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The Projective Curvature Tensors in F_n

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ABSTRACT

In the present theoretical analysis, the properties of the Finisler connection have been investigated. Linear connection in the system has been taken different from Carton's.Furthermore, it is also discussed that two quantities (vertical connections and Cartan's*C*-tensor) are identical in some cases.

Further, it is to be noted that if the vector field ξ^i is stationary, and that is $\xi^i_{;j}=0$ then the partial δ-differentiation of a tensor field is *h*-covariant derivative with respect to the Rund connection.

Keywords: *- tensor, torsion, , Cartan's C-tensor,Finisler connection,covariant derivatives*

Introduction:-

The Finsler connection $F\Gamma$ of a Finsler space F_n is a triad $(F^i_{jk}, N^i_k, C^i_{jk})$ of a V-connection F^i_{jk}, a non linear connection N^i_k and a vertical connection C^i_{jk} [11] [21]. In general, the vertical connection C^i_{jk} is different from Cartan's *C*-tensor obtained from C_{ijk} given by the equation (4.3). However, there are certain Finsler connections to be discussed, in which two quantities (vertical connections and Cartan's *C*-tensor) are identical.

If a Finsler connection is given, the *h*- and *v*-covariant derivatives of any tensor field T^i_j are defined as

(1.1) $T^i_{j|k} = \partial_k T^i_j + T^m_j F^i_{mk} - T^i_m F^m_{jk}$

and

(1.2) $T^i_{j|k} = \dot{\partial}_k T^i_j + T^m_j C^i_{mk} - T^i_m C^m_{jk}$

respectively, where

(1.3) $d_k = \partial_k - N^m_k \dot{\partial}_m$,

$\partial_k = \partial / \partial x^k$, $\dot{\partial}_k = \partial / \partial \dot{x}^k$,

$\left(_{|k}\right)$and $\left(|_k\right)$denotes the *h* and *v*-covariant derivatives respectively.

For any Finsler connection $(F^i_{jk}, N^i_k, C^i_{jk})$ we have five tensors which are expressed as follows:

(1.4) The $(h)h$-torsion tensor: $T^i_{jk} = F^i_{jk} - F^i_{kj}$,

(1.5) The $(v)V$-torsion tensor: $S^i_{jk} = C^i_{jk} - C^i_{kj}$,

(1.6) The $(h)hv$-torsion tensor: $C^i_{jk} =$ as the connection C^i_{jk},

(1.1) The $(v)h$-torsion tensor: $R^i_{jk} = d_k N^i_j - d_j N^i_k$,

(1.8) The $(v)hv$-torsion tensor: $P^i_{jk} = \dot{\partial}_k N^i_j - F^i_{kj}$.

The deflection tensor field D^i_j of a Finsler connection is given by

(1.9) $D^i_j = \dot{x}^k N^i_j - F^i_{kj}$.

When a Finsler metric is given, various Finsler connections may be defined from the metric. The well known examples are the Rund connection, the Cartan connection and the Berwald connection which are given below.

(B) <u>THE RUND CONNECTION</u>:

As in Riemannian geometry, the Christoffel's symbols of first and second kinds have been defined as [1.10]

(1.10) $\gamma_{hij}(x,\dot{x}) = \frac{1}{2}(\partial_j g_{hi} + \partial_h g_{ij} - \partial_i g_{jh})$

and

(1.11) $\gamma^h_{ij}(x,\dot{x}) = g^{hk}(x,\dot{x})\gamma_{ikj}(x,\dot{x})$.

From the definition it is clear that $\gamma_{ikj}(x,\dot{x})$ is symmetric in its extreme indices and $\gamma^h_{ij}(x,\dot{x})$ is symmetric in its lower indices and satisfy the relation

(1.12) $\partial_k g_{ij}(x,\dot{x}) = \gamma_{ijk}(x,\dot{x}) + \gamma_{jik}(x,\dot{x})$.

The symbols $\Gamma^h_{ij}(x,\dot{x})$ are defined as

(1.13) $\Gamma^h_{ij}(x,\dot{x}) = \gamma^h_{ij}(x,\dot{x}) - C^h_{im}(x,\dot{x})\gamma^m_{kj}(x,\dot{x})\dot{x}^k$

where

(1.14) $C^h_{ij}(x,\dot{x}) = g^{hk}(x,\dot{x})C_{ikj}(x,\dot{x})$

and Cartan's C-tensor C_{ikj} is defined by (1.3).

