

Algebraic and Spectral Decomposition in the Rectangular Setting: Towards a Drazin-Type Inverse

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Abstract—The Drazin inverse is a fundamental generalized inverse for square matrices and is closely related to the decomposition of a singular matrix into its invertible and nilpotent components. For a rectangular matrix, ordinary powers are not defined in general, and consequently the classical definition of the Drazin inverse cannot be applied directly. Rectangular extensions of the Drazin inverse have nevertheless been developed through compatible weight matrices and associated square products.

In this paper, we investigate an algebraic and spectral decomposition framework for a rectangular matrix $A \in \mathbb{C}^{m \times n}$ together with a compatible matrix $W \in \mathbb{C}^{n \times m}$. We define the associated square matrices

$$T = WA, \quad S = AW$$

and study their stable range and stable null-space decompositions. A common stabilization level

$$k = \max\{\text{ind}(WA), \text{ind}(AW)\}$$

is used to construct paired decompositions of the domain and codomain of A .

The fundamental intertwining identities

$$A(WA)^k = (AW)^k A$$

and

$$W(AW)^k = (WA)^k W$$

are established and used to prove that A maps the stable range of WA onto the stable range of AW . In particular, we prove that

$$A|_{\mathcal{R}((WA)^k)} : \mathcal{R}((WA)^k) \longrightarrow \mathcal{R}((AW)^k)$$

is an isomorphism. The nonzero spectral information of WA and AW is then related to the regular component of the rectangular mapping.

Based on the resulting decomposition, we introduce a decomposition-induced rectangular inverse which acts as the inverse of A on the stable regular component and vanishes on the complementary singular component. The proposed construction is explicitly distinguished from the established rectangular Drazin and weighted Drazin inverses. The purpose is to provide a structural algebraic and spectral interpretation of rectangular Drazin-type theory rather than to redefine the existing rectangular Drazin inverse.

Index Terms—Rectangular matrices, Drazin inverse, generalized inverse, weighted Drazin inverse, stable range, stable kernel, spectral decomposition, rectangular inverse, core-EP inverse.

I. INTRODUCTION

Generalized inverses constitute an important area of matrix analysis with applications in linear systems, optimization, numerical analysis, differential equations, statistics, and operator theory. The development of generalized inverses began with early work of Moore and Penrose and subsequently resulted in a broad theory for both square and rectangular matrices [1], [2], [4], [5].

For a square matrix $M \in \mathbb{C}^{n \times n}$, the Drazin inverse is particularly important when M is singular. It provides an inverse on the nonsingular part of the matrix and annihilates the nilpotent part. The theory is closely related to the stabilization of the sequences

$$\mathcal{R}(M) \supseteq \mathcal{R}(M^2) \supseteq \mathcal{R}(M^3) \supseteq \dots$$

and

$$\mathcal{N}(M) \subseteq \mathcal{N}(M^2) \subseteq \mathcal{N}(M^3) \subseteq \dots$$

The Drazin inverse and its algebraic properties were introduced by Drazin [3]. Standard treatments can be found in [4], [5].

For

$$k = \text{ind}(M),$$

the vector space admits the direct-sum decomposition

$$\mathbb{C}^n = \mathcal{R}(M^k) \oplus \mathcal{N}(M^k).$$

The restriction of M to $\mathcal{R}(M^k)$ is invertible, while its restriction to $\mathcal{N}(M^k)$ is nilpotent. This is the fundamental algebraic decomposition underlying the Drazin inverse.

The extension of these ideas to rectangular matrices presents a dimensional difficulty. Let

$$A \in \mathbb{C}^{m \times n}, \quad m \neq n.$$

Then A^2 is generally undefined. Therefore, the conventional definition of the Drazin inverse based on powers of A cannot be applied directly.

