International Journal of Scientific Research in Engineering and Management (IJSREM)
Volume: 09 Issue: 01 | Jan - 2025 SJIF Rating: 8.448 ISSN: 2582-3930

To Find the Solution of Algebraic and Transcendental Equation by Newton-Raphson Method

Ashish Gehlot

Assistant Professor, Department of Mathematics, Government College, Bajju (Bikaner), Rajasthan. Email: gehlotashish123@gmail.com

Abstract:

This paper is about solving algebraic and transcendental equations using a numerical method. Newton-Raphson method is the method named after Sir Issac Newton and Joseph Raphson, two of the most famous scientists of the time. This method is a simple way to obtain an approximation to the real-valued roots of the given equation. It is a fast method, but convergence is not guaranteed, which is the reason why several modifications of that method have been proposed. In this paper, we discuss the modifications to the Newton-Raphson method, the rate of convergence, geometrical significance, and its limitations.

Introduction:

Algebraic and Transcendental equations and their applications have been widely studied, currently, we know that if the equation is not solvable via analytical methods, we can use numerical methods to solve it. Thus, numerical methods for finding roots are continuously being developed.[1] With the rapid development of high-speed digital computers and the increasing demand for numerical answers to practical problems, Numerical methods have become very important and useful in Applied Mathematics, Engineering, Technology, etc..

For years, finding the solution to the set of equations $f(x) = (f^1, \dots, f^n)' = 0$ has been a challenge. Here, we'll look at this equation and see if we can discover a solution using the Newton-Raphson method. The Homotopy method is used to improve the convergence property of numerous methods, and it is well-known for its fast rate of convergence.[2] Newton-Raphson method is very fast and efficient as compared to other methods and also requires only one iteration and the derivative evaluation per iteration. The equipment utilized for such calculations is a scientific calculator. Because they are focused on lowering the interval between two guesses, bracketing methods that require bracketing of the root by two guesses are always convergent.[3]

Newton-Raphson Method:

If $f(x) = 0$ be a polynomial equation, then the nth approximation of the root of the equation is given by

$$x_{n+1} = x_n - \frac{f(x_{n})}{f'(x_{n})}$$

Where x_0 be the initial root and x_1, x_2, …. are its successive approximations of the root.

Proof:

Let x_0 be the approximate value of the real root of the equation $f(x) = 0$ and $x_0 + h$ be the correct value, then

$$f(x_0 + h) = 0 \qquad \dots(1)$$

By Taylor's theorem, we get

$$f(x_0+h) = f(x_0) + \frac{hf'(x_0)}{1!} + \frac{h^2 f''(x_0)}{2!} + \cdots = 0$$

Now, if h is sufficiently small, we may neglect terms of second and higher powers of h,

We obtain,

$$f(x_0) + hf'(x_0) = 0 \qquad \Rightarrow h = -\frac{f(x_0)}{f'(x_0)} \quad , \qquad f'(x_0) \neq 0 \qquad \ldots(2)$$

If the improved root is denoted by x_1, then

$$x_1 = x_0 + h$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \quad , \qquad f'(x_0) \neq 0 \qquad \text{[by (2)]} \qquad \ldots(3)$$

Now using x_1 in place of x_0 in the above equation i.e. 3 and h_2 denote the correction to be applied to x_1 for a better approximation of the root, then we have

$$h_2 = -\frac{f(x_1)}{f'(x_1)} \quad , \qquad f'(x_1) \neq 0 \qquad \ldots(4)$$

now the improved value of the root will be x_2, and then

$$x_2 = x_1 + h_2$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \quad , \qquad f'(x_1) \neq 0 \qquad \text{[by (4)]} \qquad \ldots(5)$$

similarly, successive approximate values of the roots will be $x_3, x_4, \ldots\ldots, x_{n+1}$, where

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Geometrical significance of the Newton-Raphson Method

Suppose the graph of the function $y = f(x)$ meets the *x-axis* at the point *A*.

Let x_0 be the initial starting point and P_0 (x_0, y_0) be a point on the curve. Draw the tangent at the point P_0 which meets the *x-axis* at the point T_1 whose *x*-coordinate is x_1.(Fig-1)

