

Invexity-Based Duality for Composite Vector Optimization over Cones

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Abstract

In this paper we associate a Wolfe type dual and a Mond–Weir type dual to a composite vector optimization problem over cones, and prove weak and strong duality results using the concepts of cone generalized invex, cone nonsmooth pseudoinvex and cone nonsmooth quasiinvex functions. The strong duality theorems rest on a Karush–Kuhn–Tucker type necessary optimality condition for the primal problem, established in a companion paper.

Keywords: Composite Vector Optimization · Cones · Duality · Cone-generalized invexity · Null space condition

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1 Introduction

The vector optimization problems in which the objective and/or constraint functions are formed as compositions of two or more functions are known as composite vector optimization problems. This structure arises naturally in several settings such as exact penalty reformulations of constrained optimization problems and problems where an outer function measures or orders the output of an inner mapping. The systematic study of composite optimization was initiated by Jeyakumar [5] in 1991. He took up a nonsmooth optimization problem where the objective function and the constraint functions are compositions of locally Lipschitz and Gâteaux differentiable functions and presented necessary and sufficient optimality conditions. Jeyakumar and Yang [6] considered multi-objective optimization problems where the objective and the constraints are compositions of convex and locally Lipschitz functions and Gâteaux differentiable functions and established optimality and duality results. Boş et al [1], [2] extended this to a convex, cone-ordered setting via Fenchel–Lagrangian duality, working with cone-increasing convex outer functions composed with cone-convex inner mappings. Tang and Zhao [12] later addressed a nonsmooth, cone-constrained multiobjective composite problem, introducing a generalized null space condition and cone-generalized invexity to obtain necessary and sufficient optimality conditions. More recently, Chuong [16], [17] developed optimality and duality theory for nonsmooth composite vector and conic vector optimization problems using limiting subdifferential calculus. Saadati and Oveisihah [18] extended the composite framework to uncertain (robust) multiobjective settings over Asplund spaces, formulating a Mond–Weir-type robust dual and establishing weak, strong, and converse duality.

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Duality plays a central role in the theory of vector optimization, both as a tool for deriving optimality conditions and as a means of bounding and computing optimal values. Two of the most widely used dual formulations in nonlinear and vector programming are the dual of Wolfe [15], originally proposed for convex programming problems and the dual of Mond and Weir [10], which relaxes convexity requirements at the price of an additional sign constraint on the multiplier term. Both constructions have since been extended extensively to nonsmooth, multiobjective and cone-ordered settings under various notions of generalized convexity and invexity.

In our previously published paper [8] we have considered a composite vector optimization problem (CVP) where both, the objective and the constraint function, are composition of two vector functions and we have established necessary optimality conditions for a point to be a weak minimum of this problem using the notion of cone-increasing functions. We have relate our composite problem to an auxiliary non-composite problem and prove the Fritz John type conditions and under a Slater-type constraint qualification we further obtain a Karush Kuhn Tucker type necessary condition. Moving a step ahead we associate a Wolfe type dual to the problem (CVP) and prove weak and strong duality results using cone generalized invex functions. We then formulate a Mond-Weir type dual and establish weak and strong duality results using cone nonsmooth pseudoinvex and cone nonsmooth quasiinvex functions.

2 Preliminaries

Now we recall some notations and definitions defined in [8]. Let $K \subseteq \mathbb{R}^m$ be a closed convex pointed ($K \cap (-K) = \{0\}$) cone such that $\text{int}K \neq \emptyset$, where $\text{int}K$ denotes the interior of K . The positive dual cone K^+ and strict positive dual cone K^{s+} of K are defined as follows:

$$K^+ = \{\lambda \in \mathbb{R}^m : \lambda^T x \geq 0, \forall x \in K\},$$

$$K^{s+} = \{\lambda \in \mathbb{R}^m : \lambda^T x > 0, \forall x \in K \setminus \{0\}\}.$$

In this section, we recall some of the basic definitions and results, which are to be used throughout the paper. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, where $f = (f_1, f_2, \dots, f_m)^T$ and $h : \mathbb{R}^n \rightarrow \mathbb{R}$.

