International Journal of Scientific Research in Engineering and Management (IJSREM)
Volume: 09 Issue: 05 | May - 2025 SJIF Rating: 8.586 ISSN: 2582-3930

Two Parametric Measure of Weighted Information Improvement

1. S.K. Chandbhanani,
M.Sc.(Maths),B.ed.
Gondia (Maharashtra) INDIA – 441601
Email- ahujasapana6@gmail.com

2. H.R.Trivedi
M.Sc.(Maths),PhD,
Principal M.B.Patel College Of Arts, Commerce And Science,
Sakoli,Gondia (Maharashtra) INDIA – 441601
Email- trivediharesh2@gmail.com

3. P.A.S. Naidu,
M. Sc. (Mathematics), M. Ed., P.G.D.C.S., PhD
Principal Dhote Bandhu Science College
Gondia (Maharashtra) INDIA- 441601,
Email- anjannaidu8@gmail.com

Abstract

In the present paper we have obtained measures of weighted directed divergence and measures of weighted Information improvement corresponding to Kullback-Leibler [9], Havrda and Charvat's[4], Kapur [5],[6],[7],[8], Fermi-Dirac [4] Measure of entropy and some new two parametric measures of weighted directed divergence and measures of information improvement are deduced.

Key words: - Entropy, Information improvement, Measure of directed divergence, weighted information improvement.

1. Introduction

Let P= $(p_1, p_2, p_3, \dots, p_n)$ be non-degenerate probability distribution and
Let W = $(w_1, w_2, w_3, \dots, w_n)$ be a set of weights associated with the n outcomes, then corresponding to Shannon's [10] measure of entropy

$$S(P) = -\sum_{i=1}^{n} p_i \ln p_i \qquad (1)$$

Guiasu [3] defined a measure of weighted entropy as,

$$S(P{:}W) = -\sum_{i=1}^{n} w_i p_i \ln p_i \qquad (2)$$

Naturally $\quad p_i \geq 0, w_i \geq 0$

$$\sum p_i = 1, \sum w_i = 1$$

Now if Q= $(q_1, q_2, q_3, \dots, q_n)$ be the another non-degenerate probability distribution then the well know Kullback Leibler [9] measure D_1 (P: Q) gives a measure of directed divergence of P and Q as,

Where $\quad D_1(P{:}\,Q) = \sum_{i=1}^{n} p_i \ln \frac{p_i}{q_i} \qquad (3)$

And Taneja and Tuteja [11] gave the corresponding measure of weighted directed divergence as,

$$D_1(P{:}\,Q, W) = \sum_{i=1}^{n} w_i\, p_i \ln \frac{p_i}{q_i} \qquad (4)$$

Measure (3) is a correct measure of directed divergence, since it has following properties,

(i)$D_1(P{:}Q) \geq 0$ (5)
(ii)$D_1(P{:}Q)=0$ iff $P=Q$ (6)

(iii) $D_1(P{:}Q)$ is a convex function of P and Q (7)

However $D_1(P{:}Q;W)$ is not a correct measure of weighted directed divergence because it does not satisfy (5) and (6)

To see this consider some following cases as,

Where (i) P= $\left(\frac{2}{5}, \frac{3}{5}\right)$, Q= $\left(\frac{3}{5}, \frac{2}{5}\right)$, W = $\left(\frac{3}{5}, \frac{2}{5}\right)$ (8)

Where (ii) P= $\left(\frac{1}{3}, \frac{2}{3}\right)$, Q = $\left(\frac{2}{3}, \frac{1}{3}\right)$,W=$\left(\frac{4}{5}, \frac{1}{5}\right)$ (9)

Then $D_1(P{:}Q,W) = \sum_{i=1}^{n} w_i\, p_i \ln \frac{p_i}{q_i}$

$$= \frac{1}{5}\left[3w_2 - 2w_1\right] \ln \frac{3}{2} \qquad (10)$$

Here we can say that both greater than zero and also vanish when $P \neq Q$ so it not correct to recognize $D_1(P{:}Q.W)$ as correct measure of weighted directed divergence and one object of paper is to find correct measure of weighted information improvement corresponding to six measure of entropy and new two parametric measure of weighted information improvement is obtain, in fact here we have to find more correct measure of weighted directed divergence corresponding to Csiszers's measure of directed divergence, D_2 (P: Q) $=\sum_{i=1}^{n} q_i \phi\left(\frac{p_i}{q_i}\right)$ where $\phi(.)$ is twice differentiable convex function with $\phi(1) = 0$.

The object of the present paper is to investigate the correct measures of directed divergence, correct measures of weighted directed divergence, correct measures of weighted information improvement, and also we defined properties of correct measure of directed divergence and correct measures of weighted information improvement.
In this paper, we extend the result of weighted information improvement $I_k(P{:}Q.W)$ in going from Q to R when the true distribution is P and weight function W(x) is defined by Kapur [8] is,

$$I_k(P{:}Q,R,W) = D_k(P{:}Q,W) - D_k(P{:}R,W) \qquad (11)$$

In section two we will define properties of correct measure of weighted directed divergence corresponding to Csiszers's [2] measure and properties measure of weighted information Improvement.

International Journal of Scientific Research in Engineering and Management (IJSREM)
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In fact some more correct measure of weighted directed divergence corresponding to Csiszer's [2] measure, correct measure of directed divergence and measure of weighted information improvement corresponding to Kullback Leibler [9] measure, corrected measure of weighted directed divergence corresponding to Harvda and Charvat's [4] measure and its measure of weighted information improvement, corrected measures of weighted Directed divergence and measures of weighted information improvement corresponding to Kapur's [5],[6] measure of entropy have been obtained in section three.