A fundamental solution to this problem was given by Cline and Greville, who introduced a Drazin inverse for rectangular matrices using a compatible matrix

$$W \in \mathbb{C}^{n \times m}$$

and the square products

$$AW \in \mathbb{C}^{m \times m}, \quad WA \in \mathbb{C}^{n \times n} \text{ [7].}$$

The weighted Drazin inverse and its algebraic properties were subsequently investigated in a number of works [8]–[12]. Integral representations and perturbation results have also been developed for weighted Drazin inverses [13], [14].

The theory has since expanded considerably. Weighted Drazin inverses have been used to define matrix relations and pre-orders [14], [15]. Coll, Lattanzi, and Thome introduced weighted G -Drazin inverses for rectangular matrices and studied an associated pre-order [16].

Other generalized inverses closely related to Drazin theory include the DMP inverse [17], weighted GDMP and GDR inverses [18], weak group inverses [19], and weighted core-EP inverses [20]. More recent research has extended weak group and Drazin-type constructions further, including minimal-rank weighted weak Drazin inverses [23] and the W -weighted m -weak group inverse [24].

These developments demonstrate that rectangular Drazin-type theory is already well established. Consequently, the aim of the present paper is not to introduce the first rectangular Drazin inverse. Instead, the objective is to study explicitly how the stable range–kernel decompositions of the two associated square products WA and AW induce a paired algebraic and spectral decomposition of the rectangular mapping A .

The central question is:

How does the stable decomposition of WA and AW induce a corresponding regular–singular decomposition of the rectangular transformation A ?

To answer this question, we introduce the common stabilization level

$$k = \max\{\text{ind}(WA), \text{ind}(AW)\}.$$

The main result establishes the isomorphism

$$A : \mathcal{R}((WA)^k) \longrightarrow \mathcal{R}((AW)^k).$$

This allows us to interpret A as consisting of a regular component, where it is invertible, and a singular component associated with the stable null spaces.

The main contributions of the paper are:

- 1) a paired stable range–kernel decomposition associated with WA and AW ;
- 2) a proof of the stable-range isomorphism induced by A ;
- 3) a spectral interpretation based on the common nonzero spectrum of AW and WA ;
- 4) a decomposition-induced rectangular inverse;
- 5) a precise distinction between established rectangular Drazin-type theory and the structural results developed here.

II. PRELIMINARIES

Throughout the paper,

$$A \in \mathbb{C}^{m \times n}, \quad W \in \mathbb{C}^{n \times m}.$$

Define

$$T = WA \in \mathbb{C}^{n \times n}$$

and

$$S = AW \in \mathbb{C}^{m \times m}.$$

For a matrix M , $\mathcal{R}(M)$ and $\mathcal{N}(M)$ denote its range and null space, respectively.

A. Index

Definition 1. Let $M \in \mathbb{C}^{q \times q}$. The index of M is

$$\text{ind}(M) = \min \{k \geq 0 : \text{rank}(M^{k+1}) = \text{rank}(M^k)\}.$$

The index measures the point at which the range sequence stabilizes.

B. Drazin Inverse

Definition 2. Let $M \in \mathbb{C}^{q \times q}$ and $k = \text{ind}(M)$. The Drazin inverse M^D is the unique matrix satisfying

$$\begin{aligned} M^{k+1}M^D &= M^k, \\ M^DMM^D &= M^D, \end{aligned}$$

and

$$MM^D = M^DM.$$

The Drazin inverse is a classical object [3]–[5].

C. Stable Range and Stable Null Space

Definition 3. Let $M \in \mathbb{C}^{q \times q}$ and let $k = \text{ind}(M)$. The stable range and stable null space of M are defined by

$$\mathcal{R}_\infty(M) = \mathcal{R}(M^k)$$

and

$$\mathcal{N}_\infty(M) = \mathcal{N}(M^k).$$

Theorem 1. Let $M \in \mathbb{C}^{q \times q}$ and $k = \text{ind}(M)$. Then

$$\mathbb{C}^q = \mathcal{R}(M^k) \oplus \mathcal{N}(M^k).$$

Moreover,

$$M|_{\mathcal{R}(M^k)}$$

is an isomorphism and

$$M|_{\mathcal{N}(M^k)}$$

is nilpotent.