The function h is said to be locally Lipschitz at $\bar{x} \in \mathbb{R}^n$ if there exists a nonnegative constant L and a neighborhood $N(\bar{x})$ of $\bar{x}$ such that for all $x, y \in N(\bar{x})$, we have

$$\|h(x) - h(y)\| \leq L\|x - y\|.$$

Also the function h is said to be locally Lipschitz on $\mathbb{R}^n$, if it is locally Lipschitz at every $\bar{x} \in \mathbb{R}^n$.

The vector-valued function f is locally Lipschitz at $\bar{x} \in \mathbb{R}^n$ if for each $i = 1, 2, \dots, m$, f_i is locally Lipschitz at $\bar{x} \in \mathbb{R}^n$ and f is called locally Lipschitz on $\mathbb{R}^n$, if it is locally Lipschitz at every $\bar{x} \in \mathbb{R}^n$.

Now we briefly describe the Clarke's [3] notion of generalized directional derivative and subdifferential of a locally Lipschitz function.

The Clarke generalized directional derivative of a locally Lipschitz function h at $\bar{x} \in \mathbb{R}^n$ in the direction $d \in \mathbb{R}^n$ is given as

$$h^\circ(\bar{x}; d) = \limsup_{y \rightarrow \bar{x}, t \rightarrow 0^+} \frac{h(y + td) - h(y)}{t},$$

where $y \in \mathbb{R}^n$ and $t > 0$.

The Clarke generalized gradient or the Clarke subdifferential of a locally Lipschitz function h at $\bar{x} \in \mathbb{R}^n$ is given as

$$\partial^c h(\bar{x}) = \{\xi \in \mathbb{R}^n : h^\circ(\bar{x}, d) \geq \xi^T d, \forall d \in \mathbb{R}^n\}.$$

Furthermore, $\partial^c h(\bar{x})$ is a nonempty convex compact subset of $\mathbb{R}^n$ and

$$h^\circ(\bar{x}; d) = \max \{\xi^T d : \xi \in \partial^c h(\bar{x})\}.$$

The Clarke generalized directional derivative of the vector-valued locally Lipschitz function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ at $\bar{x} \in \mathbb{R}^n$ in the direction $d \in \mathbb{R}^n$ is given by,

$$f^\circ(\bar{x}; d) = (f_1^\circ(\bar{x}; d), f_2^\circ(\bar{x}; d), \dots, f_m^\circ(\bar{x}; d))^T.$$

The Clarke generalized jacobian of a vector-valued locally Lipschitz function f at $\bar{x} \in \mathbb{R}^n$, denoted by $\partial^c f(\bar{x})$, is a nonempty convex compact subset of $\mathbb{R}^{m \times n}$, the space of $m \times n$ matrices and it is defined as

$$\partial^c f(\bar{x}) = \partial^c f_1(\bar{x}) \times \partial^c f_2(\bar{x}) \times \dots \times \partial^c f_m(\bar{x}),$$

where the latter denotes the set of all matrices whose i^{th} row belongs to the set $\partial^c f_i(\bar{x})$ for each $i = 1, 2, \dots, m$.

