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Topological Fixed-Point Principles in One, Two and Three Dimensions: Mathematical Models and the Stirring-Coffee Illustration

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Abstract
Topology studies properties of spaces and mappings that remain invariant under continuous deformation. This article develops a unified account of fixed-point ideas in dimensions one, two and three, with particular emphasis on the Brouwer Fixed-Point Theorem, winding number and topological degree. The theorem states that every continuous self-map of a closed Euclidean ball has at least one fixed point. The one-dimensional case is established directly by the Intermediate Value Theorem, while the higher-dimensional statement is explained through the non-existence of a retraction from a closed ball onto its boundary and through degree-theoretic ideas. To make the theorem concrete, explicit affine self-maps are constructed on the unit interval, disk and three-dimensional ball. Their fixed points are calculated exactly and the self-map condition is verified analytically. A three-dimensional visualization is included for the map $F(x) = x/2 + (0,0,1/4)$, whose unique fixed point is $(0,0,1/2)$. A planar curve with winding number two is also examined. Finally, the stirring-coffee illustration is discussed with care: the theorem applies to a continuous map of a closed region into itself, but it does not by itself assert that a particular molecule is stationary throughout stirring or that an interior fixed point must exist. The article therefore distinguishes rigorous topological conclusions from physical interpretations and identifies the modelling assumptions needed for applications.

Keywords: Brouwer fixed-point theorem; topology; fixed point; winding number; mapping degree; continuous mapping; fluid motion; three-dimensional visualization.

1. Introduction

Topology is the mathematical study of properties preserved by continuous transformations (Munkres, 2000). Unlike metric geometry, which emphasizes distances, angles and curvature, topology focuses on structural features such as connectedness, compactness, boundaries and the way spaces are mapped into one another. Its methods now appear in areas including dynamical systems, nonlinear analysis, data analysis, robotics, image processing and mathematical physics. A central result is the Brouwer Fixed-Point Theorem, a foundational existence theorem in nonlinear analysis and topology (Deimling, 1985; Hatcher, 2002). In its standard Euclidean form, it states that a continuous map from a closed ball into itself has at least one point that is mapped to itself. The theorem is remarkable because it guarantees existence without generally providing a formula for the fixed point. It is also dimension-independent: the same statement holds for intervals, disks, solid balls and closed balls in R^n. The motivating example of stirring coffee is useful for communicating the theorem, but it must be formulated precisely (Schikorra, 2024). A physical flow induces a map from initial particle positions to later positions. If the flow map is continuous and maps a compact region into

itself, Brouwer's theorem guarantees a fixed point of that map. This is a statement about a position unchanged after the chosen time interval; it is not a statement that a particle remains motionless at every intermediate time. Moreover, the theorem does not ensure that the fixed point lies strictly in the interior. The objectives of this paper are: (i) to present the fixed-point theorem and its one-dimensional proof; (ii) to explain the role of winding number and mapping degree; (iii) to construct explicit two- and three-dimensional examples with exact fixed points; (iv) to provide a three-dimensional geometric visualization; and (v) to assess the scope and limitations of the coffee-stirring interpretation.

2. Mathematical preliminaries

2.1 Topological space and continuity

A topological space is a pair (X, τ), where X is a non-empty set and τ is a family of subsets of X, called open sets, satisfying: (a) $\emptyset$ and X belong to τ; (b) arbitrary unions of members of τ belong to τ; and (c) finite intersections of members of τ belong to τ. A map $f:X\to Y$ between topological spaces is continuous if the inverse image of every open subset of Y is open in X.

For subsets of Euclidean space, the usual topology is generated by open balls. A function $F:K\to K$ is a self-map of K. A point $p\in K$ is a fixed point if $F(p)=p$.

2.2 Closed balls

$B^n = \{x \in \mathbb{R}^n : \|x\| \le 1\}$, $S^{n-1} = \partial B^n = \{x \in \mathbb{R}^n : \|x\| = 1\}$.

The set B^n is the closed unit ball in n-dimensional Euclidean space, and S^{n-1} is its boundary sphere. Thus $B^1=[-1,1]$, B^2 is the closed disk and B^3 is the solid unit ball.