Proof. Since

$$\text{rank}(M^{k+1}) = \text{rank}(M^k),$$

we have

$$\mathcal{R}(M^{k+1}) = \mathcal{R}(M^k).$$

Therefore, the restriction

$$M|_{\mathcal{R}(M^k)} : \mathcal{R}(M^k) \rightarrow \mathcal{R}(M^k)$$

is surjective and hence bijective in finite dimensions.

Furthermore,

$$\dim \mathcal{R}(M^k) + \dim \mathcal{N}(M^k) = q.$$

The intersection is trivial. Hence

$$\mathbb{C}^q = \mathcal{R}(M^k) \oplus \mathcal{N}(M^k).$$

If $x \in \mathcal{N}(M^k)$, then

$$M^kx = 0,$$

so the restriction of M to this subspace is nilpotent. $\square$

III. KNOWN RESULTS VERSUS RESULTS ESTABLISHED IN THIS PAPER

A clear distinction between established theory and the results of this paper is important because rectangular Drazin theory has already been developed in the literature.

A. Known Results

The following results are established in previous literature.

- 1) The Moore–Penrose inverse exists for every complex rectangular matrix [1], [2], [4].
- 2) The Drazin inverse and matrix index are classical concepts for square matrices [3], [5].
- 3) The decomposition

$$\mathbb{C}^n = \mathcal{R}(M^k) \oplus \mathcal{N}(M^k)$$

is a standard consequence of the Drazin index [3], [4].

- 4) Cline and Greville introduced a Drazin inverse for rectangular matrices [7].
- 5) Weighted Drazin inverses for rectangular matrices were developed and characterized in subsequent work [8]–[10].
- 6) Integral representations and perturbation properties of weighted Drazin inverses have been studied [11]–[13].
- 7) Weighted binary relations and pre-orders based on weighted Drazin inverses have been developed [14], [15].
- 8) Weighted G -Drazin inverses and corresponding pre-orders for rectangular matrices were introduced by Coll, Lattanzi, and Thome [16].
- 9) The DMP inverse was extended to rectangular matrices by Meng [17].
- 10) Weighted GDMP and GDR inverses were developed for rectangular matrices by Kumar, Shekhar, and Mishra [18].
- 11) Weighted core-EP inverses and their rectangular extension were studied by Ferreyra, Levis, and Thome [20].
- 12) A weak group inverse for rectangular matrices was developed using weighted core-EP theory [19].
- 13) Minimal-rank weighted weak Drazin inverses have recently been developed for rectangular matrices [23].
- 14) The W -weighted m -weak group inverse provides a further rectangular extension containing weighted Drazin-type inverses as special cases [24].

B. Results Established in This Paper

The following structural results are developed in the present paper.

- 1) We introduce the common stabilization level

$$k = \max\{\text{ind}(WA), \text{ind}(AW)\}.$$

- 2) We construct the paired decompositions

$$\mathbb{C}^n = \mathcal{R}((WA)^k) \oplus \mathcal{N}((WA)^k)$$

and

$$\mathbb{C}^m = \mathcal{R}((AW)^k) \oplus \mathcal{N}((AW)^k).$$

- 3) We establish the mapping relation

$$A\mathcal{R}((WA)^k) = \mathcal{R}((AW)^k).$$

- 4) We prove that

$$A|_{\mathcal{R}((WA)^k)}$$

is an isomorphism onto

$$\mathcal{R}((AW)^k).$$

- 5) We prove

$$A\mathcal{N}((WA)^k) \subseteq \mathcal{N}((AW)^k).$$

- 6) We formulate the associated regular–singular decomposition of the rectangular mapping.
- 7) We define a decomposition-induced rectangular inverse from this paired decomposition.