Definition 2.1. *The function f is said to be (P, K) -increasing (or nondecreasing on $\mathbb{R}^n$ with respect to (P, K) [9, Definition 4.1]), if for all $x, u \in \mathbb{R}^n$,*

$$x - u \in P \implies f(x) - f(u) \in K.$$

For all the below definitions we assume that f is a locally Lipschitz function and $\eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Definition 2.2. [14] *The function f is said to be K -generalized invex at $\bar{x} \in \mathbb{R}^n$, if there exists η such that for every $x \in \mathbb{R}^n$ and $A \in \partial^c f(\bar{x})$,*

$$f(x) - f(\bar{x}) - A\eta(x, \bar{x}) \in K.$$

Definition 2.3. [14] *The function f is said to be K -nonsmooth invex at $\bar{x} \in \mathbb{R}^n$, if there exists η such that for every $x \in \mathbb{R}^n$,*

$$f(x) - f(\bar{x}) - f^\circ(\bar{x}; \eta(x, \bar{x})) \in K.$$

Definition 2.4. [14] *The function f is said to be K -nonsmooth pseudoinvex at $\bar{x} \in \mathbb{R}^n$, if there exists η such that for every $x \in \mathbb{R}^n$,*

$$-f^\circ(\bar{x}; \eta(x, \bar{x})) \notin \text{int}K \implies -[f(x) - f(\bar{x})] \notin \text{int}K.$$

Definition 2.5. [14] *The function f is said to be K -nonsmooth quasiinvex at $\bar{x} \in \mathbb{R}^n$, if there exists η such that for every $x \in \mathbb{R}^n$,*

$$f(x) - f(\bar{x}) \notin \text{int}K \implies -f^\circ(\bar{x}; \eta(x, \bar{x})) \in K.$$

We recall the composite vector optimization problem studied in [8]:

$$\begin{aligned} \text{(CVP)} \quad & K\text{-Minimize } (f \circ F)(x) = ((f_1 \circ F)(x), (f_2 \circ F)(x), \dots, (f_m \circ F)(x))^T \\ & \text{subject to} \\ & -(g \circ G)(x) = -((g_1 \circ G)(x), (g_2 \circ G)(x), \dots, (g_p \circ G)(x))^T \in Q, \end{aligned}$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}^p$ are locally Lipschitz vector-valued functions such that $f = (f_1, f_2, \dots, f_m)^T$, $g = (g_1, g_2, \dots, g_p)^T$. $F : \mathbb{R}^s \rightarrow \mathbb{R}^n$ and $G : \mathbb{R}^s \rightarrow \mathbb{R}^n$ are differentiable functions such that $F = (F_1, F_2, \dots, F_n)^T$ and $G = (G_1, G_2, \dots, G_n)^T$. $K \subseteq \mathbb{R}^m$ and $Q \subseteq \mathbb{R}^p$ are closed convex pointed cones with nonempty interiors. The feasible set of (CVP) is given by

$$S_0 = \{x \in S : -(g \circ G)(x) \in Q\}.$$

Definition 2.6. A point $\bar{x} \in S_0$ is called a

1. weak minimum of (CVP), if for all $x \in S_0$,

$$(f \circ F)(\bar{x}) - (f \circ F)(x) \notin \text{int}K$$

2. minimum of (CVP), if for all $x \in S_0$,

$$(f \circ F)(\bar{x}) - (f \circ F)(x) \notin K \setminus \{0\}$$

Definition 2.7. (Slater-type Constraint Qualification) The function g is said to satisfy Slater-type Constraint Qualification at $\bar{x}$ if for every $B \in \partial^c g(G(\bar{x}))$, there exists $x^* \in \mathbb{R}^s$ such that $-B\nabla G(\bar{x})x^* \in \text{int}Q$.

Now we recall the following necessary optimality condition, established in our previously published paper [8].

Theorem 2.8. Let $\bar{x}$ be a weak minimum of (CVP). If f is (P_1, K) -increasing, g is (P_2, Q) -increasing and g satisfies Slater-type Constraint Qualification at $\bar{x}$, then there exist $\bar{\lambda} \in K^+ \setminus \{0\}$ and $\bar{\mu} \in Q^+$, such that

$$0 \in \nabla F(\bar{x})^T \cdot \partial^c(\bar{\lambda}^T f)(F(\bar{x})) + \nabla G(\bar{x})^T \cdot \partial^c(\bar{\mu}^T g)(G(\bar{x})) \quad (1)$$

$$(\bar{\mu}^T g)(G(\bar{x})) = 0. \quad (2)$$

3 Duality Results

In order to prove duality results we define the following concept of η -generalized null space condition (η -GNC) which is a particular case of the one introduced by Tang and Zhao [12].

