THE NEUTRIX PRODUCT OF THE DISTRIBUTIONS

x_+^{-r} AND $\delta^{(\alpha)}(x)$

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Abstract – In this paper author has obtained the neutrix product of $x_+^{-r}$and $\delta^{(\alpha)}(x)$, where α is a positive fractional number.

***Key Words*:** Neutrix limit, fractional-differentiation, sequence, function.

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1. INTRODUCTION

Neutrix N is defined by J.G. vander Corput [2] as a commutative additive group of functions $\nu(\xi)$ defined on a domain N' with values in additive group N", where further if for some ν in N, $\nu(\xi) = \gamma$ for all ξ in N', then γ= 0. The functions in N are called negligible functions. Now let N' be a set contained in a topological space with a limit point b which does not belong to N'. If $f(\xi)$ is a function defined on N' with values in N" and it is possible to find a constant β such that $f(\xi) - \beta$ is negligible in N, then β is called the neutrix limit or N-limit of f as ξ tends to b and we write

$$N - \lim_{\xi \to b} f(\xi) = \beta,$$

where β must be unique, if it exists.

Introducing the neutrix limit, Fisher [3,4] defined the neutrix product of two distributions as –

<u>Definition (1.1)</u>:- Let f and g be arbitrary distributions and let

$$g_n = g * \delta_n = \int_{-1/n}^{1/n} g(x-t)\delta_n(t)dt,$$

for n = 1,2,3, ... , where $\{\delta_n\}$ converges to dirac-delta distribution δ, and $\delta_n(x) = n\rho(nx), \rho$ is an infinitely differentiable function having the properties –

(i) $\rho(x) = 0$ for $|x| \geq 1$,
(ii) $\rho(x) \geq 0$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\int_{-1}^{1} \rho(x)dx = 1$,

We say that the neutrix product f∘g of f and g exists and equal to a distribution h if

$$N - \lim_{n\to\infty} < fg_n, \varphi > = N - \lim_{n\to\infty} < f, g_n\varphi > = < h, \varphi >,$$

for all test functions $\varphi \in K$, with support contained in the interval (a, b), where N is the neutrix having domain N' = {1,2, ... n, ...} and range N" of the real numbers with negligible functions

$$n^\lambda \ln^{r-1} n, \ln^r n,$$

for λ> 0, and r=1,2, ... and all functions f(n) for which $\lim_{n\to\infty} f(n) = 0$.

Riemann - Liouville and Wéyl-fractional integral operators are defined in [9, p.47] for $\operatorname{Re} \alpha > 0$ as -

$$(I^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt,$$

and $$(K^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t)dt.$$

In [7, p.658] the fractional differential operator is defined as -

$$I^{-\alpha} f = D^\alpha f, \quad (1.1)$$

and $$K^{-\alpha} f = (-1)^\alpha D^\alpha f. \quad (1.2)$$

These operators are adjoint, see [1],

i.e.

$$\langle I^{-\alpha} f, \varphi \rangle = \langle f, K^{-\alpha}\varphi \rangle \quad (1.3)$$

and $$\langle K^{-\alpha} f, \varphi \rangle = \langle f, I^{-\alpha}\varphi \rangle \quad (1.4)$$

In [10] the neutrix product of F(x) and $\delta^{(\alpha)}(x)$ has obtained, where F is an infinitely differentiable function in every neighbourhood of the origin.

In the present paper, we will obtain the neutrix product of $x_+^{-r}$and $\delta^{(\alpha)}(x)$, where α is a positive fractional number i.e. α= p + q, p = 1,2,3,....., and $0 \leq q < 1$. This result obviously generalizes the result obtained by Fisher [5].

2. In this section we will find the neutrix product of $x_+^{-r}$and $\delta^{(\alpha)}(x)$, First of all we will prove the following theorem :

<u>Theorem (2.1)</u> - Let *f* be a distribution and *f*(-*x*) = -*f*(*x*), for all *x* in an open interval (-*a*, *a*). If *f*(*x*) and all its derivatives vanish at *x* = 0, then the neutrix product $\delta^{(\alpha)}$ with f exists and

$$\delta^{(\alpha)} \circ f = 0$$

Proof - Since f(-x) = - f(x) for all x in the interval (-a, a), then

$$f_n(x) = f(x) * \delta_n(x) = \int_{-1/n}^{1/n} f(x-t)\delta_n(t)dt$$

It follows that $f_n(-x) = -f_n(x)$, in all open intervals $\left(-\frac{1}{2}a, \frac{1}{2}a\right)$, when n > 2/*a*.
Since f_n is continuous, $f_n(0) = 0$, when n > 2/a, thus