The contribution is therefore structural and decomposition-based, rather than a claim that the rectangular Drazin inverse itself is being introduced for the first time.

IV. FUNDAMENTAL INTERTWINING IDENTITIES

The following identities are fundamental.

Lemma 1. For every integer $k \geq 1$,

$$A(WA)^k = (AW)^k A$$

and

$$W(AW)^k = (WA)^k W.$$

Proof. For $k = 1$,

$$A(WA) = (AW)A$$

and

$$W(AW) = (WA)W.$$

Assume

$$A(WA)^k = (AW)^k A.$$

Then

$$\begin{aligned} A(WA)^{k+1} &= [A(WA)^k](WA) \\ &= (AW)^k A(WA) \\ &= (AW)^k (AW)A \\ &= (AW)^{k+1} A. \end{aligned}$$

The second identity follows similarly:

$$\begin{aligned} W(AW)^{k+1} &= [W(AW)^k](AW) \\ &= (WA)^k W(AW) \\ &= (WA)^k (WA)W \\ &= (WA)^{k+1} W. \end{aligned}$$

Thus both identities hold for all $k \geq 1$. $\square$

Corollary 1. For every $k \geq 1$,

$$A\mathcal{R}((WA)^k) \subseteq \mathcal{R}((AW)^k).$$

Proof. Let

$$x = (WA)^k z.$$

Then

$$Ax = A(WA)^k z = (AW)^k Az,$$

and hence

$$Ax \in \mathcal{R}((AW)^k).$$

□

V. PAIRED ALGEBRAIC DECOMPOSITION

Let

$$k = \max\{\text{ind}(WA), \text{ind}(AW)\}.$$

Define

$$V_r = \mathcal{R}((WA)^k), \quad V_s = \mathcal{N}((WA)^k),$$

and

$$U_r = \mathcal{R}((AW)^k), \quad U_s = \mathcal{N}((AW)^k).$$

Theorem 2. *The following direct-sum decompositions hold:*

$$\mathbb{C}^n = V_r \oplus V_s$$

and

$$\mathbb{C}^m = U_r \oplus U_s.$$

Proof. Since

$$k \geq \text{ind}(WA),$$

the range and null-space sequences of WA have stabilized at the k th power. Hence

$$\mathbb{C}^n = \mathcal{R}((WA)^k) \oplus \mathcal{N}((WA)^k).$$

Similarly,

$$k \geq \text{ind}(AW)$$

gives

$$\mathbb{C}^m = \mathcal{R}((AW)^k) \oplus \mathcal{N}((AW)^k).$$

Therefore,

$$\mathbb{C}^n = V_r \oplus V_s$$

and

$$\mathbb{C}^m = U_r \oplus U_s.$$

□

Proposition 1. *The rectangular mapping A satisfies*

$$A(V_r) \subseteq U_r$$

and

$$A(V_s) \subseteq U_s.$$

Proof. The first inclusion follows from the intertwining identity.

Let $x \in V_s$. Then

$$(WA)^k x = 0.$$

Using

$$(AW)^k A = A(WA)^k,$$

we obtain

$$(AW)^k Ax = A(WA)^k x = 0.$$

Therefore,

$$Ax \in \mathcal{N}((AW)^k) = U_s.$$

VI. STABLE-RANGE ISOMORPHISM THEOREM

We now establish the principal structural result.

Theorem 3 (Stable-Range Isomorphism). *Let*

$$A \in \mathbb{C}^{m \times n}, \quad W \in \mathbb{C}^{n \times m},$$

and

$$k = \max\{\text{ind}(WA), \text{ind}(AW)\}.$$

Then

$$A|_{V_r} : V_r \rightarrow U_r$$

is an isomorphism.

Proof. We first prove surjectivity.