For a vector X^i the components $\frac{\delta X^i}{\delta t}$ defined by

(1.15) $$\frac{\delta X^i}{\delta t} = \frac{d X^i}{d t} + \Gamma^i_{jk}(x,\dot{x}) X^j \frac{dx^k}{dt}$$

form the contra variant components of a vector. The process of differentiation given by (1.15) is called 'δ-differentiation'.

In particular, this process gives a well defined parallel displacement. The vector $X^i + dX^i$ of $T_n(x^i + dx^i)$ is said to be obtained from the vector X^i of $T_n(x^i)$ by parallel displacement if $\delta X^i = 0$. Hence, for such a displacement, we have [1.12]

(1.16) $$dX^i = -\Gamma^i_{jk}(x,\dot{x}) X^j dx^k$$

The partial δ- derivative with respect to x^k in the direction $\dot{x}^i$ of the arbitrary tensor $T^i_j(x,\xi)$ is defined by the formula [21]

(1.11) $$T^i_{j;k} = \partial_k T^i_j + \dot{\partial}_h T^i_j \partial_k \xi^h + T^m_j \Gamma^{*i}_{mk}(x,\dot{x}) - T^i_m \Gamma^{*m}_{jk}(x,\dot{x}),$$

where the coefficients $\Gamma^{*m}_{jk}(x,\dot{x})$ is given by

(1.18) $$\Gamma^{*m}_{jk}(x,\dot{x}) = g^{ih}(x,\dot{x}) \Gamma^{*m}_{jk}(x,\dot{x})$$

and

(1.19) $$\Gamma^*_{jhk}(x,\dot{x}) = \gamma_{jhk}(x,\dot{x}) - [C_{khi}(x,\dot{x})\Gamma^i_{jm}(x,\dot{x}) +$$

$$+ C_{hji}(x,\dot{x})\Gamma^i_{km}(x,\dot{x}) - C_{jki}(x,\dot{x})\Gamma^i_{hm}(x,\dot{x})]\dot{x}^m.$$

The symbol Γ^{*i}_{jk} is symmetric in its lower indices *j* and *k*, while Γ^i_{jk} is no-symmetric in *j* and *k*. Also, we have

(1.20) $$\Gamma^{*i}_{jk}\dot{x}^j\dot{x}^k = \Gamma^i_{jk}\dot{x}^j\dot{x}^k = \gamma^i_{jk}\dot{x}^j\dot{x}^k,$$

(1.21) $$\Gamma^i_{jk}\dot{x}^k = \Gamma^{*i}_{jk}\dot{x}^k,$$

(1.22) $$\Gamma^i_{jk}\dot{x}^j = \gamma^i_{jk}\dot{x}^j.$$

The partial δ-derivative of the metric tensor $g_{ij}(x,\xi)$ in the direction $\dot{x}^i$ in view of (1.11) is given by

(1.23) $$g_{ij}(x,\xi);_k = \partial_k g_{ij}(x,\xi) + 2C_{ijh}(x,\xi)\partial_k \xi^h$$

$$- g_{hj}(x,\xi)\Gamma^{*h}_{ik}(x,\dot{x}) - g_{ih}(x,\xi)\Gamma^{*h}_{jk}(x,\dot{x})$$

If, in particular, $\dot{x}^i = \xi^i$, the above equation reduces to

(1.24) $g_{ij}(x,\xi);_k = 2C_{ijk}(x,\xi)\,\xi^h_{;k}$.

We see that the partial δ-derivative of the metric tensor g_{ij} does not vanish in general. Therefore, further developments of theory of Finsler spaces will differ considerably from the established results of Riemannian geometry in which the covariant derivative of the metric tensor vanishes.

Further, it is to be noted that if the vector field ξ^i is stationary, and that is $\xi^i_{;j} = 0$ then the partial δ-differentiation of a tensor field is *h*-covariant derivative with respect to the Rund connection $(\Gamma^{*i}_{jk}, G^i_j, 0)$ where Γ^{*i}_{jk} is V-connection defined by the equation (1.19) and G^i_j is defined by

(1.25) $G^i_j(x,\dot{x}) = \dot{\partial}_j G^i$, $2G^i(x,\dot{x}) = \gamma^i_{jk}\dot{x}^j\dot{x}^k$

and the vertical connection C^i_{jk} vanishes in this triad. Hence the *v*-covariant derivative of a tensor field is identical to the partial derivative with respect to the element of support $\dot{x}^i$ [1.12] [1.16].