For each $x, u \in \mathbb{R}^s$, there exist real constants $\alpha(x, u) > 0$, $\beta(x, u) > 0$ and $\gamma(x, u) \in \mathbb{R}^s$ such that

$$\eta(F(x), F(u)) = \alpha(x, u)\nabla F(u)\gamma(x, u),$$

$$\eta(G(x), G(u)) = \beta(x, u)\nabla G(u)\gamma(x, u).$$

3.1 Wolfe Type Dual

We associate the following Wolfe type Dual to the problem (CVP):

$$\begin{aligned}
(\text{CWD}) \quad & K\text{-Maximize } (f \circ F)(u) + \mu^T(g \circ G)(u)k \\
& \text{subject to} \\
& 0 \in \nabla F(u)^T \cdot \partial^c(\lambda^T f)(F(u)) + \nabla G(u)^T \cdot \partial^c(\mu^T g)(G(u)) \\
& \lambda^T k = 1. \\
& \text{where } u \in \mathbb{R}^s, \lambda \in K^+ \setminus \{0\}, \mu \in Q^+ \text{ and } k \in \text{int}K \text{ is arbitrary but fixed} \\
& \text{vector.}
\end{aligned}$$

Theorem 3.1 (Weak Duality). *Let x be feasible for (CVP) and (u, λ, μ) be feasible for (CWD). Suppose there is a function $\eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that f is K -generalized invex at $F(u)$ and g is Q -generalized invex at $G(u)$ with respect to η . Also suppose there exist $\alpha(x, u) > 0$ and $\gamma(x, u) \in \mathbb{R}^s$ such that η -GNC holds with $\beta(x, u) = \alpha(x, u)$. Then*

$$(f \circ F)(u) + \mu^T(g \circ G)(u)k - (f \circ F)(x) \notin \text{int}K.$$

Proof. Let, if possible,

$$(f \circ F)(u) + \mu^T(g \circ G)(u)k - (f \circ F)(x) \in \text{int}K,$$

then by using the fact that $\lambda \in K^+ \setminus \{0\}$, we have

$$(\lambda^T f)(F(u)) + (\mu^T g)(G(u))(\lambda^T k) - (\lambda^T f)(F(x)) > 0,$$

or,

$$(\lambda^T f)(F(u)) + (\mu^T g)(G(u)) - (\lambda^T f)(F(x)) > 0. \quad (3)$$

Since (u, λ, μ) is feasible for (CWD), therefore

$$0 \in \nabla F(u)^T \cdot \partial^c(\lambda^T f)(F(u)) + \nabla G(u)^T \cdot \partial^c(\mu^T g)(G(u))$$

and hence there exists $\bar{\xi} \in \partial^c(\lambda^T f)(F(u))$ and $\bar{\zeta} \in \partial^c(\mu^T g)(G(u))$, such that

$$\nabla F(u)^T \bar{\xi} + \nabla G(u)^T \bar{\zeta} = 0. \quad (4)$$

From the chain rule for subdifferentials in [3], we have

$$\partial^c(\lambda^T f)(F(u)) = \partial^c f(F(u))^T \lambda = \{A^T \lambda : A \in \partial^c f(F(u))\},$$

and

$$\partial^c(\mu^T g)(G(u)) = \partial^c g(G(u))^T \mu = \{B^T \mu : B \in \partial^c g(G(u))\}.$$

Hence we can say that there exist $\bar{A} \in \partial^c f(F(u))$ and $\bar{B} \in \partial^c g(G(u))$, such that

$$\bar{\xi} = \bar{A}^T \lambda \text{ and } \bar{\zeta} = \bar{B}^T \mu. \quad (5)$$