Let

$$y \in U_r.$$

Then there exists $z \in \mathbb{C}^m$ such that

$$y = (AW)^k z.$$

For $k \geq 1$,

$$(AW)^k = A(WA)^{k-1}W.$$

Hence

$$y = A(WA)^{k-1}Wz.$$

Define

$$x = (WA)^{k-1}Wz.$$

Then

$$x \in \mathcal{R}((WA)^{k-1}).$$

Since k is a stabilization level for WA ,

$$\mathcal{R}((WA)^{k-1}) = \mathcal{R}((WA)^k).$$

Thus

$$x \in V_r$$

and

$$Ax = y.$$

Therefore,

$$A(V_r) = U_r.$$

We now prove injectivity.

Suppose

$$x \in V_r$$

and

$$Ax = 0.$$

Then

$$W Ax = 0,$$

so

$$(WA)x = 0.$$

However,

$$x \in \mathcal{R}((WA)^k),$$

and the restriction

$$WA|_{\mathcal{R}((WA)^k)}$$

□ is invertible because k is at least $\text{ind}(WA)$.

Therefore,

$$x = 0.$$

Hence the restriction

$$A|_{V_r} : V_r \rightarrow U_r$$

is both injective and surjective. Thus it is an isomorphism. $\square$

Corollary 2. *The stable regular components have equal dimension:*

$$\dim V_r = \dim U_r.$$

Proof. This follows immediately from the stable-range isomorphism. $\square$

VII. SPECTRAL DECOMPOSITION

The algebraic decomposition has a corresponding spectral interpretation.

Theorem 4. *Let*

$$A \in \mathbb{C}^{m \times n}, \quad W \in \mathbb{C}^{n \times m}.$$

Then

$$\sigma(AW) \setminus \{0\} = \sigma(WA) \setminus \{0\},$$

including algebraic multiplicities.

Proof. Let

$$\lambda \neq 0$$

be an eigenvalue of AW . Then there exists $y \neq 0$ such that

$$AWy = \lambda y.$$

If $Wy = 0$, then

$$AWy = 0,$$

contradicting

$$\lambda y \neq 0.$$

Hence

$$Wy \neq 0.$$

Applying W to the eigenvalue equation gives

$$WAWy = \lambda Wy.$$

Thus

$$WA(Wy) = \lambda(Wy),$$

so λ is an eigenvalue of WA .

The reverse implication follows by interchanging A and W .

Equality of algebraic multiplicities follows from the standard identity

$$\det(\lambda I_m - AW) = \lambda^{m-n} \det(\lambda I_n - WA)$$

for $\lambda \neq 0$, with the appropriate interpretation when m and n differ. $\square$

Define

$$\sigma_{\text{nz}}(M) = \sigma(M) \setminus \{0\}.$$

Then

$$\sigma_{\text{nz}}(AW) = \sigma_{\text{nz}}(WA).$$

The nonzero spectrum therefore corresponds to the regular components

$$V_r \quad \text{and} \quad U_r.$$

The zero spectral part corresponds to the nilpotent components contained in

$$V_s \quad \text{and} \quad U_s.$$

This gives the schematic decomposition

$$\begin{array}{ccc} \mathbb{C}^n & = & V_r \oplus V_s \\ \downarrow A & & \downarrow \\ \mathbb{C}^m & = & U_r \oplus U_s. \end{array}$$

The restriction

$$A : V_r \rightarrow U_r$$

is invertible, while

$$A : V_s \rightarrow U_s$$

represents the singular component.

VIII. DECOMPOSITION-INDUCED RECTANGULAR INVERSE

The preceding decomposition naturally suggests an inverse construction.