2.3 Brouwer Fixed-Point Theorem

Theorem 1 (Brouwer). Let K be a non-empty compact convex subset of $\mathbb{R}^n$. Every continuous map $F:K\to K$ has at least one fixed point. In particular, every continuous self-map $F:B^n\to B^n$ has a point $p\in B^n$ satisfying $F(p)=p$.

The theorem is an existence theorem. It does not state that the fixed point is unique, stable, or located in the interior. Uniqueness requires additional assumptions, for example a contraction condition.

3. One-dimensional case

Let $f:[0,1]\to[0,1]$ be continuous. Define $g(x)=f(x)-x$. Since f and the identity function are continuous, g is continuous on $[0,1]$. Because $f(0)\in[0,1]$, $g(0)=f(0)\ge 0$. Similarly, $f(1)\in[0,1]$, so $g(1)=f(1)-1\le 0$. By the Intermediate Value Theorem, there exists $c\in[0,1]$ such that $g(c)=0$. Consequently $f(c)=c$. This proves the one-dimensional Brouwer theorem.

3.1 Exact analytical example

Consider $f(x)=0.6x+0.2$ on $[0,1]$. Since $f([0,1])=[0.2,0.8]\subset[0,1]$, this is a continuous self-map. Solving $f(c)=c$ gives $0.6c+0.2=c$, hence $c=0.5$. Direct substitution gives $f(0.5)=0.5$. Since $|f'(x)|=0.6<1$, the map is a contraction and this fixed point is unique.

4. Two-dimensional case: disk maps and winding number

4.1 Brouwer theorem on the disk

Let $D=\{(x,y)\in\mathbb{R}^2 : x^2+y^2\le 1\}$. If $F:D\to D$ is continuous, the Brouwer theorem guarantees at least one $p\in D$ for which $F(p)=p$. The result concerns the whole closed disk, including its boundary.

4.2 An explicit disk self-map

$F(x,y) = (x/2 + 1/4, y/2)$.

For every $(x,y)\in D$, the triangle inequality gives $\|F(x,y)\|\le(1/2)\|(x,y)\|+1/4\le 3/4<1$. Hence $F(D)\subset D$. The fixed-point equations are $x=x/2+1/4$ and $y=y/2$, giving $p=(1/2,0)$. The point belongs to D and $F(1/2,0)=(1/2,0)$. Moreover, the linear part has operator norm $1/2$, so the map is a contraction and the fixed point is unique.

4.3 Winding number

Let $\gamma:[0,2\pi]\to\mathbb{R}^2\setminus\{0\}$ be a continuous closed curve, with $\gamma(0)=\gamma(2\pi)$. Its winding number about the origin is the total change in a continuous choice of argument divided by 2π:

$\mathrm{wind}(\gamma,0) = [\arg \gamma(2\pi) - \arg \gamma(0)]/(2\pi) \in \mathbb{Z}$.

For a differentiable curve $\gamma(t)=(x(t),y(t))$ that avoids the origin, an equivalent integral formula is

$\mathrm{wind}(\gamma,0) = (1/2\pi) \int_0^{2\pi} [x(t)y'(t) - y(t)x'(t)]/[x(t)^2+y(t)^2]\, dt$.

The integer is invariant under a homotopy of closed curves that avoids the reference point (Munkres, 2000). Reversing the orientation changes the sign.

4.4 Computed winding-number example

Take $\gamma(t)=(\cos 2t, \sin 2t)$, $0\le t\le 2\pi$. A continuous argument is $\theta(t)=2t$, so the total argument change is 4π. Therefore $\text{wind}(\gamma,0)=4\pi/(2\pi)=2$. The curve makes two counterclockwise turns around the origin, as shown in Figure 2.