In section four some another class of measures of directed divergence and measures of weighted information improvement corresponding to Kapur [5],Kullback Leibler [9] ,Havrda and Charvat [4] ,Kapur [8,][6],[5] measures of entropy have been deduced by using theorem's state in section four .

In section five we will obtain new two parametric measures of weighted directed divergence and its two parametric measures of weighted information improvement and its limiting cases.

In section six we will give some concluding remarks of measure of weighted directed divergence and applications of measures of weighted directed divergence and measures of weighted information improvement.

The references which we used in present paper are given in section seven

2. Preliminaries

2.1 Correct Measure Of Weighted Directed Divergence Corresponding To Csiszer's [3] Measure

Since $\phi(x)$ is convex function, $\phi'(x) = 0$ is always increasing function also $\phi(1) = 0$ at x =1 and $\phi(x)$ may or may not be positive, negative or zero is defined (From figure 2.1(a), 2.2(b), 2.3(c)

Case I -

In the case (a) and (b) $\phi(\frac{p_i}{q_i})$ can be positive or negative $D_1(P:Q)$ is always ≥ 0, but $D_1(P:Q,W)$ can be negative or positive

Case II - In case 3.1(c) $\phi(\frac{p_i}{q_i})$ is always ≥ 0 and vanishes only when $p_i = q_i$ so that in this case Kapur [6] has obtained accurate measure of weighted directed divergence as ,

$D_2(P:Q,W) = \sum_{i=1}^{n} w_i q_i \phi\left(\frac{p_i}{q_i}\right)$ (12)

It is ≥ 0 and vanishes when $p_i = q_i \ \forall\, i$

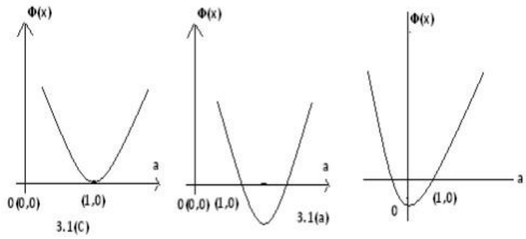
Now the accurate measure of weighted directed divergence corresponding to Csiszer's[2] measure of directed divergence is given by (12), provided

$\phi(x)$ is convex function twice differentiate (13)

$\phi(x) = 0$ (14)

$\phi'(x) = 0$ (15)

The condition (15) is the additional condition we impose on $\phi(x)$ has graph $\phi(\frac{p_i}{q_i})$ is always ≥ 0 and vanishes only when $p_i = q_i$



2.2 Measure Of Weighted Information Improvement

Now suppose during the course of investigation Q revised to R = $\{r_1, r_2, \dots, r_n\}$ then measure of information improvement is given by

I(P:Q,R) = D(P:Q) – D(P:R) (16)

The measure $I_k(P:Q:R,W)$ of weighted information improvement is going from Q to R when the true distribution P is and the weight function W(x) is associated then Kapur [8] is defined as,

$I_k(P:Q:R,W) = D_k(P:Q,W) - D_k(P:R,W)$

Where,

R = $\{r_1, r_2, \dots, r_n\}, r_i \geq 0, \sum_{i=1}^{n} r_i = 1$ (17)

3. Correct Measures Of Directed Divergence And Weighted Information Improvement

3.1 Correct Measure Of Directed Divergence And Measure Of Weighted Information Improvement Corresponding To Kullback Leibler [9] Measure

Let

$\phi(x) = x \ln x - x + 1$ (18)

$\phi'(x) = \ln x$

$\phi''(x) = \frac{1}{x}$

Hence, $\phi(1) = 0$

$\phi'(1) = 0$

$\phi(x)$ is convex function and it is twice differentiable so that, by(12) the correct measure of weighted directed divergence corresponding to Kullback Leibler [9] measures is as follows,

$$D_3(P:Q,W) = \sum_i^n w_i\{ p_i \ln (\frac{p_i}{q_i}) - p_i + q_i\} \quad (19)$$

$$D_3(P:Q,W) = \sum_i^n w_i\{ p_i \ln (\frac{p_i}{r_i}) - p_i + r_i\} \quad (20)$$

From (17),

$I_3(P:Q:R,W) = D_3(P:Q,W) - D_3(P:R,W)$

$$I_3(P:Q:R,W) = \sum_i^n w_i\{ p_i \ln (\frac{r_i}{q_i}) + q_i - r_i \} \quad (21)$$

$I_3(P:Q:R,W)$ is the measure of weighted information improvement corresponding to Kullback Leibler [9] measure of entropy.

3.2 Corrected Measure Of Weighted Directed Divergence Corresponding To Harvda And Charvat's [4] Measure Of Entropy And Its Measure Of Weighted Information Improvement

Take

$$\phi(x) = (\frac{x^\alpha - \alpha x + \alpha - 1}{\alpha(\alpha-1)}), \alpha \neq 0, 1 \quad (22)$$

$$\phi'(x) = \frac{1}{\alpha-1}\{(\alpha-1)x^{\alpha-2}\}, \alpha \neq 0, 1 \quad (23)$$

$$\phi''(x) = x^{\alpha-2} \quad (24)$$

From it is clear that (24) $\phi(x)$ is twice differentiate convex function and from (22) and (23) it is seen that $\phi(1) = 0$ and $\phi'(1) = 0$
Again by (12) the correct measure of directed divergence and measured weighted information improvement corresponding to Harvda and charvats [4] measure of entropy

$$D_4(P:Q,W) = \frac{1}{\alpha(\alpha-1)} \sum_{i=1}^n w_i\,[p_i^\alpha q_i^{1-\alpha} - \alpha p_i + \alpha q_i - q_i] \quad (25)$$

$$D_4(P:R,W) = \frac{1}{\alpha(\alpha-1)} \sum_{i=1}^n w_i[p_i^\alpha r_i^{1-\alpha} - \alpha p_i + \alpha\, r_i - r_i] \quad (26)$$