Definition 4. *Let*

$$A_r = A|_{V_r} : V_r \rightarrow U_r.$$

Since A_r is an isomorphism, define

$$A_r^{-1} : U_r \rightarrow V_r.$$

The decomposition-induced rectangular inverse associated with (A, W) is the mapping

$$A_{\text{dec}}^{-1} : \mathbb{C}^m \rightarrow \mathbb{C}^n$$

defined by

$$A_{\text{dec}}^{-1}u = \begin{cases} A_r^{-1}u, & u \in U_r, \\ 0, & u \in U_s. \end{cases}$$

Since

$$\mathbb{C}^m = U_r \oplus U_s,$$

every vector $u \in \mathbb{C}^m$ can be uniquely represented as

$$u = u_r + u_s,$$

where

$$u_r \in U_r, \quad u_s \in U_s.$$

Therefore,

$$A_{\text{dec}}^{-1}u = A_r^{-1}u_r.$$

Theorem 5. *The decomposition-induced inverse satisfies*

$$AA_{\text{dec}}^{-1} = P_{U_r}$$

and

$$A_{\text{dec}}^{-1}A = P_{V_r},$$

where P_{U_r} and P_{V_r} are the projections associated with

$$\mathbb{C}^m = U_r \oplus U_s$$

and

$$\mathbb{C}^n = V_r \oplus V_s.$$

Proof. Let

$$u = u_r + u_s.$$

Then

$$A_{\text{dec}}^{-1}u = A_r^{-1}u_r.$$

Consequently,

$$AA_{\text{dec}}^{-1}u = AA_r^{-1}u_r = u_r.$$

Hence

$$AA_{\text{dec}}^{-1} = P_{U_r}.$$

Similarly, let

$$x = x_r + x_s,$$

where

$$x_r \in V_r, \quad x_s \in V_s.$$

Since

$$Ax_r \in U_r$$

and

$$Ax_s \in U_s,$$

we obtain

$$A_{\text{dec}}^{-1}Ax = A_{\text{dec}}^{-1}(Ax_r + Ax_s) = x_r.$$

Thus

$$A_{\text{dec}}^{-1}A = P_{V_r}.$$

□

Corollary 3. *The decomposition-induced inverse is an outer inverse of A :*

$$A_{\text{dec}}^{-1}AA_{\text{dec}}^{-1} = A_{\text{dec}}^{-1}.$$

Proof. Since

$$A_{\text{dec}}^{-1}A = P_{V_r}$$

and

$$\mathcal{R}(A_{\text{dec}}^{-1}) \subseteq V_r,$$

we obtain

$$A_{\text{dec}}^{-1}AA_{\text{dec}}^{-1} = P_{V_r}A_{\text{dec}}^{-1} = A_{\text{dec}}^{-1}.$$

□

Remark 1. *The inverse*

$$A_{\text{dec}}^{-1}$$

should not automatically be identified with the W -weighted Drazin inverse. The latter is an established rectangular generalized inverse characterized by algebraic conditions involving A , W , and the associated Drazin inverse [7], [9], [15]. The present inverse is defined from the paired stable range–kernel decomposition.

Determining conditions under which these two constructions coincide is an important direction for future research.

IX. STRUCTURAL INTERPRETATION

The decomposition established above can be summarized by the following diagram:

$$\begin{array}{ccc} \mathbb{C}^n & = & V_r \oplus V_s \\ \downarrow A & & \downarrow \\ \mathbb{C}^m & = & U_r \oplus U_s. \end{array}$$

The regular components satisfy

$$A(V_r) = U_r$$

and

$$A|_{V_r} : V_r \rightarrow U_r$$

is an isomorphism.

The singular components satisfy

$$A(V_s) \subseteq U_s.$$

Therefore, with respect to bases adapted to these decompositions, the rectangular mapping has a block structure of the form

$$A \sim \begin{bmatrix} A_r & A_{rs} \\ 0 & A_s \end{bmatrix},$$

where

$$A_r : V_r \rightarrow U_r$$

is invertible and

$$A_s : V_s \rightarrow U_s$$

represents the singular component.

Under the decomposition-induced inverse, only the regular block is inverted:

$$A_{\text{dec}}^{-1} \sim \begin{bmatrix} A_r^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

This resembles the classical Drazin philosophy for square matrices, where the invertible part is inverted and the nilpotent part is annihilated [3], [5].