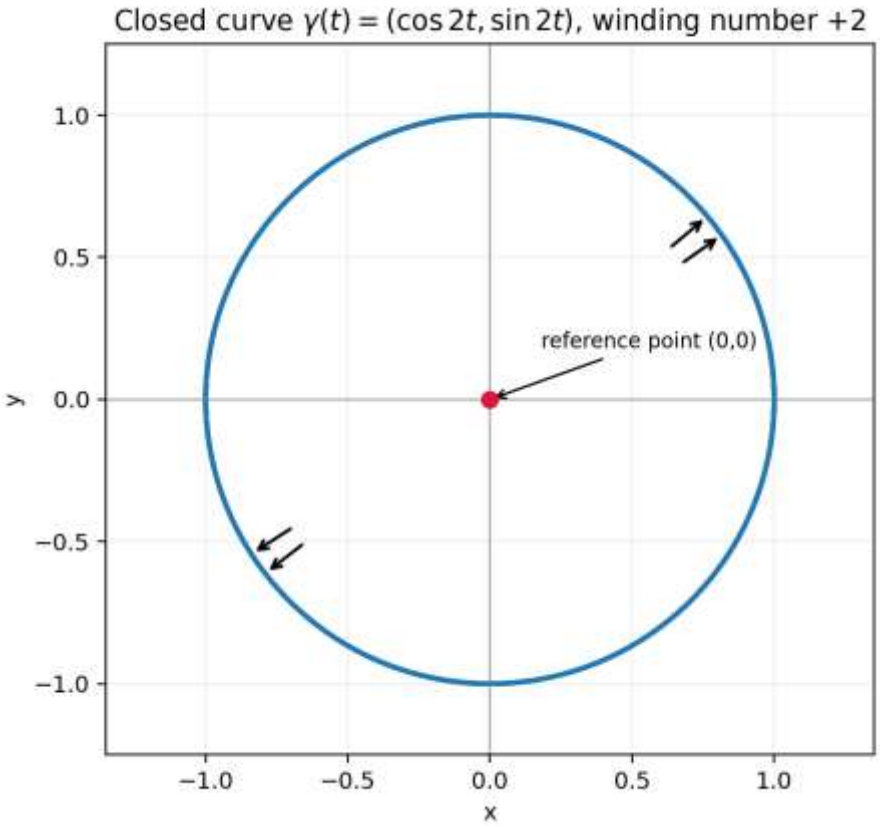


Figure 2. A planar closed curve with winding number +2 about the origin.

5. Three-dimensional case: ball maps and mapping degree

5.1 Brouwer theorem in three dimensions

Let $B^3=\{(x,y,z)\in\mathbb{R}^3: x^2+y^2+z^2\le 1\}$. For every continuous self-map $F:B^3\to B^3$, there exists $p\in B^3$ such that $F(p)=p$. The three-dimensional result is not a different theorem; it is the $n=3$ instance of the same dimension-independent statement.

5.2 Explicit three-dimensional analytical model

$F(x,y,z) = (x/2, y/2, z/2 + 1/4)$.

For any $q=(x,y,z)\in B^3, ||F(q)||\le(1/2)||q||+1/4\le 3/4$. Thus F maps B^3 into itself and is continuous. To determine its fixed point, solve

$$x=x/2,\quad y=y/2,\quad z=z/2+1/4.$$

The equations yield $x=0, y=0$ and $z=1/2$. Hence $p=(0,0,1/2)$, and direct substitution confirms $F(p)=p$. The Euclidean distance between the images of any two points satisfies $||F(q_1)-F(q_2)||=(1/2)||q_1-q_2||$. Therefore F is a contraction with constant 1/2, and Banach's contraction principle gives uniqueness as well as existence.