Again from (17)

$I_4(P:Q:R,W) = D_4(P:Q,W) - D_4(P:R,W)$

$$I_4(P:Q:R,W) = \frac{1}{\alpha(\alpha-1)} \sum_{i=1}^n w_i[\, p_i^\alpha (q_i^{1-\alpha} - r_i^{1-\alpha}) + (\alpha-1)(q_i - r_i)] \quad (27)$$

$\therefore$ $I_4(P:Q:R,W)$ is the measure of weighted information improvement corresponding to Havrda and Charvats [4] measure of entropy

Now we will obtain the particular cases of $I_4(P:Q:R,W)$ as if $\alpha \to 1$ in (25) we get (19) Kullback –Zeibler [7] measure

as $\alpha \to 0$ measure (25) approaches (19)

$$\therefore D_4(P:Q,W) = D_3(P:Q,W) \quad (28)$$

And $D_4(P:R,W) = D_3(P:R,W)$ (29)

And so $I_4(P:Q:R,W) = I_3(P:Q:R,W)$ (30)

3.3 Corrected Measure Of Weighted Directed Divergence And Measure Of Weighted Information Improvement Corresponding To Kapur's [5], [6] Measure Of Entropy

$$\phi(x) = x \ln x - \frac{(1+ax)}{a} \ln (1+ax) + x \ln (1+a) + \frac{(1+a)}{a} \ln (1+a) - \ln (1+a),\ a \geq 0 \quad (31)$$

$$\phi'(x) = \ln x - \ln (1+ax) + \ln (1+a) \quad (32)$$

$$\phi''(x) = \frac{1}{x} - \frac{a}{1+ax} \quad (33)$$

It is clearly seen from (32) and (33) $\phi(x)$ is a twice differentiable convex function $\phi(1) = 0$ and $\phi'(1) = 0$
By using (12) the correct measure of directed divergence and measure of weighted information improvement corresponding to Kapur [6] and [7] measure of entropy as,

$$D_5(P:Q,W) = \sum_{i=1}^n w_i\, p_i \ln(\frac{p_i}{q_i}) + \sum_{i=1}^n w_i\, (p_i - q_i) \ln (1+a) + \frac{(1+a)}{a} \ln (1+a) \sum_{i=1}^n w_i q_i - \frac{1}{a}\sum_{i=1}^n w_i(q_i + ap_i) \ln (1+\frac{ap_i}{q_i}) \quad (34)$$

$$D_5(P:R,W) = \sum_{i=1}^n w_i\, p_i \ln (\frac{p_i}{r_i}) + \sum_{i=1}^n w_i\, (p_i - r_i) \ln (1+a) + \frac{(1+a)}{a} \ln (1+a) \sum_{i=1}^n w_i r_i - \frac{1}{a} \sum_{i=1}^n w_i\, (r_i + ap_i) \ln (1+\frac{ap_i}{r_i}) \quad (35)$$

$I_5(P:Q:R,W) = D_5(P:Q,W) - D_5(P:R,W)$

$$= \sum_{i=1}^n w_i\, \{p_i \ln (\frac{r_i}{q_i}) + (r_i - q_i) \ln (1+a) + \frac{1+a}{a} \ln (1+a)\ (q_i - r_i) - \frac{1}{a} [(q_i + ap_i) \ln (1+\frac{ap_i}{q_i}) - (r_i+ap_i)\ln(1+\frac{ap_i}{r_i})]\} \quad (36)$$

We can noted that
(i)When $a \to 0$ (34) becomes (19)
$D_5(P:Q,W) = D_3(P:Q,W)$
So Naturally $D_5(P:R,W) = D_3(P:R,W)$
Hence $I_5(P:Q:R,W) = I_3(P:Q:R,W)$
(ii) from (25) $D_{4,1-\alpha}(P:Q,W) = D_{4,\alpha}(Q:P,W)$

So naturally $I_{4,1-\alpha}(P:Q,W) = I_{4,\alpha}(Q:P,W)$ (37)

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4.Some Another Class Of Measure Of Weighted Directed Divergence And Its Measure Of Weighted Information Improvement

It is clear that $\sum_{i=1}^{n} q_i\phi\left(\frac{p_i}{q_i}\right)$ is a directed divergence but $\sum_{i=1}^{n} w_i q_i\phi\left(\frac{p_i}{q_i}\right)$ need not to be a measure of directed divergence . However, we have got following theorems :

Theorem 4.1 :- if $\phi(.)$ is a convex twice differentiable function such that $\phi(1) = 0$, then $\sum_{i=1}^{n} w_i \{q_i\ \phi\left(\frac{p_i}{q_i}\right) + (1-q_i)\ \phi\left(\frac{1-p_i}{1-q_i}\right)\}$

It is a valid measure of weighted directed divergence, as generalization of above measure, consider the measure $D(P:Q:W)= \sum_{i=1}^{n} w_i\ \{q_i\ \phi\left(\frac{p_i}{q_i}\right) + (1+aq_i)\phi\left(\frac{1+ap_i}{1+aq_i}\right)\}$

At a= -1

$D(P:Q:W) = \sum_{i=1}^{n} w_i\, q_i\, \phi + \sum_{i=1}^{n} w_i\, (1-q_i)\, \phi\left(\frac{1-p_i}{1-q_i}\right)$