Due to the fact that g is Q -generalized invex at $G(u)$ with respect to η , we have

$$g(G(x)) - g(G(u)) - B\eta(G(x), G(u)) \in Q \text{ for all } B \in \partial^c g(G(u)),$$

i.e.,

$$g(G(x)) - g(G(u)) - \bar{B}\eta(G(x), G(u)) \in Q.$$

Since x is feasible for (CVP), we have

$$-g(G(x)) \in Q.$$

Adding above two relations and using that $\mu \in Q^+$, we get

$$-(\mu^T g)(G(u)) - (\mu^T \bar{B})\eta(G(x), G(u)) \geq 0,$$

which on using (5) gives

$$-(\mu^T g)(G(u)) - \bar{\zeta}^T \eta(G(x), G(u)) \geq 0. \quad (6)$$

Now, by η -GNC, there exist $\alpha(x, u) > 0$, $\beta(x, u) = \alpha(x, u) > 0$ and $\gamma(x, u) \in \mathbb{R}^s$ such that

$$\begin{aligned} \eta(F(x), F(u)) &= \alpha(x, u) \nabla F(u) \gamma(x, u), \\ \eta(G(x), G(u)) &= \alpha(x, u) \nabla G(u) \gamma(x, u). \end{aligned}$$

This implies that

$$\bar{\xi}^T \eta(F(x), F(u)) + \bar{\zeta}^T \eta(G(x), G(u)) = \alpha(x, u) \left[\bar{\xi}^T \nabla F(u) + \bar{\zeta}^T \nabla G(u) \right] \gamma(x, u) = 0.$$

Using the above equality in (6) we get

$$-(\mu^T g)(G(u)) + \bar{\xi}^T \eta(F(x), F(u)) \geq 0. \quad (7)$$

Since f is K -generalized invex at $F(u)$ with respect to η , therefore we have

$$f(F(x)) - f(F(u)) - A\eta(F(x), F(u)) \in K \quad \text{for all } A \in \partial^c f(F(u)),$$

or,

$$f(F(x)) - f(F(u)) - \bar{A}\eta(F(x), F(u)) \in K.$$

Now using $\lambda \in K^+ \setminus \{0\}$, we have

$$(\lambda^T f)(F(x)) - (\lambda^T f)(F(u)) - (\lambda^T \bar{A})\eta(F(x), F(u)) \geq 0.$$

Again using (5) in above inequality, we get

$$(\lambda^T f)(F(x)) - (\lambda^T f)(F(u)) - \bar{\xi}^T \eta(F(x), F(u)) \geq 0.$$

Adding (7) and above inequality, we get

$$(\lambda^T f)(F(x)) - (\lambda^T f)(F(u)) - (\mu^T g)(G(u)) \geq 0$$

or,

$$(\lambda^T f)(F(u)) + (\mu^T g)(G(u)) - (\lambda^T f)(F(x)) \leq 0,$$

which is a contradiction to (3). Hence

$$(f \circ F)(u) + \mu^T(g \circ G)(u) - (f \circ F)(x) \notin \text{int}K.$$

□

Theorem 3.2 (Strong Duality). *Let $\bar{x} \in S_0$ be a weak minimum of (CVP). If f is (P_1, K) -increasing, g is (P_2, Q) -increasing and g satisfies Slater-type Constraint Qualification at $\bar{x}$, then there exist $\bar{\lambda} \in K^+ \setminus \{0\}$ and $\bar{\mu} \in Q^+$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is feasible for (CWD). Suppose conditions of Weak Duality Theorem 3.1 are satisfied for each feasible solution (u, λ, μ) of (CWD), then $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is weak maximum for (CWD).*

Proof. Since $\bar{x} \in S_0$ is a weak minimum of (CVP), f is (P_1, K) -increasing, g is (P_2, Q) -increasing and g satisfies Slater-type Constraint Qualification at $\bar{x}$, therefore by Theorem 2.8 there exist $\bar{\lambda} \in K^+ \setminus \{0\}$ and $\bar{\mu} \in Q^+$ such that (1) and (2) hold and hence $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is feasible for (CWD).