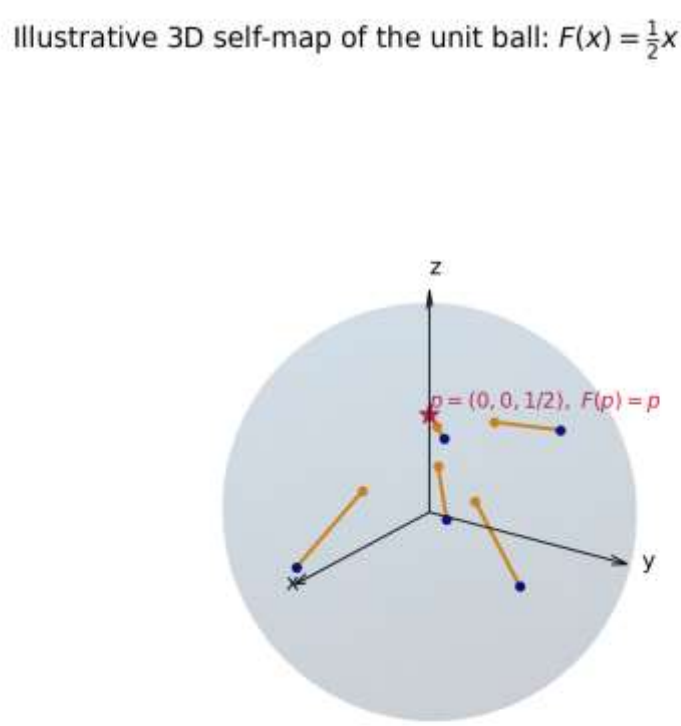


Figure 2. Three-dimensional unit ball and illustrative affine self-map. Blue points indicate sample initial positions, orange points their images, and the red star marks the analytically computed fixed point.

5.3 Mapping degree

For a bounded domain $\Omega \subset \mathbb{R}^n$, a continuous map $G:\bar{\Omega} \to \mathbb{R}^n$, and a point $a \notin G(\partial\Omega)$, the Brouwer degree $\deg(G,\Omega,a)$ is an integer that records the oriented number of preimages of a, counted with multiplicity. When G is sufficiently smooth and a is a regular value,

$$deg(G, \Omega, a) \; = \; \Sigma_\{x \in G^{-1}(a)\} \, sign(det \, DG(x)).$$

The degree is invariant under admissible homotopies and is non-zero only if a belongs to the image of G (Deimling, 1985; Hatcher, 2002). In particular, if $\deg(G,\Omega,a) \neq 0$, then $G(x)=a$ has at least one solution in Ω. For sphere maps $S^n \to S^n$, the degree is an integer homotopy invariant. The identity map has degree +1. The degree is not simply an informal count of how many times a solid ball is covered; orientation and multiplicity are essential.

5.4 Relation between winding number and degree

The winding number is the planar degree of a map from a circle to a circle, after identifying the circle with S^1. Mapping degree generalizes this integer-valued invariant to maps between higher-dimensional spheres and to suitable maps between oriented domains. In dimension two, the winding number records signed turns around a point; in dimension three, degree records signed covering behaviour of a sphere map.

6. Three-dimensional stirring model and physical interpretation

Let $\Omega \subset \mathbb{R}^3$ represent the coffee volume at the initial time and let $\Phi_T:\Omega \to \Omega$ denote the particle flow map over a fixed time interval T. For an initial particle position a, $\Phi_T(a)$ is the position of that same material particle at time T. If the velocity field is sufficiently regular to generate a continuous flow map, and if the region is invariant over the interval so that $\Phi_T(\Omega) \subset \Omega$, then Brouwer's theorem gives at least one $a \in \Omega$ satisfying $\Phi_T(a)=a$.

This statement means that at least one particle returns to its initial spatial position at time T. It does not mean that the particle has zero velocity throughout [0,T], nor that it stays at the same location at every intermediate time. A particle may move along a nontrivial closed trajectory and return after one period. In a cup with a stationary no-slip wall,

boundary particles may already be fixed by the flow map; therefore the theorem alone does not guarantee a non-boundary fixed point.

A rigorous physical model must specify the flow domain, boundary conditions, regularity of the velocity field, and the time interval. If particles can leave the region, if the domain changes, or if the map is not continuous, the stated Brouwer argument cannot be applied directly. Thus the coffee example is a valuable illustration of a topological existence principle, but it is not a quantitative fluid-mechanics prediction.

7. Scientific results and discussion

This section reports exact analytical results from the explicitly defined maps in Sections 3–5. These are mathematical model results, not experimental measurements or numerical simulations of real coffee. Their purpose is to verify the hypotheses of the fixed-point theorem and demonstrate how the dimension changes the geometry without changing the theorem's logical structure.