Theorem 4.2:- If ϕ (.) is a convex twice differentiable function such that $\phi(1) =0$, $\phi'(1) =0$ then $\sum_{i=1}^{n} w_i \{q_i\ \phi\left(\frac{p_i}{q_i}\right) + (1+aq_i)\ \phi\left(\frac{1+ap_i}{1+aq_i}\right)\}$

is a valid measure of weighted directed divergence As generalization of $D_2(P:Q,W)$ Kapur [8] has considered the measure when each $p_i = q_i\ \forall\ i$

$$D_6(P:Q,W) = \sum_{i=1}^{n} w_i\ q_i\ \phi\left(\frac{p_i}{q_i}\right)+ \sum_{i=1}^{n} w_i\ (1+aq_i)\ \phi\left(\frac{1+ap_i}{1+aq_i}\right) \quad (38)$$

(38) Provided

(i) $\phi(x)$ is twice differentiable convex function

(ii) $\phi(1) = 0$

(iii) $\phi'(1) = 0$

Then $D_6(P:Q,W)$ is convex function of $(p_1,p_2,\ldots\ldots p_n)$ which has minimum value zero when each $p_i=q_i$

So that $D_6(P:Q,W) \geq 0$ and vanishes iff P = Q

When we put a = 0 in (38)

We get $D6(P:Q,W) = D_2(P:Q,W)$ (39)

$\therefore D6(P{:}\,Q, W)$is a valid measure of weighted directed divergence

(39) is true for a > 0 or a < 0

If a < 0, since we want $1 + ap_i > 0$ and $a \geq -1$ by using (12) taking limit as $\alpha \to 1$ or any other suitable measure we can get a large number of correct measure of weighted directed divergence and weighted information improvement

4.3 Some Another Class Of Measure Of Weighted Directed Divergence And Its Measure Of Weighted Information Improvement Corresponding To Kullback Leibler [9] Measure Of Entropy

Let $\phi(x) = \ln x - x + 1$ (40)

$\phi'(x) = 1/x - 1$ (41)

$\phi''(x) = - 1/x^2$ (42)

It is easily seen from (41), (42) ,

$\phi(x)$ is twice differentiable convex function

$\phi(1) =0$

$\phi'(1) = 0$

Hence (40) satisfy the properties discussed in section 4.2

$\therefore$ (38) Gives the following measure of weighted directed divergence as

$$D_7(P:Q,W) = \sum_{i=1}^{n} w_i\ [p_i\ \ln\left(\frac{p_i}{q_i}\right)- p_i+q_i] + \sum_{i=1}^{n} w_i\ [(1+ap_i)\ln\left(\frac{1+ap_i}{1+aq_i}\right)- ap_i + aq_i] \quad (43)$$

$$D_7(P:R,W) = \sum_{i=1}^{n} w_i\ [p_i\ \ln\left(\frac{p_i}{r_i}\right)- p_i + r_i] + \sum_{i=1}^{n} w_i\ \{(1+ap_i)\ln\left(\frac{1+ap_i}{1+ar_i}\right) - ap_i + ar_i\} \quad (44)$$

(43) and (44) gives measure of weighted information improvement as,

$$I_7(P:Q:R,W) = D_7\ (P:Q,W) - D_7\ (P:R,W)$$

$$= \sum_{i=1}^{n} w_i\ [p_i\ \ln\left(\frac{r_i}{q_i}\right) + (q_i - r_i)] + \sum_{i=1}^{n} w_i\,[(1+ap_i)\ \ln\left(\frac{1+ar_i}{1+aq_i}\right)+ a(q_i - r_i)] \quad (45)$$

$I_7(P:Q:R,W)$ is the measure of weighted information improvement corresponding to Kullback Leibler [9] measure of entropy.

If we put a = 0 in (43) and (45) we get (19) and (21), if we put a = -1 in (43) and (45) we get the following measures as,

$$D_7(P:Q,W)=\sum_{i=1}^{n} w_i\, \{p_i \ln\left(\frac{p_i}{q_i}\right) + (1-p_i)\ln\left(\frac{1-p_i}{1-q_i}\right)\}$$
(46)

Similarly,

$D_7(P:R,W)=\sum_{i=1}^{n} w_i \{p_i \ln(\frac{p_i}{r_i}) + (1-p_i)\ln(\frac{1-p_i}{1-r_i})\}$ (47)

Which is measure of weighted directed divergence corresponding to Fermi-Dirac measure of directed divergence and its measure of weighted information improvement is as,

When a = -1 in (45) we will get weighted information improvement measure as,

$I_7(P:Q:R,W) = \sum_{i=1}^{n} w_i\ [p_i \ln(\frac{r_i}{q_i}) + (1-p_i) \ln(\frac{1-r_i}{1-q_i})]$ (48)

4.4 The Measure Of Weighted Directed Divergence Corresponding To Havrda And Charvat [4] Measure Of Entropy And Its Measure Of Weighted Information Improvement

Let $\phi(x) = \frac{x^{\alpha}-\alpha x+\alpha-1}{\alpha(\alpha-1)}$, $\alpha \neq 0,\ \ \alpha \neq 1$ or $\alpha > 1$ (49)

$\phi'(x) = \frac{x^{\alpha-1}-1}{\alpha-1}$ (50)

$\phi''(x) = x^{\alpha-2}$ (51)

It is easily seen from (50) and (51)

$\phi(1) = 0, \quad \phi'(1) = 0$

Since $\phi(x)$ satisfies (13), (14) and (15) so (38) gives

$D_8(P:Q,W) = \frac{1}{\alpha(\alpha-1)} \sum_{i=1}^{n} w_i\ \{p_i^{\alpha} q_i^{1-\alpha} - \alpha p_i + \alpha q_i - q_i) + (1+ap_i)^{\alpha} (1+aq_i)^{1-\alpha} - \alpha a(p_i - q_i) - q_i\}$ (52)