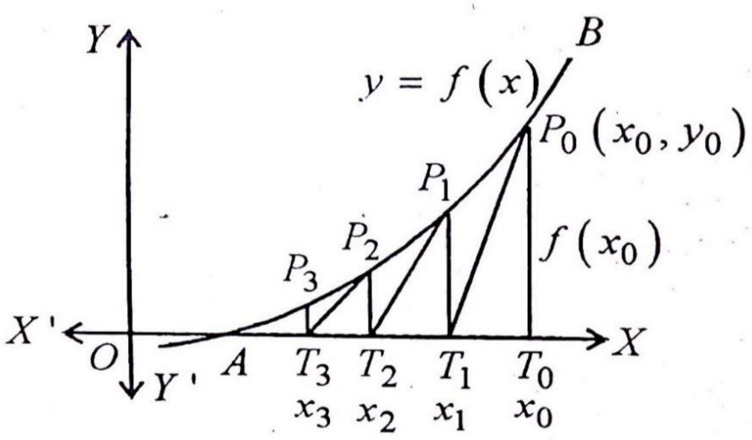

Figure - 1

$$OT_1 = OT_0 - T_1T_0$$

$$x_1 = x_0 - y_0 \cot(\angle P_0T_1T_0)$$

$$x_1 = x_0 - \frac{f(x_0)}{\tan(\angle P_0T_1T_0)}$$

But,

$$tan(\angle P_0T_1T_0) = \text{the slope of the tangent at } P_0 \text{ i.e. } f'(x_0)$$

Therefore

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Similarly, to get the second approximate value x_2, draw a tangent at the point P_1which meets the *x-axis* at the point T_2.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Proceeding in the same way, we get

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Where x_n and x_{n+1} have their usual meanings.

Rate of convergence of Newton-Raphson Method:

Let a be the correct value of the root and the difference of x_n from a be ε_n

i.e.

$$x_n = a + \varepsilon_n \qquad \text{and} \qquad x_{n+1} = a + \varepsilon_{n+1} \qquad \ldots(1)$$

By the Newton-Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \ldots(2)$$

$$a + \varepsilon_{n+1} = a + \varepsilon_n - \frac{f(a+\varepsilon_n)}{f'(a+\varepsilon_n)}$$

$$\varepsilon_{n+1} = \varepsilon_n - \frac{f(a+\varepsilon_n)}{f'(a+\varepsilon_n)} \qquad \ldots(3)$$

Now by Taylor's theorem,

$$f(a+\varepsilon_n) = f(a) + \frac{\varepsilon_n f'(a)}{1!} + \frac{\varepsilon_n^2 f''(a)}{2!} + \cdots$$

$$\because f(a) = 0$$

$$f(a+\varepsilon_n) = \varepsilon_n \left(f'(a) + \frac{\varepsilon_n f''(a)}{2!} \right) \qquad \ldots(4)$$

and $$f'(a+\varepsilon_n) = f'(a) + \varepsilon_n f''(a) \qquad \ldots(5)$$

using (4) and (5) in (3), we have

$$\varepsilon_{n+1} = \varepsilon_n - \varepsilon_n \frac{f'(a)+\frac{1}{2}\varepsilon_n f''(a)}{f'(a)+\varepsilon_n f''(a)}$$

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2 f''(a)}{2(f'(a)+\varepsilon_n f''(a))}$$

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2 f''(a)}{2f'(a)\left(1+\varepsilon_n \frac{f''(a)}{f'(a)}\right)}$$

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2 f''(a)}{2f'(a)}\left(1+\varepsilon_n \frac{f''(a)}{f'(a)}\right)^{-1}$$

$$\varepsilon_{n+1} \approx \frac{\varepsilon_n^2 f''(a)}{2f'(a)} \qquad \ldots(6)$$

From (6), it is clear that the successive error is proportional to the square of the previous error.

Hence the order of convergence of Newton Raphson method is quadratic.[4,5]

Modified Newton-Raphson Formula:

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)}$$

where the equation $f(x) = 0$ has a pair of double roots in the neighbourhood of $x = x_n$.

Newton-Raphson Method for Multiple Roots:

If $x = a$ be a root of k multiplicity of the equation $f(x) = 0$, then the Newton-Raphson Formula for the root of k multiplicity is

$$x_{n+1} = x_n - \frac{kf(x_n)}{f'(x_n)}$$

Results

1. We find the real roots of the equation $3x = cosx + 1$ up to four decimal places by the Newton-Raphson method.[6]

Let $f(x) = 3x - cosx - 1$

$f(0) = -2 \quad and \quad f(1) = 3 - 0.5403.1 = 1.4597$

So a root of $f(x) = 0$ lies between 0 and 1 (by Descrate's rule), it is nearer to 1. So, let us take $x_0 = 0.6$

Also, $f'(x) = 3 + sinx$

∴ Newton's iteration formula is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{3x_n - cosx_n - 1}{3 + sinx_n}$$

$$x_{n+1} = \frac{x_n sinx_n + cosx_n + 1}{3 + sinx_n}$$

Putting $n = 0$, the first approximation x_1 is given by

$$x_1 = \frac{x_0 sinx_0 + cosx_0 + 1}{3 + sinx_0}$$

$$x_1 = \frac{0.6sin0.6 + cos0.6 + 1}{3 + sin0.6}$$

$$x_1 = 0.6071$$

Putting $n = 1$, the first approximation x_2 is given by

$$x_2 = \frac{x_1 sinx_1 + cosx_1 + 1}{3 + sinx_1}$$

$$x_2 = \frac{0.6071\sin(0.6071) + \cos(0.6071) + 1}{3 + \sin(0.6071)}$$

$$x_1 = 0.6071$$

2. We use the Newton-Raphson method and its modifications to estimate the root of $f(x) = -x^4 + 6x^2 + 11$ employing an initial guess of $x_0 = 1$.[7]
By Newton-Raphson method for multiple roots

$$x_{n+1} = x_n - \frac{kf(x_n)}{f'(x_n)}$$

In the following table, we present the results obtained by applying several iterations taking $m = 2,3$ and 4.