$Domain$	$Self-map$	$Verified\ bound$	$Fixed\ point$	$Uniqueness$
$[0,1]$	$f(x) = 0.6x + 0.2$	$f([0,1]) = [0.2,0.8]$	(0.5)	$Yes;\ contraction\ 0.6$
$D \subset \mathbb{R}^2$	$F(x,y) = (x/2 + 1/4, y/2)$	$\|\|F(q)\|\| \leq 0.75$	$(0.5,0)$	$Yes;\ contraction\ 0.5$
$B^3 \subset \mathbb{R}^3$	$F(x) = x/2 + (0,0,1/4)$	$\|\|F(q)\|\| \leq 0.75$	$(0,0,0.5)$	$Yes;\ contraction\ 0.5$

Table 1. Exact fixed points and self-map verification for the illustrative affine models.

7.1 Interpretation of the analytical results

- Existence is dimension-independent: the interval, disk and solid ball examples all satisfy the fixed-point conclusion.
- The self-map condition is essential. In each example, the image is shown to remain inside the original closed domain.
- The fixed points are obtained by solving linear equations, so the reported coordinates are exact rather than approximations.
- The contraction estimates establish uniqueness for these selected examples. Brouwer's theorem by itself guarantees existence only and does not imply uniqueness.
- For the planar curve γ(t)=(cos 2t,sin 2t), the winding number is exactly +2, illustrating an integer invariant that is unchanged by deformations avoiding the origin.

7.2 What these results do not establish

The affine maps are deliberately simple test cases. They do not constitute a computational fluid-dynamics simulation, a laboratory experiment, or evidence that a specific stirred beverage has a stationary interior particle. No measured velocity, viscosity, Reynolds number, temperature field or particle-tracking data were supplied in the source project. Such claims would require a separately specified physical model and validated numerical or experimental data.

8. Applications and scope

8.1 Nonlinear equations and equilibrium

Fixed-point theorems are used to establish existence of solutions to nonlinear equations by rewriting a problem as x=F(x). The theorem can guarantee an equilibrium without giving a closed-form solution. Additional tools, such as contraction mappings, may establish uniqueness and convergence of iterative schemes.

8.2 Dynamical systems

For a continuous dynamical system, fixed points represent equilibria. For a time-T map, a fixed point represents a state that returns to itself after time T; it may correspond to a periodic orbit rather than a stationary equilibrium.

8.3 Robotics, image analysis and computational topology

Winding numbers and degree-like invariants can support point-in-region tests, orientation analysis and the classification of mappings. In robotics and computer graphics, winding-number methods are used in geometric

containment and boundary-processing algorithms. In imaging and data analysis, topological methods can help identify connected components, loops and persistent structural features.

9. Limitations and modelling requirements

- Continuity must be established; a discontinuous update rule is outside the theorem's hypotheses.
- The map must send the entire closed domain into itself.
- The theorem gives existence, not a formula, uniqueness, stability or an interior location.
- A fixed point of a time-T flow map is not necessarily a stationary particle at all times.
- Mapping degree requires suitable domain, boundary and regularity conditions; it is not interchangeable with an unqualified count of coverings.
- The explicit affine examples are analytical demonstrations and should not be presented as empirical scientific data.

10. Conclusion

The Brouwer Fixed-Point Theorem provides a unified existence principle for continuous self-maps of closed balls in every finite dimension. Its one-dimensional form follows from the Intermediate Value Theorem, while the higher-dimensional theory is connected to the impossibility of retracting a ball onto its boundary and to the invariance properties of topological degree. Winding number provides the corresponding planar integer invariant for closed curves.

The exact affine examples in one, two and three dimensions verify the theorem's hypotheses and produce fixed points at 1/2, (1/2,0) and (0,0,1/2), respectively. The three-dimensional figure makes the geometric structure visible. The stirring-coffee illustration links topology with fluid motion, provided the flow map is continuous and maps the chosen region into itself. Careful modelling is necessary: the theorem guarantees a return point for the time map, not a particle that remains motionless throughout the stirring process. These distinctions make the application mathematically accurate and suitable for further scientific investigation.

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