$D_8(P:R,W)= \frac{1}{\alpha(\alpha-1)} \sum_{i=1}^{n} w_i\ \{p_i^{\alpha} r_i^{1-\alpha} - \alpha p_i + \alpha r_i - r_i + (1+ap_i)^{\alpha} (1+ar_i)^{1-\alpha} - \alpha a(p_i - r_i) - r_i\}$ (53)

$I_8(P:Q:R,W) = D_8(P:Q,W) - D_8(P:R,W)$

$I_8(PQ:R,W)=\frac{1}{\alpha(\alpha-1)} \sum_{i=1}^{n} w_i\{p_i^{\alpha}\ (q_i^{1-\alpha} - r_i^{1-\alpha})+(\alpha+\alpha a-2)(q_i - r_i) + (1+ap_i)^{\alpha}[(1+aq_i)^{1-\alpha} - (1+ar_i)^{1-\alpha}\ ]\}$ (54)

$I_8(P:Q:R,W)$ is the measure of weighted information improvement corresponding to Havrda and Charvats [4] measure of entropy.

We put a = -1 in (52) we get

$D_8(P:Q,W) =$

$\frac{1}{\alpha(\alpha-1)} \sum_{i=1}^{n} w_i[p_i^{\alpha}q_i^{1-\alpha} + (1-p_i)^{\alpha}(1-q_i)^{1-\alpha} - 2q_i]$ (55)

Similarly, $D_8(P:R,W) = \frac{1}{\alpha(\alpha-1)} \sum_{i=1}^{n} w_i[p_i^{\alpha}r_i^{1-\alpha} + (1 - p_i)^{\alpha}(1 - r_i)^{1-\alpha}-2r_i]$ (56)

And (54) becomes

$I_8(P:Q:R,W) = \frac{1}{\alpha(\alpha-1)} \sum_{i=1}^{n} w_i\{p_i^{\alpha}\ (q_i^{1-\alpha}- r_i^{1-\alpha}\ ) + (1-p_i)^{\alpha}\ [(1-q_i)^{1-\alpha} - (1-r_i)^{1-\alpha} - 2(q_i - r_i)]\}$ (57)

4.5 Measure Of Weighted Directed Divergence And Its Measure Of Weighted Information Improvement Corresponding To Kapur [8] Measure Of Entropy.

If inequality constraint is imposed on probabilities i.e.

$a_i \leq p_i \leq b_i\ , \qquad i = 1,2,\dots,n$ (58)

the following measure of weighted directed divergence is introduce by Kapur [8]

$D_9(P:Q,W) = \sum_{i=1}^{n} w_i\ \{(p_i - a_i)\ln(\frac{p_i-a_i}{q_i-a_i}) + (b_i - p_i) \ln(\frac{b_i-p_i}{b_i-q_i})]\}$ (59)

$D_9(P:R,W) = \sum_{i=1}^{n} w_i\{(p_i - a_i)\ln(\frac{p_i-a_i}{r_i-a_i}) + (b_i - p_i) \ln(\frac{b_i-p_i}{b_i-r_i})]\}$ (60)

From (59) and (60) it is easily seen to be a convex function of $p_1,p_2,\dots,p_n$ whose minimum value zero arises when each $p_i = q_i$

∴ (59) It is used as valid measure of weighted directed divergence.

Therefore the measure of weighted information improvement corresponding to $D_9(P:Q,W)$ is as follows,

$I_9(P:Q:R,W) = D_9(P:Q,W) - D_9(P:R,W)$

$I_9(P:Q:R,W) = \sum_{i=1}^{n} w_i\{(p_i-a_i) \ln(\frac{r_i-a_i}{q_i-a_i}) + (b_i - p_i) \ln(\frac{b_i-r_i}{b_i-q_i})\}$ (61)

If we put $a_i = 0$ and $b_i = 1$ in (61), i.e. $0\leq p_i \leq 1$ which is natural constraints

∴ (61) Becomes

$I_9(P:Q:R,W)=\sum_{i=1}^{n} w_i\{p_i \ln(\frac{r_i}{q_i}) + (1 - p_i) \ln(\frac{1-r_i}{1-q_i})\}$ (62)

Which is same as weighted information improvement $I_7(P:Q:R,W)$ corresponding to

Kullback Leibler [9] measure of entropy at $a_i = 0$ and $b_i=1$

I_9 (P:Q:R,W) = I_7 (P:Q:R,W) (63)

4.6 Measure of weighted directed divergence and its measure of weighted information improvement corresponding to Kapur[5],[6] measure of entropy

Consider the Kapur[5],[6] parametric measure of entropy
Let ,
$\phi(x) = x \ln x - 1/a\,(1+ax)\ln(1+ax) + x\ln(1+a) + 1/a\,(1+a)\ln(1+a) - \ln(1+a),\quad a \geq 0$ (64)

It is already seen in (32) and (33) that it satisfies conditions $\phi(x)$ is twice differentiable convex function and $\phi(1) = 0$, $\phi'(1) = 0$

∴ its measure of weighted directed divergence given by(38)

$D_{10}(P:Q,W) = \sum_{i=1}^{n} w_i \{p_i \ln(\frac{p_i}{q_i}) + (p_i - q_i)\ln(1+a) + \frac{(1+a)}{a}\ln(1+a)\, q_i - \frac{1}{a}(q_i + ap_i)\ln(1+\frac{ap_i}{q_i})\} + \sum_{i=1}^{n} w_i \{(1+ap_i)\ln(\frac{1+ap_i}{1+aq_i}) + \frac{(1+a)}{a}\ln(1+a)(1+aq_i) + a(p_i - q_i)\ln(1+a) - \frac{1}{a}[(1+aq_i)+a(1+ap_i)]\ln(1+a(\frac{1+ap_i}{1+aq_i}))\}$ (65)