	k = 2	k = 3	k = 4
x_1	-3	-5	-7
x_2	-2.5555	-1.8363	-0.4906
x_3	-2.9729	-23.6285	-8.6581
.	.	.	.
.	.	.	.
.	.	.	.
x_{997}	-2.7358	-3.0060	-1.23×10^{-8}
x_{998}	-2.7311	-2.3267	2.98×10^{-8}
x_{999}	-2.7358	-4.2198	1.23×10^{-8}
x_{1000}	-2.7311	-1.8282	-2.98×10^{-8}

Table 1: Iteration for Newton-Raphson method for multiple roots (k)

For Newton's method with $k = 2$, we have a slow convergent method. For $k = 3$, we have an unstable non-convergent method, and for $k = 4$, we get a method such that, for large values n the result jumps between

$$x_{4n} = -2.98 \times 10^{-8} = -x_{4n+2}$$
$$x_{4n+1} = -1.23 \times 10^{-8} = -x_{4n+3}$$

Hence, for $f(x) = -x^4 + 6x^2 + 11$ with initial guess $x_0 = 1$, the Newton-Raphson method and the modifications considered in this paper do not have a good performance. However, we can use a modification of the Newton-Raphson method with cubic convergence given by the recurrence relation

$$x_{n+1} = x_n - \frac{4f(x_n)}{f'(x_n) + 3f'\left(\frac{x_n + 2\rho_n}{3}\right)}$$

where,

$$\rho_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

It is easy to verify that if $\rho_n = x_n$, we get the classical Newton-Raphson method. As and interesting fact, by apply the above recurrence to $f(x) = -x^4 + 6x^2 + 11$
with initial guess $x_0 = 1$, we obtain the root $x_8 = 2.7335$.

Limitations of The Newton-Raphson Method:

1. If $f'(x_n) = 0$ for some *n*, the method can no longer be applied.
2. If $f(x)$ has no real root, then there is no indication by the method and the iteration may simply oscillate. For example, compute the Newton-Raphson iteration for $f(x) = x^2 - 4x + 5$.
3. If the equation $f(x) = 0$ has more than one root and we are specific about capturing a particular root (say the smallest positive root). Then we have to be careful in choosing the initial guess. If the initial guess is far from the expected root, then there is a danger that the iteration converges to another root of the equation. This usually happens when the slope $f'(x_0)$ is small and the tangent line to the curve $y = f(x)$ is nearly horizontal.

Conclusion:
The Newton-Raphson method is a widely used numerical technique for approximating the roots of both algebraic and transcendental equations. One of its key advantages is its fast quadratic convergence. However, this method also has notable limitations. It is highly dependent on a good initial guess, can fail at points where the derivative is zero, and is sensitive to multiple roots or oscillatory behaviour in cases that do not converge.

There are several modifications to the Newton-Raphson method that enhance its performance under certain conditions, such as cubic convergence or addressing issues with multiple roots. This paper highlights the importance of understanding these limitations, carefully selecting initial guesses, and utilizing appropriate modifications to obtain accurate and reliable results in practical applications.

Reference:

1. A. Ozyapici, Z. Sensoy and T. Karanfiller. Effective root-finding methods for nonlinear equations based on multiplicative calculi. J. Math, pp. 1-7, 2016.
2. A. Saba, U.A. Qurrat. Newton Raphson Method, International Journal of Scientific and Engineering Research, Volume 6, Issue 7, July 2015.
3. VM Vishal, RS Dhruv. Review on Newton Raphson Method And Applications, Internation Research Journal Of Modernization in Engineering Technology and Science, Volume 04, Issue 01, January 2022.
4. S. Chapra and R. Canale. Numerical methods for engineers, Mc Graw Hill, 7th edition, 2015.
5. A. Faul. A concise introduction to numerical analysis. CRC Press, 2016.
6. BS. Grewal. Numerical Methods in engineering and science. Edition 3,2014.
7. JG. Triana Laverde. On The Newton Raphson Method and its Modifications. Correo Electronico, Volume 14, pp- 75-80, December 2023.
8. M. Allen and E. Isaacson. Numerical analysis for applied science. John Wiley and Sons, 2nd edition, 2019.
9. T. Heister, L. Rebholz and F. Xue. Numerical analysis. De Gruyter, 2019.
10. T. Sauer. Numerical analysis. Pearson, 3rd edition, 2018.