(C) <u>THE CARTAN CONNECTION</u>:

In 1934, E. Cartan [5] published his monograph 'Les espaces de Finsler' and fixed his method to determine the notion of connection in the geometry of Finsler spaces. Although the aim of Cartan's axioms is to determine both the fundamental tensor *g* and the connection from the Finsler metric, it seems that some of his axioms are rather artificial and are introduced after foreseeing the result. In 1966, his method was reconsidered by M. Matsumoto [11] and determined uniquely the Cartan connection by assuming the following axioms [13] [11]:

(1.26) (a) The connection is *h*-metrical, i.e.

$$g_{ij|k} = 0,$$

(b) The connection is *v*-metrical, i.e.

$$g_{ij}\big|_k = 0,$$

(c) The (*h*)*h*-torsion tensor field T^i_{jk} vanishes, i.e.

$$T^i_{jk} = F^i_{jk} - F^i_{kj} = 0,$$

(d) The (*v*)*v*-torsion tensor field S^i_{jk} vanishes, i.e.

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$$S^i_{jk} = C^i_{jk} - C^i_{kj} = 0,$$

(c) The deflection tensor field D^i_j vanishes, i.e.

$$D^i_j = \dot{x}^h F^i_{hj} - N^i_j = 0.$$

The components of the Cartan connection $C\Gamma$ is denoted by $(\Gamma^{*i}_{jk}, G^i_j, C^i_{jk})$. The axioms (1.26b) and (1.26d) in view of the equaiton (1.2), give

(1.21) $C^i_{jk} = \frac{1}{2} g^{ih} \dot{\partial}_h g_{jk}.$

This shows that the vertical connection and Cartan's *C*-tensor are identical.jkm

Further, from the axioms (1.26a) and (1.26c), in view of relation (1.21) anikd (1.1), we get

(1.28) $F_{ijk} = g_{jh} F^h_{ik} = \gamma_{ijk} - C_{ijm} N^m_k - C_{jkm} N^m_i + C_{kim} N^m_j.$

Contracting the equation (1.28) with $\dot{x}^i g^{jh}$ and thereafter applying the axiom (1.26e), we get

(1.29) $N^h_k = \gamma^h_{ik} \dot{x}^i - C^h_{km} N^m_i \dot{x}^i.$

Again, contracting this equation with $\dot{x}^k$, we get

(1.30) $N^h_k \dot{x}^k = \gamma^h_{ik} \dot{x}^i \dot{x}^k.$

Substituting (1.29) and (1.30) in (1.28), we get $F_{ijk} = \Gamma^*_{ijk}$ where Γ^*_{ijk} is defined by the equation (1.19). thus, the Cartan *V*-connection and the Rund *V*-connection are identical. After substituting from (1.30) in (1.29), the Cartan non-linear connection is given by

(1.31) $N^i_j = \gamma^i_{kj} \dot{x}^k - C^i_{jm} \gamma^m_{hp} \dot{x}^h \dot{x}^p = G^i_j = \Gamma^{*i}_{oj}.$

The Cartan vertical connection C^i_{jk} is given by (1.21).

It is easy to verify from the axioms (1.26a), (1.26e) and the equation (4.1) that

(1.32) (a) $\dot{x}^i_{|h} = 0,$ (b) $F_{|h} = 0$ and (c) $l^i_{|h} = 0,$

where l^i is unit vector in the direction of element of support $\dot{x}^i$ i.e. $l^i = \dot{x}^i / F(x, \dot{x})$. Since C^i_{jk} is an indicatory tensor, then from (1.2), we have

(1.33) (a) $\dot{x}^i\big|_h = \delta^i_h,$ (b) $F\big|_i = \frac{\partial F}{\partial \dot{x}^i} = l_i,$ where $l_i = g_{ij} l^j.$

It may also be verified that

(1.34) (a) F (b) $l_{i|j} = 0,$ (c) $l_i\big|_j = \overline{F}^1 h_{ij},$

(1.35) (a) $h_{ij|k} = 0,$ (b) $h_{ij}\big|_k = -\overline{F}^1 (l_i h_{jk} + l_i h_{ki}),$

where h_{ij} are components of the angular metric tensor defined by

(1.36) $h_{ij} = g_{ij} - l_i l_j$ and $h^i_j = g^{ik} h_{jk}.$

(D) THE BERWALD CONNECTION:

L. Berwald defined a connection coefficient defined by

(1.31) $G^i_{jk}(x, \dot{x}) = \dot{\partial}_j \dot{\partial}_k G^i,$

where $2G^i(x, \dot{x}) = \gamma^i_{jk} \dot{x}^j \dot{x}^k.$

He defined the covariant derivative in a manner analogous to that of Cartan, the only difference being that Γ^{*i}_{jk} are replaced by $G^i_{jk}.$

Thus, the covariant derivative of a mixed tensor $T^i_j(x, \dot{x})$ in the sense of Berwald is defined by

(1.38) $T^i_{j(k)} = \partial_k T^i_j - \dot{\partial}_m T^i_j - G^m_k + T^m_j + T^m_j G^i_{mk} - T^i_m G^m_{jk}.$

The function $G^i(x, \dot{x})$ are positively homogeneous of degree 2 in their directional arguments $\dot{x}^i$ and G^i_j is given by the equation (1.25).

Thus, the Berwald connection $B\Gamma$ of a Finsler space F_n is a triad $(G^i_{jk}, G^i_j, C^i_{jk} = 0)$ where G^i_{jk} and G^i_j are Berwald's *V*-connection and non-linear connection respectively. The vertical connection vanishes in case of Berwald triad [2] [11].

The relation between Berwald's and Cartan's *V*-connections $\dot{x}^j$ and Γ^{*i}_{jk} is given by [5].

(1.39) $G^i_{jk} = \Gamma^{*i}_{jk} + P^i_{jk}$

where

(1.40) $P^i_{jk}(x, \dot{x}) = C^i_{jk|o} = \dot{\partial}_k \Gamma^{*i}_{jp} \dot{x}^p = \dot{\partial}_j \Gamma^{*i}_{kp} \dot{x}^p.$

Also, we can get

(1.41) $G^i_{jk} \dot{x}^j = \Gamma^{*i}_{jk} \dot{x}^j.$

Further, the Berwald's covariant derivative of the metric tensor g_{ij} is given by [2].

(1.42) $g_{ij(k)} = -2P_{ijk}$ and therefore $g_{ij(k)}\dot{x}^i = 0$

where

(1.43) $P_{ijk} = g_{jk}P^h_{ik} = C_{i\,jk|o}$.

This tensor P^i_{jk} is a symmetric and is the indicatory tensor. Also we have the following relations:

(1.44) $F_{(i)} = 0,\ l^i_{(j)} = 0,\ l_{i(j)} = 0,\ h^i_{j(k)} = 0,\ h_{ij(k)} = -2P_{ijk}$.

Taking $G^i_{hjk} = \dot{\partial}^h G^i_{jk}$, the following relations hold:

(1.45) (a) $G^i_{jkh}\dot{x}^j = 0$, (b) $g_{jk}G^h_{ik} = G_{ijk}$ and (c) $\dot{\partial}_h G^i_{jk} = G^i_{jkh}$.

Those Finsler spaces for which the function G^i_{jk} are independent of the directional arguments $\dot{x}^j$ are called 'affinely connected spaces'. The affinely connected spaces are characterized by the condition $C_{i\,jk|o} = 0$. It therefore follows that

(1.46) $G^i_{jk} = \Gamma^{*i}_{jk}$

for an affinely connected Finsler space.

CONCLUSIONS

When a Finsler metric is given, various Finsler connections may be defined from the metric. The well known examples are the Rund connection, the Cartan connection and the Berwald connection. form the contra variant components of a vector. The process of differentiation is called 'δ-differentiation'.

In particular, this process gives a well defined parallel displacement. The vector $X^i + dX^i$ of $T_n(x^i + dx^i)$ is said to be obtained from the vector X^i of $T_n(x^i)$ by parallel displacement if $\delta X^i = 0$. Hence, for such a displacement.We see that the partial δ-derivative of the metric tensor g_{ij} does not vanish in general. Therefore, further developments of theory of Finsler spaces will differ considerably from the established results of Riemannian geometry in which the covariant derivative of the metric tensor vanishes.

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