$D_{10}(P:R,W) = \sum_{i=1}^{n} w_i \{p_i \ln(\frac{p_i}{r_i}) + (p_i - r_i)\ln(1+a) + \frac{(1+a)}{a}\ln(1+a)\, r_i - \frac{1}{a}(r_i + ap_i)\ln(1+\frac{ap_i}{r_i})\} + \sum_{i=1}^{n} w_i \{(1+ap_i)\ln(\frac{1+ap_i}{1+ar_i}) + \frac{(1+a)}{a}\ln(1+a)(1+ar_i) + a(p_i - r_i)\ln(1+a) - \frac{1}{a}[(1+ar_i)+a(1+ap_i)]\ln(1+a(\frac{1+ap_i}{1+ar_i}))$ (66)

I_{10} (P:Q:R.W) = D_{10}(P:Q,W) – D_{10}(P:R,W)

$= \sum_{i=1}^{n} w_i \{p_i \ln(\frac{r_i}{q_i}) + (1+a)(r_i - q_i)\ln(1+a) - \frac{1}{a}[(q_i + ap_i)\ln(1+\frac{ap_i}{q_i}) + (r_i + ap_i)\ln(1+\frac{ap_i}{r_i})] + (1+ap_i)\ln(\frac{1+ar_i}{1+aq_i}) + (1+\frac{1}{a})(1+a)\ln(1+a)(q_i - r_i) - \frac{1}{a}[(1+aq_i)+a(1+ap_i)]\ln(1\ a(\frac{1+ap_i}{1+aq_i})) - \frac{1}{a}[(1+ar_i)+a(1+ap_i)]\ln(1+a(\frac{1+ap_i}{1+ar_i}))\}$ (67)

I_{10} (P:Q:R.W) is the measure of weighted information improvement corresponding to Kapur [5],[6] measure of entropy.

5. New Two Parametric Measures

5.1 New Two Parametric Measure Of Weighted Directed Divergence And Weighted Information Improvement.

Now we will obtain the two parametric weighted directed divergence as well as the two parametric measure of weighted information improvement.

(I) Let,
$\phi(x) = x \ln x - a/b\,(1+ b/a.x)\ln(1 + b/a.x) + a/b\,(1+b/a)\ln(1+b/a) + x\ln(1+ b/a) - \ln(1+ b/a),$
$b \geq 0,\ a>0$ (68)
$\phi'(x) = \ln x - \ln(1+b/a.x) + \ln(1+b/a)$ (69)

$\phi''(x) = 1/x - b/a\ .\ 1/(1+b/a.x)$ (70)
Hence (68) is the twice differentiable convex function
$\phi(1) = 0$ and $\phi'(1) = 0$, (68) satisfied (13), (14) and (15)
∴ (12) gives the measure of weighted directed divergence as follows

$D_{12}(P:Q:W) = \sum_{i=1}^{n} w_i\, p_i \ln(\frac{p_i}{q_i}) + \sum_{i=1}^{n} w_i (p_i - q_i)\ln(1+\frac{b}{a}) + \frac{a}{b}(1+\frac{b}{a})\ln(1+\frac{b}{a})\sum_{i=1}^{n} w_i\, q_i - \frac{a}{b}\sum w_i (q_i + \frac{b}{a}p_i)\ln(1+\frac{b}{a}\frac{p_i}{q_i}),\ b \geq -1,\ a>0$ (71)

$D_{12}(P:R,W) = \sum_{i=1}^{n} w_i\, p_i \ln(\frac{p_i}{r_i}) + \sum_{i=1}^{n} w_i (p_i - r_i)\ln(1+\frac{b}{a}) + \frac{a}{b}(1+\frac{b}{a})\ln(1+\frac{b}{a})\sum_{i=1}^{n} w_i\, r_i - \frac{a}{b}\sum w_i (r_i + \frac{b}{a}p_i)\ln(1+\frac{b}{a}\frac{p_i}{r_i}),\ b \geq -1,\ a>0$ (72)

I_{12}(P:Q:R,W) = D_{12}(P:Q,W) – D_{12}(P:R,W)

$I_{12}(P:Q:R,W) = \sum_{i=1}^{n} w_i \{p_i \ln(\frac{r_i}{q_i}) + \ln(1+\frac{b}{a})(r_i - q_i) + \frac{a}{b}(1+\frac{b}{a})\ln(1+\frac{b}{a})(q_i - r_i) - \frac{a}{b}(q_i + \frac{b}{a}p_i)\ln(1+\frac{b}{a}\frac{p_i}{q_i}) + \frac{a}{b}(r_i + \frac{b}{a}p_i)\ln(1+\frac{b}{a}\frac{p_i}{r_i}),\ b \geq -1,\ a>0$ (73)

I_{12}(P:Q:R,W) is the two parametric measure of weighted information improvement

When a=1 and b=a reduces as,

∴ D_5(P:Q,W) = D_{11}(P:Q,W), D_5 (P:R,W) = D_{11} (P:Q,W)

And Naturally

I_5(P:Q:R,W) = I_{12}(P:Q:R,W) (79)
from (79)it is prove that I_{11}(P:Q:R,W) is corresponds to Kapur's[5],[6] measure at limit a=1 and b = a

International Journal of Scientific Research in Engineering and Management (IJSREM)
Volume: 09 Issue: 05 | May - 2025 SJIF Rating: 8.586 ISSN: 2582-3930

(II) Again here we will obtain new two parametric weighted directed divergence and two parametric measure of weighted information improvement

$\phi(x) = x \ln x - a/b\,(1+ b/a.x) \ln (1 + b/a.x) + a/b$

again (68) becomes

$(1+b/a)\ln (1+b/a) + x \ln (1+ b/a) - \ln (1+ b/a)$, $b\geq 0,\ a>0$

Hence (68) is the twice differentiable convex function

$\phi(1) = 0$ and $\phi'(1) = 0$, (68) satisfied (13), (14) and (15)