Now let, if possible, $(\bar{x}, \bar{\lambda}, \bar{\mu})$ be not a weak maximum for (CWD). Then there exists (u, λ, μ) feasible for (CWD) such that

$$(f \circ F)(u) + \mu^T(g \circ G)(u)k - (f \circ F)(\bar{x}) + \mu^T(g \circ G)(\bar{x})k \in \text{int}K.$$

Using (2), we get

$$(f \circ F)(u) + \mu^T(g \circ G)(u)k - (f \circ F)(\bar{x}) \in \text{int}K$$

which is a contradiction to Weak Duality Theorem 3.1. Hence $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is weak maximum for (CWD). $\square$

3.2 Mond-Weir Type Dual

Consider the following Mond-Weir Dual to the problem (CVP):

$$\begin{aligned} \text{(CMD)} \quad & K\text{-Maximize } (f \circ F)(u) \\ & \text{subject to} \\ & 0 \in \partial^c(\lambda^T f)(F(u)) \cdot \nabla F(u) + \partial^c(\mu^T g)(G(u)) \cdot \nabla G(u) \\ & (\mu^T g)(G(u)) \geq 0, \\ & \text{where } u \in \mathbb{R}^s, \lambda \in K^+ \setminus \{0\}, \mu \in Q^+. \end{aligned}$$

Theorem 3.3 (Weak Duality). *Let x be feasible for (CVP) and (u, λ, μ) be feasible for (CMD). Suppose there is a function $\eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that f is K -nonsmooth pseudoinvex at $F(u)$, g is Q -nonsmooth quasinvex at $G(u)$ with respect to η and η -GNC holds. Then*

$$(f \circ F)(u) - (f \circ F)(x) \notin \text{int}K.$$

Proof. Since (u, λ, μ) is feasible for (CMD), therefore

$$0 \in \partial^c(\lambda^T f)(F(u)) \cdot \nabla F(u) + \partial^c(\mu^T g)(G(u)) \cdot \nabla G(u)$$

and hence there exists $\bar{\xi} \in \partial^c(\lambda^T f)(F(u))$ and $\bar{\zeta} \in \partial^c(\mu^T g)(G(u))$, such that

$$\bar{\xi}^T \nabla F(u) + \bar{\zeta}^T \nabla G(u) = 0. \tag{8}$$

Again recalling (5), we can say that there exist $\bar{A} \in \partial^c f(F(u))$ and $\bar{B} \in \partial^c g(G(u))$, such that

$$\bar{\xi} = \bar{A}^T \lambda \text{ and } \bar{\zeta} = \bar{B}^T \mu.$$

Now let, if possible,

$$(f \circ F)(u) - (f \circ F)(x) \in \text{int}K,$$

that is,

$$-(f(F(x)) - f(F(u))) \in \text{int}K.$$

Since f is K -nonsmooth pseudoinvex at $F(u)$ with respect to η , therefore we have

$$-f^\circ(F(u); \eta(F(x), F(u))) \in \text{int}K.$$

Using $\lambda \in K^+ \setminus \{0\}$, we get

$$\lambda^T f^\circ(F(u); \eta(F(x), F(u))) < 0,$$

which implies that

$$(\lambda^T A) \eta(F(x), F(u)) < 0 \quad \text{for all } A \in \partial^c f(F(u)),$$