$$\delta^{(\alpha)} \circ f = 0.$$

Theorem (2.2) - The neutrix product $\{x_+^{-r} \circ \delta^{(\alpha)}(x)\}$ and $\delta^{(\alpha)}(x) \circ x_+^{-r}$ exist and

$$x_+^{-r} \circ \delta^{(\alpha)}(x) = \frac{(-1)^r\Gamma(\alpha+1)}{2\Gamma(\alpha+r+1)}\delta^{(\alpha+r)}(x) \qquad (2.3)$$

$$\delta^{(\alpha)}(x) \circ x_+^{-r} = 0, \qquad (2.4)$$

for r= 1,2 ....
Proof – For φ ϵ K, we have

$$\langle x_+^{-1}, \varphi(x)\rangle = \int_0^\infty x^{-1}[\varphi(x) - \varphi(0)H(1-x)]dx,$$

where H(x) denotes the Heavi-side's unit function and so

$$\langle x_+^{-1}, \delta_n^{(\alpha)}(x)\varphi(x)\rangle = \int_0^1 x^{-1}\begin{bmatrix}\delta_n^{(\alpha)}(x)\varphi(x)\\ -\delta_n^{(\alpha)}(0)\varphi(0)\end{bmatrix}dx$$

$$= \int_0^{1/n} x^{-1}\delta_n^{(\alpha)}(x)[\varphi(x)-\varphi(0)]dx$$

$$+ \varphi(0)\int_0^{1/n} x^{-1}[\delta_n^{(\alpha)}(x) - \delta_n^{(\alpha)}(0)]dx$$

$$- \delta_n^{(\alpha)}(0)\varphi(0)\int_{1/n}^1 x^{-1}dx$$

$$= I_1 + I_2 + I_3$$

Now

$$I_1 = \int_0^{1/n}\delta_n^{(\alpha)}(x)\left[\sum_{m=0}^{\alpha+1}\frac{x^{m-1}}{\Gamma(m+1)}\varphi^{(m)}(0) + \frac{x^{\alpha+1}}{\Gamma(\alpha+3)}\varphi^{(\alpha+2)}(\xi x)\right]dx,$$

where $0 \le \xi \le 1$. (by [8, p.40])

Substituting nx = t we have

$$I_1 = \sum_{m=0}^{\alpha}\frac{n^{\alpha+1-m}}{\Gamma(m+1)}\varphi^{(m)}(0)\int_0^1 t^{m-1}\rho^{(\alpha)}(t)dt + \frac{\varphi^{(\alpha+1)}(0)}{\Gamma(\alpha+2)}\int_0^1 t^\alpha\rho^{(\alpha)}(t)dt + \frac{n^{-1}}{\Gamma(\alpha+3)}\int_0^1 t^{\alpha+1}\rho^{(\alpha)}(t)\varphi^{(\alpha+2)}\left(\frac{\xi t}{n}\right)dt,$$

Since $n^{\alpha+1-m}\int_0^1 t^{m-1}\rho^{(\alpha)}(t)dt$ is negligible on neutrix limit

N or zero for $\alpha > m$, and

$$\int_0^1 t^\alpha\rho^{(\alpha)}(t)dt = \frac{1}{2}(-1)^\alpha\Gamma(\alpha+1),$$

$$\text{and } n^{-1}\int_0^1 t^{\alpha+1}\rho^{(\alpha)}(t)\varphi^{(\alpha+2)}\left(\frac{\xi t}{n}\right)dt = O\left(\frac{1}{n}\right)$$

It follows that

$$N - \lim_{n\to\infty} I_1 = (-1)^a\frac{\Gamma(\alpha+1)\varphi^{(\alpha+1)}(0)}{2\Gamma(\alpha+2)}$$

$$= (-1)^\alpha\frac{1}{2(\alpha+1)}\varphi^{(\alpha+1)}(0)$$

$$= \frac{-(-1)^{\alpha+1}}{2(\alpha+1)}\varphi^{(\alpha+1)}(0)$$

$$= -\frac{1}{2(\alpha+1)}\langle\delta^{(\alpha+1)}, \varphi\rangle$$

Again

$$I_2 = \varphi(0)\int_0^{\frac{1}{n}} x^{-1}[\delta_n^{(\alpha)}(x) - \delta_n^{(\alpha)}(0)]dx$$

$$= n^{\alpha+1}\varphi(0)\int_0^1 t^{-1}\left[\rho^{(\alpha)}(t) - \rho^{(\alpha)}(0)\right]dt$$