∴ (38) gives the measure of weighted directed divergence as follows

$$D_{13} (P:Q,W) =\sum_{i=1}^{n} w_i \{p_i \ln(\frac{p_i}{q_i}) + (p_i - q_i) \ln (1+\frac{b}{a}) + \frac{(a+b)}{b} \ln (1+\frac{b}{a})\, q_i - \frac{a}{b}(q_i + \frac{b}{a}p_i) \ln (1+ \frac{b}{a}\frac{p_i}{q_i})\} + \sum_{i=1}^{n} w_i \{(1+\frac{b}{a}p_i) \ln (\frac{a+bp_i}{a+bq_i}) + \frac{(a+b)}{b} \ln (1 + \frac{b}{a}) (1 + \frac{b}{a} q_i) + \frac{b}{a}(p_i - q_i) \ln (1+\frac{b}{a}) - \frac{a}{b}[(1+\frac{b}{a}q_i)+ \frac{b}{a} (1+\frac{b}{a}p_i)] \ln (1 + \frac{b}{a} (\frac{a+bp_i}{a+bq_i}))\} \quad (80)$$

$$D_{13}(P:R,W) = \sum_{i=1}^{n} w_i \{p_i \ln(\frac{p_i}{r_i}) + (p_i - r_i) \ln (1+\frac{b}{a}) + \frac{(a+b)}{b} \ln (1+\frac{b}{a})\, r_i - \frac{a}{b}(r_i + \frac{b}{a}p_i) \ln (1+ \frac{b}{a}\frac{p_i}{r_i})\} + \sum_{i=1}^{n} w_i \{(1+\frac{b}{a}p_i) \ln (\frac{a+bp_i}{a+br_i}) + \frac{(a+b)}{b} \ln (1 + \frac{b}{a}) (1 + \frac{b}{a} r_i) + \frac{b}{a}(p_i - r_i) \ln (1+\frac{b}{a}) - \frac{a}{b}[(1+\frac{b}{a}r_i)+ \frac{b}{a} (1+\frac{b}{a}p_i)] \ln (1 + \frac{b}{a} (\frac{a+bp_i}{a+br_i})) \quad (81)$$

$I_{13} (P:Q:R.W) = D_{10}(P:Q,W) - D_{10}(P:R,W)$

$$= \sum_{i=1}^{n} w_i \{p_i \ln (\frac{r_i}{q_i}) +(1+\frac{b}{a})(r_i - q_i) \ln (1+\frac{b}{a})-\frac{a}{b} [(q_i +\frac{b}{a} p_i) \ln (1+\frac{b/ap_i}{q^i}) +(r_i+\frac{b}{a}p_i)\ln(1+\frac{b/ap_i}{r_i})] + (1+\frac{b}{a}p_i) \ln (\frac{a+br_i}{a+bq_i}) + (1 + \frac{a}{b}) (1 + \frac{b}{a}) \ln (1 + \frac{b}{a}) (q_i - r_i) - \frac{a}{b}[(1+\frac{b}{a}q_i)+ \frac{b}{a} (1+\frac{b}{a}p_i)] \ln (1 + \frac{b}{a} (\frac{a+bp_i}{a+bq_i})) - \frac{a}{b}[(1+\frac{b}{a}r_i)+ \frac{b}{a} (1+\frac{b}{a}p_i)] \ln (1 + \frac{b}{a} (\frac{a+bp_i}{a+br_i}))\} \quad (82)$$

$I_{13}(P:Q:R,W)$ is the two parametric measure of weighted information improvement

When a=1 and b=a reduces as,

∴ $D_{10}(P:Q,W) = D_{13}(P:Q,W)$, $D_{10} (P:R,W) = D_{13} (P:Q,W)$

And Naturally

$I_{10}(P:Q:R,W) = I_{13}(P:Q:R,W)$ (83)

from (83) we can say $I_{13}(P:Q:R,W)$ is corresponds to Kapur's[5],[6] measure at limit a=1 and b = a

(III) Let

$$\phi(x) = x \ln x - \frac{a}{b} (1+\frac{b}{a}x) \ln (1+ \frac{b}{a}x) + \frac{a}{b} (1+\frac{b}{a}) \ln (1+\frac{b}{a})+x \ln (1+\frac{b}{a}x) \qquad b \geq -1,\ a\neq 0 \quad (84)$$

$$\phi''(x) = 1/x - 1/(1+\frac{b}{a}x)\,\frac{b}{a} \quad (85)$$

Hence (84) satisfied the condition (13), and (15) as $\phi(x)$ is twice differentiable convex function $\phi(1) = 0$,

according to the theorem 4.2 we can obtain two parametric measure of weighted directed divergence and is improvement measure so, first we find the measures by (38)

$$D_{14} (P:Q,W) = \sum_{i=1}^{n} w_i \{p_i \ln \frac{p_i}{q_i} - \frac{a}{b} (q_i +\frac{b}{a} p_i) \ln (1+\frac{b}{a}\frac{p_i}{q_i}) + \frac{a}{b}(1 + \frac{b}{a}) \ln (1+\frac{b}{a})\, q_i + p_i \ln (1+\frac{b}{a})+ (1+\frac{b}{a} p_i) \ln (\frac{(1+\frac{b}{a}p_i}{(1+\frac{b}{a}q_i})+ \frac{a}{b} (1+ \frac{b}{a}) \ln (1+\frac{b}{a}) (1+\frac{b}{a} q_i)+ (1+\frac{b}{a} p_i) \ln (1+\frac{b}{a}) - \frac{a}{b} [(1+\frac{b}{a}.q_i) +\frac{b}{a} (1+\frac{b}{a} p_i)] \ln (1+\frac{b}{a} (\frac{(1+\frac{b}{a}p_i}{(1+\frac{b}{a}q_i})]\} \quad (86)$$