The difference is that in the rectangular setting the regular part is a mapping between two different spaces:

$$V_r \rightarrow U_r.$$

X. EXAMPLES

A. Example 1: Full Stable Domain

Consider

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{3 \times 2}$$

and

$$W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \in \mathbb{C}^{2 \times 3}.$$

Then

$$WA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

and

$$AW = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Both have index 1. Hence

$$k = 1.$$

We obtain

$$V_r = \mathbb{C}^2, \quad V_s = \{0\},$$

and

$$U_r = \text{span}\{e_1, e_2\}, \quad U_s = \text{span}\{e_3\}.$$

The restriction

$$A|_{V_r} : \mathbb{C}^2 \rightarrow \text{span}\{e_1, e_2\}$$

is an isomorphism.

Therefore,

$$A_{\text{dec}}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Indeed,

$$AA_{\text{dec}}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = P_{U_r}$$

and

$$A_{\text{dec}}^{-1}A = I_2 = P_{V_r}.$$

B. Example 2: A Nontrivial Singular Component

Consider

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$W = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then

$$WA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$AW = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus

$$\text{ind}(WA) = 1, \quad \text{ind}(AW) = 1.$$

Consequently,

$$k = 1.$$

The stable components are

$$V_r = \text{span}\{e_1, e_3\}, \quad V_s = \text{span}\{e_2\},$$

and

$$U_r = \mathbb{C}^2, \quad U_s = \{0\}.$$

The restriction

$$A|_{V_r} : V_r \rightarrow U_r$$

is an isomorphism.

The decomposition-induced inverse is

$$A_{\text{dec}}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

This example illustrates that the regular part may be lower-dimensional than the original domain.

XI. RELATION TO EXISTING RECTANGULAR DRAZIN-TYPE THEORY

The proposed framework is closely related to the established theory of rectangular generalized inverses.

Cline and Greville provided the foundational rectangular Drazin construction [7]. Their work establishes a unique matrix satisfying a system of identities involving a rectangular matrix, a compatible weight, and the corresponding square product.

The weighted Drazin inverse subsequently became an important tool for rectangular matrix theory. Integral representations and computational characterizations were investigated by Wei and others [9]–[11].

Weighted binary relations and pre-orders provide an algebraic framework for comparing rectangular matrices through weighted Drazin inverses [14], [15].

Weighted G -Drazin inverses extend the theory further and establish connections with other partial orders [16].

The DMP inverse combines Drazin-type information with the Moore–Penrose inverse [17]. Weighted GDMP and GDR inverses provide additional rectangular generalizations [18].

The core-EP approach provides another decomposition-oriented framework. Ferreyra, Levis, and Thome extended the core-EP concept to rectangular matrices through weighted core-EP decompositions [20]. Weak group inverses were subsequently considered for rectangular matrices [19].

More recent results include minimal-rank weighted weak Drazin inverses [23] and the W -weighted m -weak group inverse [24].

The present framework differs primarily in the starting point. Instead of defining another generalized inverse through a system of equations, we begin with the paired decompositions

$$\mathbb{C}^n = V_r \oplus V_s$$

and

$$\mathbb{C}^m = U_r \oplus U_s$$

generated by WA and AW .

The key structural relation is

$$A(V_r) = U_r.$$

Thus, the proposed construction explicitly identifies the regular component of the rectangular transformation.

XII. DISCUSSION

The rectangular setting requires two vector spaces:

$$\mathbb{C}^n \quad \text{and} \quad \mathbb{C}^m.$$

The products

$$WA : \mathbb{C}^n \rightarrow \mathbb{C}^n$$

and

$$AW : \mathbb{C}^m \rightarrow \mathbb{C}^m$$

provide square operators on these two spaces.

The common stabilization level

$$k = \max\{\text{ind}(WA), \text{ind}(AW)\}$$

allows both spaces to be decomposed simultaneously.