This gives

$$N - \lim_{n\to\infty} I_2 = 0$$

Similarly

$$I_3 = -\delta_n^{(\alpha)}(0)\varphi(0)\int_{1/n}^1 x^{-1}dx$$

$$= -\rho^{(\alpha)}(0)\varphi(0)n^{\alpha+1}\ln n,$$

and so $\quad N - \lim_{n\to\infty} I_3 = 0$

It follows that

$$N - \lim_{n\to\infty}\left\langle x_+^{-1}, \delta_n^{(\alpha)}(x)\varphi(x)\right\rangle = -\frac{1}{2(\alpha+1)}\langle\delta^{(a+1)}(x), \varphi(x)\rangle,$$

for all test function φ .
Thus the neutrix product $x_+^{-1} \circ \delta^{(\alpha)}(x)$ exists and

$$x_+^{-1} \circ \delta^{(\alpha)}(x) = -\frac{1}{(\alpha+1)}\delta^{(a+1)}(x).$$

Equation (2.3) therefore holds for r = 1, Now assume that Equation (2.3) holds for some r, then by [5, theorem (2), p.1441] the neutrix product $x_+^{-r-1} \circ \delta^{(\alpha)}(x)$ exists and

$$-rx_+^{-r-1} \circ \delta^{(\alpha)}(x) = \frac{(-1)^r\Gamma(\alpha+1)}{2\Gamma(\alpha+r+1)}\delta^{(\alpha+r+1)} - x_+^{-r} \circ \delta^{(\alpha+1)}(x)$$

$$= \frac{(-1)^r r\,\Gamma(\alpha+1)}{2\Gamma(\alpha+r+2)}\delta^{(\alpha+r+1)}(x)$$

This gives

$$x_+^{-r-1} \circ \delta^{(\alpha)}(x) = \frac{(-1)^{r+1}\Gamma(\alpha+1)}{2\Gamma(\alpha+r+2)}\delta^{(\alpha+r+1)}(x)$$

Hence equation (2.3) follow by induction.

We now consider the neutrix product $\delta^{(\alpha)}(x) \circ x_+^{-r}$ for r= 1, 2, ...

Since

$$(x_+^{-r})_n = x_+^{-r} * \delta_n$$

$$= \frac{(-1)^{r-1}}{(r-1)!} \int_{-1/n}^{x} \ln(x-s)\, \delta_n^{(r)}(s) ds,$$

it follows for arbitrary test function φ

$$\langle \delta(x), (x_+^{-r})_n \varphi(x) \rangle = \frac{(-1)^{r-1}\varphi(0)}{(r-1)!} \int_{-1/n}^{0} \ln(-s)\, \delta_n^{(r)}(s) ds.$$

Making the substitution $ns = -t$, we have

$$\int_{-1/n}^{0} \ln(-s) \delta_n^{(r)}(s) ds = (-1)^r n^r \int_0^1 \ln\left(\frac{t}{n}\right) \rho^{(r)}(t) dt$$

$$= (-1)^r n^r \int_0^1 \ln t\, \rho^{(r)}(t)\, dt$$

$$-(-1)^r n^r \ln n \int_0^1 \rho^{(r)}(t) dt,$$

which is either negligible on neutrix limits or zero for r = 1,2,3 ..... . It follows that

$$N - \lim_{n\to\infty} \langle \delta(x), (x_+^{-r})_n \varphi(x) \rangle = 0$$

for all test function φ.
Thus the neutrix product $\delta(x) \circ x_+^{-r}$ exists and

$$\delta(x) \circ x_+^{-r} = 0.$$

Thus equation (2.4) holds when $\alpha = 0$.
Now we consider the neutrix product $\delta^{(\alpha)}(x) \circ x_+^{-r}$

$$\langle \delta^{(\alpha)}(x), (x_+^{-r})_n \varphi(x) \rangle = \langle I^{-\alpha} \delta(x), (x_+^{-r})_n \varphi(x) \rangle$$

$$= \langle \delta(x), K^{-\alpha}\{(x_+^{-r})_n \varphi(x)\} \rangle$$

[By equation (1.3)]

$$= \left\langle \delta(x), \sum_{r=0}^{\infty} {}^{\alpha}C_r K^{-(\alpha-r)} (x_+^{-r})_n \varphi^{(r)}(x) \right\rangle$$

$$= \sum_{r=0}^{\infty} {}^{\alpha}C_r \varphi^{(r)}(0) \{K^{-(\alpha-r)}(x_+^{-r})_n\}_{x=0},$$

which is again zero or negligible in N. Hence

$$N - \lim_{n\to\infty} \delta^{(\alpha)}(x) \circ x_+^{-r} = 0$$

i.e. equation (2.4) holds for every positive fractional number α

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