$$D_{14}(P:R,W) =\sum_{i=1}^{n} w_i \{p_i \ln \frac{p_i}{r_i} - \frac{a}{b} (r_i +\frac{b}{a} p_i) \ln (1+\frac{b}{a}\frac{p_i}{r_i})+ \frac{a}{b}(1+ \frac{b}{a}) \ln (1+\frac{b}{a})\, r_i + p_i \ln (1+\frac{b}{a})+ (1+\frac{b}{a} p_i) \ln (\frac{(1+\frac{b}{a}p_i}{(1+\frac{b}{a}r_i})+ \frac{a}{b} (1+ \frac{b}{a}) \ln (1+\frac{b}{a}) (1+\frac{b}{a} r_i)+ (1+\frac{b}{a} p_i) \ln (1+\frac{b}{a}) - \frac{a}{b} [(1+\frac{b}{a}.r_i) +\frac{b}{a} (1+\frac{b}{a} p_i)]\ln (1+\frac{b}{a} (\frac{(1+\frac{b}{a}p_i}{(1+\frac{b}{a}r_i})]\} \quad (87)$$

$I_{14}(P:Q:R,W) = D_{13}(P:Q,W) - D_{13}(P:R,W)$

$$= \sum_{i=1}^{n} w_i \{p_i\ln \frac{r_i}{q_i} - \frac{a}{b} (r_i - q_i)\ln (\frac{(1+\frac{b}{a}\frac{p_i}{r_i}}{(1+\frac{b}{a}\frac{p_i}{q_i}}) + \frac{a}{b} (1+ \frac{b}{a}) \ln (1+\frac{b}{a}) (q_i - r_i) +(1+\frac{b}{a} p_i) \ln (\frac{(1+\frac{b}{a}r_i}{(1+\frac{b}{a}q_i}) + (1+\frac{b}{a}) (q_i - r_i)+ \frac{a}{b} [(1+\frac{b}{a}.r_i) +\frac{b}{a} (1+\frac{b}{a} p_i)]\ln (1+\frac{b}{a} (\frac{(1+\frac{b}{a}p_i}{(1+\frac{b}{a}r_i})] - \frac{a}{b} [(1+\frac{b}{a}.q_i) +\frac{b}{a} (1+\frac{b}{a} p_i)] \ln (1+\frac{b}{a} (\frac{(1+\frac{b}{a}p_i}{(1+\frac{b}{a}q_i})]\} \quad (88)$$

$I_{14}(P:Q:R,W)$ is the weighted information improvement of (80)

(IV) Now here we will obtain another two parametric measures of weighted directed divergence and measure of weighted information improvement

Take, $\phi(x) = x \ln x - \frac{a}{b}(1+\frac{b}{a}x) \ln (1+\frac{b}{a}x) + \frac{a}{b}(1+\frac{b}{a}) \ln (1+\frac{b}{a})$, $b \geq -1, a \neq 0$ (89)

$\phi'(x) = \ln x - \ln (1+\frac{b}{a}x)$ (90)

$\phi''(x) = 1/x - \frac{b}{a}\left(\frac{1}{1+\frac{b}{a}x}\right)$ (91)

Hence $\phi(x)$ is convex and twice differentiable function and $\phi(1) = 0$
∴ the two parametric measure of weighted directed divergence by using theorem 4.2

$D(P:Q:W) = \sum_{i=1}^{n} w_i \{p_i \ln \frac{p_i}{q_i} - \frac{a}{b}(q_i + \frac{b}{a} p_i) \ln (1+\frac{b}{a}.\frac{p_i}{q^i}) + \frac{a}{b}(1+\frac{b}{a}) \ln (1+\frac{b}{a})\, q_i + (1+\frac{b}{a} p_i) \ln \left(\frac{(1+\frac{b}{a} p_i}{(1+\frac{b}{a} q_i}\right) + \frac{a}{b}(1+\frac{b}{a}) \ln (1+\frac{b}{a})(1+\frac{b}{a} q_i) + -\frac{a}{b}[(1+\frac{b}{a}.q_i) + \frac{b}{a}(1+\frac{b}{a} p_i)] \ln (1+\frac{b}{a}\left(\frac{(1+\frac{b}{a} p_i}{(1+\frac{b}{a} q_i}\right)]\}$

6. Conclusion:-

1. All the outcomes weighted directed divergence measure and weighted information improvement measure of one parametric and two parameter mention in section three,four and five are equally important and open the further space of research, the corrected measures of weighted directed divergence as well as the measures of weighted information improvement are interconnected and out of which all probability distribution satisfying constraints.

2. D_1 (P:Q) has minimum value for distribution P Which outcomes are not equally important for this we can use the modified principal of minimum weighted cross entropy, by using this, out of all probability distribution satisfying given constraints, D_1 (P:Q,W) is minimize for distribution P

3. In the present paper we have given correct directed divergence, weighted directed divergence and information improvement measures for discrete random variable probability distribution and corresponding to this measure we can obtained probability distribution of continuous random variable

4. The term weighted directed divergence is also called as directed divergence with respected to W(x) or useful directed divergence or qualitative -quantitative measure of directed divergence.

By using all these weighted directed divergence we can obtain symmetric directed divergence of P and Q and measure of generalized information improvement.

5. Here we have not discussed here how to assign the weights partly because such an assignment will depend on the specific application we have in view. Thus the weights may depend on the penalty to be paid due to our being able to meet the demands, these may be due to different returns from different outcomes in stock market or the different losses caused by earthquakes or different intensities and so on.

7. **References**

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