ddot{z})$$

By the Routh −Hurwitz criteria, E_3 is considered as locally asymptotically stable if $T_1>0, T_2>0, T_3>0$ and $T_1T_2>T_3$. It is very difficult to explain these equationsin ecological terms, however for these expressions, numerical examples are taken and graphs are plotted to represent the dynamical behavior of the system about the equilibrium E_3.

5 Numerical simulation

In this simulation section, we have demonstrated the dynamical behavior of the effect of toxicant on a given three species food-chain system for the models with the help of numerical simulations to facilitate the interpretation of our mathematical findings. All the figures of this section describe the stability behavior of all the equilibrium points of the model and plotted graphs with the help of the software MATLAB.

- **For the equilibrium point** $E_1(x, 0, 0)$**:**

  We have chosen the following values of parameters for $E_1(x, 0, 0)$:

  $r = 2; k = 0.2; \alpha_1 = 0.04; \alpha_2 = 0.8; b_1 = 0.05; c_1 = 0.05; c_2 = 0.1;$

  $c_3 = 0.19; d_1 = 0.02; d_2 = 0.08; d_3 = 0.09; e_1 = 0.2; e_2 = 0.2.$

  It is found that under the above set of parameters, the equilibrium point $E_1(x, 0, 0)$

  x=0.1970, y=0.0000, z=0.0000

  is locally asymptotically stable (see Fig.1).

- **For the equilibrium point** $E_2(x, y, 0)$

  We have chosen the following values of parameters for $E_2(x, y, 0)$:

  $r = 2; k = 0.2; a_1 = 0.5; a_2 = 0.3; b_1 = 0.05; c_1 = 0.05; c_2 = 0.07;$

  $c_3 = 0.19; d_1 = 0.02; d_2 = 0.08; d_3 = 0.09; e_1 = 0.9; e_2 = 0.2.$

  It is found that under the above set of parameters, the equilibrium point $E_2(x, y, 0)$

  x=0.1923, y=0.0933, z=0.0000

  is locally asymptotically stable (see Fig.2).

- **For the equilibrium point** $E_3(x, y, z)$

  We choose the following values of parameters for $E_3(x, y, z)$:$r = 1.8; k =$

  $2; a_1 = 0.8; a_2 = 1; b_1 = 0.3; c_1 = 0.09; c_2 = 0.09;$

  $c_3 = 0.21; d_1 = 0.08; d_2 = 0.1; d_3 = 0.1; e_1 = 0.3; e_2 = 0.2.$

  It is found that under the above set of parameters, the equilibrium point $E_3(x, y, z)$

  x=1.0565, y=0.6040, z=0.0991

is locally asymptotically stable (see Fig.3).

- **For the equilibrium point** $E_3(x, y, z)$

We choose the following values of parameters for $E_3(x, y, z)$:$r = 1.8; k = 2; a_1 = 0.8; a_2 = 1; c_1 = 0.09; c_2 = 0.09;$
$c_3 = 0.21; d_1 = 0.08; d_2 = 0.1; d_3 = 0.1; e_1 = 0.3; e_2 = 0.2.$

It is found that under the above set of parameters, the equilibrium point $E_3(x, y, z)$without toxicant

x=1.2183, y=0.6416, z=0.1348

is locally asymptotically stable (see Fig.4).

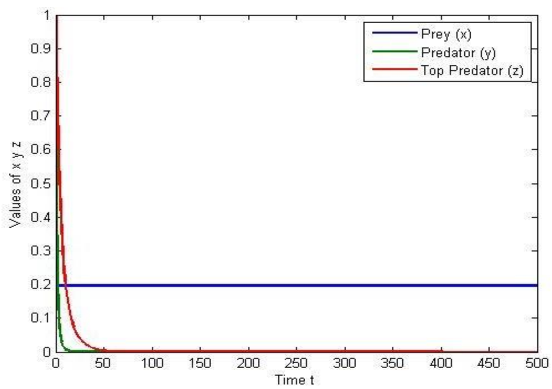

Figure 1. Time series graph for E_1 with toxicant, showing the stability behavior

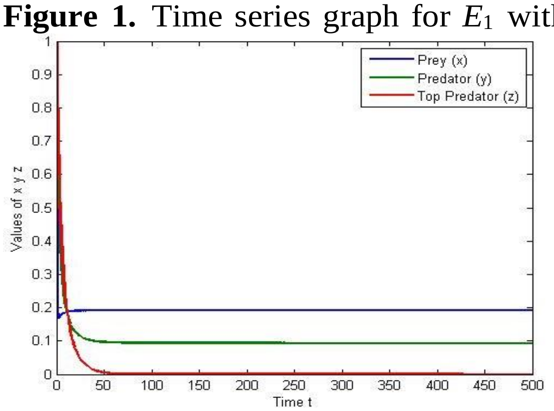

Figure 2. Time series graph for E_2 with toxicant, showing the stability behavior

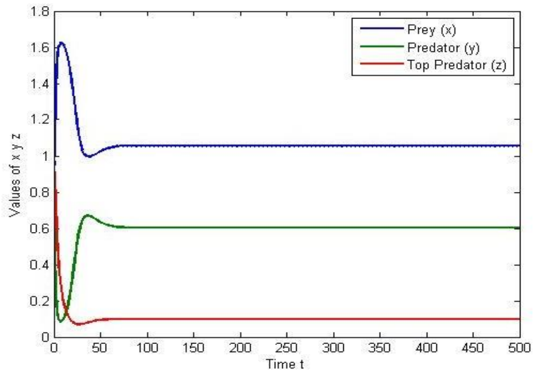

Figure 3. Time series graph for E_3 with toxicant, showing the stability behavior

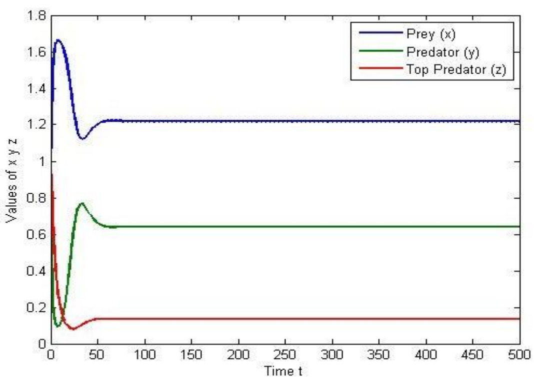

Figure 4. Time series graph for E_3 without toxicant, showing the stability behavior

Table 1. Values of x, y, z when with toxicant and without toxicant at different equilibriumpoints.

Points	x	y	z
$E_1(x, 0, 0)$	0.1970	0.0000	0.0000
$E_2(x, y, 0)$	0.1923	0.0933	0.0000
$E_3(x, y, z)$ with toxic	1.0565	0.6040	0.0991
$E_3(x, y, z)$ without toxic	1.2183	0.6416	0.1348

Conclusion

In this mathematical model paper, a three populations prey predator model is proposed to study the effect of toxicant on the system. It is considered that Prey release toxins as their defense. For the equilibrium point of E_1, it has been observed at the stability point that only the prey population will survive (Fig.1) and both the predator's populations would tend to extinction. For the equilibrium point of E_2, it has been observed at the stability point that the prey and predator populations will survive

(Fig.2) and the top

predator population would tend to extinction. For the interior equilibrium point E_3 of the model, it is locally stable and showing the co-existence of all the prey and predator species with and without toxicant (Fig.3, 4). It has also been noted from the calculations and simulations (see Table 1) that when prey releases the toxicant then the predator populations reduces. Also, when prey doesn't release toxicant then predator population increases (see Fig.4).

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