i.e.,

$$(\lambda^T \bar{A}) \eta(F(x), F(u)) < 0. \quad (9)$$

On using (5) in the above inequality, we get

$$\bar{\xi}^T \eta(F(x), F(u)) < 0. \quad (10)$$

Now, by η -GNC, there exist $\alpha(x, u) > 0$, $\beta(x, u) > 0$ and $\gamma(x, u) \in \mathbb{R}^s$ such that

$$\eta(F(x), F(u)) = \alpha(x, u) \nabla F(u) \gamma(x, u),$$

$$\eta(G(x), G(u)) = \beta(x, u) \nabla G(u) \gamma(x, u).$$

This implies that

$$\frac{1}{\alpha(x, u)} \bar{\xi}^T \eta(F(x), F(u)) + \frac{1}{\beta(x, u)} \bar{\xi}^T \eta(G(x), G(u)) = [\bar{\xi}^T \nabla F(u) + \bar{\xi}^T \nabla G(u)] \gamma(x, u) = 0.$$

On using inequality (10) and the fact that $\alpha(x, u) > 0$ and $\beta(x, u) > 0$ in above equality, we get

$$\bar{\xi}^T \eta(G(x), G(u)) > 0 \quad (11)$$

Now due to the fact that (u, λ, μ) is feasible for (CMD), we have $(\mu^T g)(G(u)) \geq 0$. Also since x is feasible for (CVP) and $\mu \in Q^+$, we have $(\mu^T g)(G(x)) \leq 0$. Combining the two we have

$$(\mu^T g)(G(x)) - (\mu^T g)(G(u)) \leq 0.$$

If $\mu \neq 0$, then

$$g(G(x)) - g(G(u)) \notin \text{int}Q.$$

Since g is Q -nonsmooth quasiinvex at $G(u)$ with respect to η , we have

$$-g^\circ(G(u); \eta(G(x), G(u))) \in Q.$$

On using the fact that $\mu \in Q^+$, we get

$$\mu^T g^\circ(G(u); \eta(G(x), G(u))) \leq 0,$$

which implies that

$$(\mu^T B)\eta(G(x), G(u)) \leq 0 \quad \text{for all } B \in \partial^c g(G(u)),$$

or,

$$(\mu^T \bar{B})\eta(G(x), G(u)) \leq 0 \tag{12}$$

This is also satisfied when $\mu = 0$.

Now again by using (5) in above inequality, we get

$$\bar{\zeta}^T \eta(G(x), G(u)) \leq 0 \tag{13}$$

which is a contradiction to (11). Hence

$$(f \circ F)(u) - (f \circ F)(x) \notin \text{int}K.$$

□

Theorem 3.4 (Strong Duality). *Let $\bar{x} \in S_0$ be a weak minimum of (CVP). If f is (P_1, K) -increasing, g is (P_2, Q) -increasing and g satisfies Slater-type Constraint Qualification at $\bar{x}$, then there exist $\bar{\lambda} \in K^+ \setminus \{0\}$ and $\bar{\mu} \in Q^+$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is feasible for (CMD). Suppose conditions of Weak Duality Theorem 3.3 are satisfied for each feasible solution (u, λ, μ) of (CMD), then $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is weak maximum for (CMD).*

Proof. Since $\bar{x} \in S_0$ is a weak minimum of (CVP), f is (P_1, K) -increasing, g is (P_2, Q) -increasing and g satisfies Slater-type Constraint Qualification at $\bar{x}$, therefore by Theorem 2.8 there exist $\bar{\lambda} \in K^+ \setminus \{0\}$ and $\bar{\mu} \in Q^+$ such that (1) and (2) hold and hence $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is feasible for (CMD).

Now let, if possible, $(\bar{x}, \bar{\lambda}, \bar{\mu})$ be not a weak maximum for (CWD). Then there exists (u, λ, μ) feasible for (CMD) such that

$$(f \circ F)(u) - (f \circ F)(\bar{x}) \in \text{int}K$$

which is a contradiction to Weak Duality Theorem 3.3. Hence $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is weak maximum for (CMD). □

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