The resulting regular components

$$V_r = \mathcal{R}((WA)^k)$$

and

$$U_r = \mathcal{R}((AW)^k)$$

are linked by the original rectangular matrix A .

The theorem

$$A|_{V_r} : V_r \rightarrow U_r$$

being an isomorphism is therefore a natural rectangular analogue of the invertible component in the classical Drazin decomposition.

The spectral relation

$$\sigma(AW) \setminus \{0\} = \sigma(WA) \setminus \{0\}$$

supports this interpretation. The nonzero spectral information is shared between the two associated square products, whereas the zero spectral structure may differ because the dimensions of AW and WA may be different.

An important limitation is that the decomposition depends on the compatible matrix W . Hence the construction is not an intrinsic inverse of A alone. This dependence is consistent with the weighted rectangular generalized-inverse literature [7], [9], [15].

A significant research question is therefore to determine conditions on W under which

$$A_{\text{dec}}^{-1}$$

coincides with a known W -weighted Drazin-type inverse.

Another direction is to investigate whether a canonical choice of W can be obtained from the geometry of A , for example through rank-revealing factorizations, singular-value decompositions, or operator-theoretic constructions.

XIII. LIMITATIONS AND FUTURE WORK

The present work has several limitations.

First, the decomposition depends on the auxiliary matrix W . Different compatible matrices may generate different stable decompositions.

Second, although the decomposition-induced inverse satisfies outer-inverse properties, its equality with a particular weighted Drazin, DMP, GDMP, core-EP, or weak-group inverse has not been established in full generality.

Third, the present analysis is finite-dimensional and is carried out over $\mathbb{C}$. Extensions to bounded operators between different Banach or Hilbert spaces require additional functional-analytic assumptions.

Future research will address the following problems:

- 1) Determine necessary and sufficient conditions for

$$A_{\text{dec}}^{-1} = A^{D,W}.$$

- 2) Characterize the dependence of the decomposition on W .
- 3) Develop canonical choices of W .
- 4) Establish perturbation bounds for

$$A_{\text{dec}}^{-1}.$$

- 5) Investigate numerical algorithms for computing the stable components.
- 6) Extend the theory to operators between Banach and Hilbert spaces.
- 7) Study relationships with weighted core-EP, DMP, GDMP, and weak-group inverses.

XIV. CONCLUSION

This paper developed an algebraic and spectral decomposition framework for rectangular matrices using compatible square products.

Given

$$A \in \mathbb{C}^{m \times n}, \quad W \in \mathbb{C}^{n \times m},$$

we considered

$$WA \quad \text{and} \quad AW.$$

At the common stabilization level

$$k = \max\{\text{ind}(WA), \text{ind}(AW)\},$$

we obtained the paired decompositions

$$\mathbb{C}^n = \mathcal{R}((WA)^k) \oplus \mathcal{N}((WA)^k)$$

and

$$\mathbb{C}^m = \mathcal{R}((AW)^k) \oplus \mathcal{N}((AW)^k).$$

The central theorem proves that

$$A : \mathcal{R}((WA)^k) \longrightarrow \mathcal{R}((AW)^k)$$

is an isomorphism.

This identifies a regular component of the rectangular mapping on which A possesses a genuine inverse. The complementary component corresponds to the singular structure.

Based on this decomposition, we defined the decomposition-induced rectangular inverse

$$A_{\text{dec}}^{-1}.$$

This construction follows the fundamental philosophy of the Drazin inverse—invert the regular part and annihilate the singular part—while respecting the fact that a rectangular matrix maps between two different vector spaces.

The paper has explicitly distinguished these structural results from the established rectangular Drazin theory initiated by Cline and Greville and subsequently extended through weighted Drazin, weighted G -Drazin, DMP, core-EP, weak-group, and minimal-rank weighted weak Drazin constructions.

The framework provides a basis for further investigation of rectangular Drazin-type inverses and their algebraic and spectral decompositions.

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